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Weak convergence theorems for two asymptotically quasi-nonexpansive non-self mappings in uniformly convex Banach spaces

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Abstract

The purpose of this paper is to establish some weak convergence theorems of modified two-step iteration process with errors for two asymptotically quasi-nonexpansive non-self mappings in the setting of real uniformly convex Banach spaces if E satisfies Opial's condition or the dual E^* of E has the Kedec-Klee property. Our results extend and improve some known corresponding results from the existing literature.

Keywords: Asymptotically quasi-nonexpansive non-self mappings, common fixed point, the modified two-step iteration process with errors for non-self maps, uniformly convex Banach space, weak convergence.
2010 MSC: 47H09, 47H10, 47J25.

1. Introduction

Let K be a nonempty subset of a real Banach space E. Let $T\colon K \to K$ be a mapping, then we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of two mappings S and T will be denoted by $F = F(S) \cap F(T)$. A mapping $T\colon K \to K$ is said to be:

(i) asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \;\; \leq \;\; k_n \|x - y\| \tag{1.1}$$

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for all x, $y \in K$ and $n \geq 1$,

(ii) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \in [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - p\| \leq k_n \|x - p\| \tag{1.2}$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$,

(iii) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\| \tag{1.3}$$

for all x, $y \in K$ and $n \geq 1$.

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [8] as an important generalization of the class of nonexpansive maps (i.e., mappings $T: K \to K$ such that $\|Tx - Ty\| \leq \|x - y\|$, $\forall\, x, y \in K$) who proved that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

Iterative techniques for approximating fixed points of nonexpansive mappings and asymptotically nonexpansive mappings have been studied by various authors (see e.g., [2]-[4],[7],[10],[12],[15]-[18], [20]-[25]) using the Mann iteration method (see e.g.,[13])/the modified Mann iteration method or the Ishikawa iteration method (see e.g.,[9])/the modified Ishikawa iteration method. (See, also [19] and [26]).

In 1978, Bose [1] proved that if K is a bounded closed convex nonempty subset of a uniformly convex Banach space E satisfying Opial's [14] condition and $T: K \to K$ is an asymptotically nonexpansive mapping, then the sequence $\{T^n x\}$ converges weakly to a fixed point of T provided T is asymptotically regular at $x \in K$, i.e., $\lim_{n\to\infty} \|T^n x - T^{n+1} x\| = 0$. Passty [16] and also Xu [27] proved that the requirement that E satisfies Opial's condition can be replaced by the condition that E has a Frechet differentiable norm. Furthermore, Tan and Xu [22, 23] later proved that the asymptotic regularity of T can be weakened to the weakly asymptotic regularity of T at x, i.e., $\omega - \lim_{n\to\infty} \|T^n x - T^{n+1} x\| = 0$.

In [20, 21], Schu introduced a modified Mann process to approximate fixed points of asymptotically nonexpansive self-maps defined on nonempty closed convex and bounded subset of a Hilbert space H. In 1994, Rhoades [18] extended the Schu's result to uniformly convex Banach space using a modified Ishikawa iteration scheme.

In all the above results, the operator T remains a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space E. If, however, the domain of T, $D(T)$, is a proper subset of E (and this is the case in several applications), and T maps $D(T)$ into E, then the iteration processes of Mann and Ishikawa studied by these authors; and their modifications introduced by Schu may fail to be well defined.

The aim of this paper is to establish some weak convergence theorems for two asymptotically quasi-nonexpansive non-self mappings in the framework of real uniformly convex Banach spaces. Our results extend, improve and unify some known corresponding results from the existing literature.

2. Preliminaries

Let E be a real normed linear space. The modulus of convexity of E is the function $\delta_E \colon (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) \;=\; \inf\Big\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1,\, \epsilon = \|x-y\|\Big\}$$

E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$.

A subset K of E is said to be a retract of E if there exists a continuous map $P \colon E \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P \colon E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map P is a retraction, then $Py = y$ for all y in the range of P.

Definition 2.1. Let E be a real normed linear space, K a nonempty subset of E. Let $P \colon E \to K$ be the nonexpansive retraction of E onto K. A map $T \colon K \to E$ is said to be:

(i) asymptotically nonexpansive [5] if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that for all $x, y \in K$, the following inequality holds:

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \;\le\; k_n\|x-y\|,\ \forall\, n \ge 1, \tag{2.1}$$

(ii) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that for all $x, y \in K$ and $x^* \in F(T)$, the following inequality holds:

$$\|T(PT)^{n-1}x - T(PT)^{n-1}x^*\| \;\le\; k_n\|x-x^*\|,\ \forall\, n \ge 1, \tag{2.2}$$

(iii) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in K$,

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \;\le\; L\|x-y\|,\ \forall\, n \ge 1. \tag{2.3}$$

Let K be a nonempty closed convex subset of a uniformly convex Banach space E. The iteration scheme: $x_1 \in K$ and

$$\begin{aligned} x_{n+1} &= P(a_n x_n + b_n T_1(PT_1)^{n-1}y_n + c_n l_n),\ \forall\, n \ge 1,\\ y_n &= P(\bar{a}_n x_n + \bar{b}_n T_2(PT_2)^{n-1}x_n + \bar{c}_n m_n),\ \forall\, n \ge 1, \end{aligned} \tag{2.4}$$

where $l_n, m_n \in K$ and $\{l_n\}_{n=1}^{\infty}$, $\{m_n\}_{n=1}^{\infty}$ are bounded, $a_n+b_n+c_n = 1 = \bar{a}_n+\bar{b}_n+\bar{c}_n$, $0 \le a_n, b_n, c_n, \bar{a}_n, \bar{b}_n, \bar{c}_n \le 1$, for all $n \ge 1$, $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \bar{c}_n < \infty$, and P is as in definition 2.1, is called modified Ishikawa iteration scheme with errors in the sense of Xu [28] for two mappings.

Remark 2.2. If T is a self map, then P becomes the identity map so that (2.1), (2.2) and (2.3) coincide with (1.1), (1.2) and (1.3) respectively. Moreover, iteration scheme (2.4) reduces to the modified Ishikawa iteration scheme with errors.

Now, we study the iteration scheme which is independent of (2.4) is as follows:

$$\begin{aligned} x_{n+1} &= P(a_n T_1(PT_1)^{n-1}x_n + b_n T_2(PT_2)^{n-1}y_n + c_n l_n),\ \forall\, n \ge 1,\\ y_n &= P(\bar{a}_n x_n + \bar{b}_n T_1(PT_1)^{n-1}x_n + \bar{c}_n m_n),\ \forall\, n \ge 1, \end{aligned} \tag{2.5}$$

where $l_n, m_n \in K$ and $\{l_n\}_{n=1}^{\infty}$, $\{m_n\}_{n=1}^{\infty}$ are bounded, $a_n+b_n+c_n = 1 = \bar{a}_n+\bar{b}_n+\bar{c}_n$, $0 \le a_n, b_n, c_n, \bar{a}_n, \bar{b}_n, \bar{c}_n \le 1$, for all $n \ge 1$, $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \bar{c}_n < \infty$, and P is as in definition 2.1, is called modified Ishikawa type iteration scheme with errors in the sense of Xu [28] for two mappings.

In the sequel, we shall need the following lemmas.

Lemma 2.3. *(See [24]). Let $\{r_n\}$,$\{s_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying*

$$r_{n+1} \leq (1+s_n)r_n + t_n, \ \forall\, n \geq 1.$$

If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} r_n$ exists. In particular, if $\{r_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} r_n = 0$.

Lemma 2.4. *(See [21]). Let E be a uniformly convex Banach space and $0 < a \leq t_n \leq b < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in E satisfying $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$, $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.5. *(See [11]) Let E be a real reflexive Banach space with its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $p,\, q \in w_w(x_n)$ (where $w_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$). Suppose $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$. Then $p = q$.*

Lemma 2.6. *(See [2]) Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing continuous convex function $\phi\colon [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for any Lipschitzian mapping $T\colon K \to E$ with the Lipschitz constant $L \geq 1$, and element $\{x_j\}_{j=1}^{n}$ in K and any nonnegative number $\{t_j\}_{j=1}^{n}$ with $\sum_{j=1}^{n} t_j = 1$, the following inequality holds:*

$$\|T\Big(\sum_{j=1}^{n} t_j x_j\Big) - \sum_{j=1}^{n} t_j T x_j\| \leq L\phi^{-1}\Big\{\max_{1\leq j,\, k\leq n}(\|x_j - x_k\| - \frac{1}{L}\|Tx_j - Tx_k\|)\Big\}.$$

We recall that a Banach space E is said to satisfy Opial's condition [14] if, for any sequence $\{x_n\}$ in E, $x_n \to x$ weakly implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

A Banach space E has the Kadec-Klee property [6] if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ it follows that $\|x_n - x\| \to 0$.

3. Main Results

In this section, we establish some weak convergence theorems of the iteration scheme (2.5) by using Opial condition and Kadec-Klee property in the framework of real uniformly convex Banach space. First we need the following lemma to prove our main results of this paper.

Lemma 3.1. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T_1, T_2\colon K \to E$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive non-self mappings with sequences $\{k_n\}$, $\{h_n\} \subset [1,\infty)$ such that $F = \cap_{i=1}^{2} F(T_i) \neq \phi$. Suppose $N_1 = \lim_n k_n \geq 1$ and $N_2 = \lim_n h_n \geq 1$ such that $\sum_{n=1}^{\infty}(k_n h_n - 1) < \infty$. From arbitrary $x_1 \in K$, the sequence $\{x_n\}$ defined iteratively by (2.5) with the restrictions $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n \bar{c}_n < \infty$. Let $\{a_n\}$ and $\{\bar{a}_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. Then we have the following:*

(a) $\lim_{n\to\infty} \|x_n - x^*\|$ *exists for all* $x^* \in F$.

(b) $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$ *and* $\lim_{n\to\infty} \|x_n - T_2 x_n\| = 0$.

Proof. For all $x^* \in F$, we set

$$M_1 = \max\{\sup_{n\geq 1} \|l_n - x^*\|, \sup_{n\geq 1} \|m_n - x^*\|\}.$$

Then from (2.5), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|P(a_nT_1(PT_1)^{n-1}x_n + b_nT_2(PT_2)^{n-1}y_n + c_nl_n) - Px^*\| \\
&\leq \|a_nT_1(PT_1)^{n-1}x_n + b_nT_2(PT_2)^{n-1}y_n + c_nl_n - x^*\| \\
&\leq a_n\|T_1(PT_1)^{n-1}x_n - x^*\| + b_n\|T_2(PT_2)^{n-1}y_n - x^*\| \\
&\quad +c_n\|l_n - x^*\| \\
&\leq a_nk_n\|x_n - x^*\| + b_nh_n\|y_n - x^*\| + c_nM_1 \qquad (3.1)
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - x^*\| &= \|P(\bar{a}_nx_n + \bar{b}_nT_1(PT_1)^{n-1}x_n + \bar{c}_nm_n) - Px^*\| \\
&\leq \|\bar{a}_nx_n + \bar{b}_nT_1(PT_1)^{n-1}x_n + \bar{c}_nm_n - x^*\| \\
&\leq \bar{a}_n\|x_n - x^*\| + \bar{b}_n\|T_1(PT_1)^{n-1}x_n - x^*\| \\
&\quad +\bar{c}_n\|m_n - x^*\| \\
&\leq \bar{a}_n\|x_n - x^*\| + \bar{b}_nk_n\|x_n - x^*\| + \bar{c}_nM_1 \\
&\leq [\bar{a}_n + \bar{b}_n]k_n\|x_n - x^*\| + \bar{c}_nM_1 \\
&= [1 - \bar{c}_n]k_n\|x_n - x^*\| + \bar{c}_nM_1 \\
&\leq k_n\|x_n - x^*\| + \bar{c}_nM_1 \qquad (3.2)
\end{aligned}$$

which implies that

$$\begin{aligned}
\|y_n - x^*\| &\leq k_n\|x_n - x^*\| + \bar{c}_nM_1 \\
&\leq k_nh_n\|x_n - x^*\| + \bar{c}_nM_1. \qquad (3.3)
\end{aligned}$$

Using (3.1) and (3.3), we obtain that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq a_nk_n\|x_n - x^*\| + b_nh_n[k_nh_n\|x_n - x^*\| + \bar{c}_nM_1] + c_nM_1 \\
&\leq a_nk_nh_n\|x_n - x^*\| + b_nk_nh_n[k_nh_n\|x_n - x^*\| + \bar{c}_nM_1] + c_nM_1 \\
&\leq (a_n + b_n)k_n^2h_n^2\|x_n - x^*\| + [b_nk_nh_n\bar{c}_n + c_n]M_1 \\
&= (1 - c_n)k_n^2h_n^2\|x_n - x^*\| + [b_nk_nh_n\bar{c}_n + c_n]M_1 \\
&\leq k_n^2h_n^2\|x_n - x^*\| + (b_n\bar{c}_n + c_n)k_nh_nM_1 \\
&\leq k_n^2h_n^2\|x_n - x^*\| + (b_n\bar{c}_n + c_n)M_2 \\
&= [1 + (k_n^2h_n^2 - 1)]\|x_n - x^*\| + A_n \qquad (3.4)
\end{aligned}$$

where

$$M_2 = \sup_{n\geq 1}\{k_nh_n\}M_1, \quad A_n = (b_n\bar{c}_n + c_n)M_2.$$

By putting $\lambda_n = (k_n^2h_n^2 - 1)$, the inequality (3.4) can be written as follows

$$\|x_{n+1} - x^*\| \leq (1 + \lambda_n)\|x_n - x^*\| + A_n. \qquad (3.5)$$

By hypothesis of the theorem, we find

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_n &= \sum_{n=1}^{\infty}(k_n^2h_n^2-1)\\ &= \sum_{n=1}^{\infty}(k_nh_n+1)(k_nh_n-1)\\ &\leq (N_1N_2+1)\sum_{n=1}^{\infty}(k_nh_n-1)<\infty.\end{aligned}$$

Since by assumptions of the theorem $\sum_{n=1}^{\infty}c_n<\infty$ and $\sum_{n=1}^{\infty}b_n\bar{c}_n<\infty$, it follows that $\sum_{n=1}^{\infty}A_n<\infty$ and $\sum_{n=1}^{\infty}\lambda_n<\infty$, thus by Lemma 2.3, we have $\lim_{n\to\infty}\|x_n-x^*\|$ exists. Let $\lim_{n\to\infty}\|x_n-x^*\|=r$ for some $r\geq 0$. From (3.3), we have

$$\|y_n-x^*\| \leq k_nh_n\|x_n-x^*\|+\bar{c}_nM_1,\ \forall\, n\geq 1.$$

Taking $limsup_{n\to\infty}$ in both sides, we obtain

$$\limsup_{n\to\infty}\|y_n-x^*\| \leq \limsup_{n\to\infty}\|x_n-x^*\|=\lim_{n\to\infty}\|x_n-x^*\|=r. \tag{3.6}$$

Since T_1 is asymptotically quasi-nonexpansive non-self mapping, we have

$$\|T_1(PT_1)^{n-1}x_n-x^*\| \leq k_n\|x_n-x^*\|,\ \forall n\geq 1.$$

Taking $limsup_{n\to\infty}$ in both sides, we obtain

$$\limsup_{n\to\infty}\|T_1(PT_1)^{n-1}x_n-x^*\| \leq r. \tag{3.7}$$

In a similar way, we have

$$\|T_2(PT_2)^{n-1}y_n-x^*\| \leq h_n\|y_n-x^*\|,\ \forall\, n\geq 1.$$

By using (3.6), we obtain

$$\limsup_{n\to\infty}\|T_2(PT_2)^{n-1}y_n-x^*\| \leq r. \tag{3.8}$$

Also, it follows from

$$\begin{aligned}r &= \lim_{n\to\infty}\|x_{n+1}-x^*\|\\ &= \lim_{n\to\infty}\|a_nT_1(PT_1)^{n-1}x_n+b_nT_2(PT_2)^{n-1}y_n+c_nl_n-x^*\|\\ &= \lim_{n\to\infty}\|a_n[(T_1(PT_1)^{n-1}x_n-x^*)+\frac{c_n}{2a_n}(l_n-x^*)]\\ &\quad +b_n[(T_2(PT_2)^{n-1}y_n-x^*)+\frac{c_n}{2b_n}(l_n-x^*)]\|\\ &= \lim_{n\to\infty}\|a_n[(T_1(PT_1)^{n-1}x_n-x^*)+\frac{c_n}{2a_n}(l_n-x^*)]\\ &\quad +(1-a_n)[(T_2(PT_2)^{n-1}y_n-x^*)+\frac{c_n}{2b_n}(l_n-x^*)]\|\end{aligned}$$

and Lemma 2.4 that

$$\lim_{n\to\infty}\|T_1(PT_1)^{n-1}x_n-T_2(PT_2)^{n-1}y_n+(\frac{c_n}{2a_n}-\frac{c_n}{2b_n})(l_n-x^*)\|=0. \tag{3.9}$$

Since $\lim_{n\to\infty}\|(\frac{c_n}{2a_n}-\frac{c_n}{2b_n})(l_n-x^*)\|=0$, we obtain that

$$\lim_{n\to\infty}\|T_1(PT_1)^{n-1}x_n-T_2(PT_2)^{n-1}y_n\|=0. \tag{3.10}$$

Now

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|a_n T_1(PT_1)^{n-1}x_n + b_n T_2(PT_2)^{n-1}y_n + c_n l_n - x^*\| \\ &= \|(T_1(PT_1)^{n-1}x_n - x^*) + b_n(T_2(PT_2)^{n-1}y_n \\ &- T_1(PT_1)^{n-1}x_n) + c_n(l_n - T_1(PT_1)^{n-1}x_n)\| \\ &\le \|T_1(PT_1)^{n-1}x_n - x^*\| + b_n\|T_2(PT_2)^{n-1}y_n - T_1(PT_1)^{n-1}x_n\| \\ &+ c_n\|l_n - T_1(PT_1)^{n-1}x_n\|\end{aligned}$$

yields that

$$r \le \liminf_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x^*\|$$

so that (3.7) gives

$$\lim_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x^*\| = r. \tag{3.11}$$

On the other hand,

$$\begin{aligned}\|T_1(PT_1)^{n-1}x_n - x^*\| &\le \|T_1(PT_1)^{n-1}x_n - T_2(PT_2)^{n-1}y_n\| + \|T_2(PT_2)^{n-1}y_n - x^*\| \\ &\le \|T_1(PT_1)^{n-1}x_n - T_2(PT_2)^{n-1}y_n\| + h_n\|y_n - x^*\|\end{aligned}$$

so we have

$$r \le \liminf_{n\to\infty} \|y_n - x^*\|. \tag{3.12}$$

By using (3.6) and (3.12), we obtain

$$\lim_{n\to\infty} \|y_n - x^*\| = r. \tag{3.13}$$

Thus

$$\begin{aligned}r &= \lim_{n\to\infty} \|y_n - x^*\| \\ &= \lim_{n\to\infty} \|\bar{a}_n x_n + \bar{b}_n T_1(PT_1)^{n-1}x_n + \bar{c}_n m_n - x^*\| \\ &= \lim_{n\to\infty} \|\bar{b}_n[(T_1(PT_1)^{n-1}x_n - x^*) + \frac{\bar{c}_n}{2\bar{b}_n}(m_n - x^*)] \\ &\quad + \bar{a}_n[x_n - x^*) + \frac{\bar{c}_n}{2\bar{a}_n}(m_n - x^*)]\| \\ &= \lim_{n\to\infty} \|\bar{b}_n[(T_1(PT_1)^{n-1}x_n - x^*) + \frac{\bar{c}_n}{2\bar{b}_n}(m_n - x^*)] \\ &\quad + (1 - \bar{b}_n)[x_n - x^*) + \frac{\bar{c}_n}{2\bar{a}_n}(m_n - x^*)]\|.\end{aligned}$$

Using (3.11), (3.13) and Lemma 2.4, the above inequality gives

$$\lim_{n\to\infty} \|T_1(PT_1)^{n-1}x_n - x_n\| = 0. \tag{3.14}$$

Now

$$\|y_n - x_n\| \le \bar{b}_n\|T_1(PT_1)^{n-1}x_n - x_n\| + \bar{c}_n\|m_n - x_n\|. \tag{3.15}$$

Using (3.14) and by hypothesis of the theorem in (3.15), we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.16}$$

Also note that

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|a_nT_1(PT_1)^{n-1}x_n+b_nT_2(PT_2)^{n-1}y_n+c_nl_n-x_n\|\\ &= \|(1-b_n-c_n)T_1(PT_1)^{n-1}x_n+b_nT_2(PT_2)^{n-1}y_n+c_nl_n-x_n\|\\ &\leq \|T_1(PT_1)^{n-1}x_n-x_n\|+b_n\|T_2(PT_2)^{n-1}y_n-T_1(PT_1)^{n-1}x_n\|\\ &\quad +c_n\|l_n-T_1(PT_1)^{n-1}x_n\|\\ &\quad \to 0 \text{ as } n\to\infty, \end{aligned} \tag{3.17}$$

so that

$$\|x_{n+1}-y_n\| \leq \|x_{n+1}-x_n\|+\|y_n-x_n\| \to 0 \text{ as } n\to\infty. \tag{3.18}$$

Furthermore, from

$$\begin{aligned}\|x_{n+1}-T_2(PT_2)^{n-1}y_n\| &\leq \|x_{n+1}-x_n\|+\|x_n-T_1(PT_1)^{n-1}x_n\|\\ &\quad +\|T_1(PT_1)^{n-1}x_n-T_2(PT_2)^{n-1}y_n\|\end{aligned}$$

we find that

$$\lim_{n\to\infty}\|x_{n+1}-T_2(PT_2)^{n-1}y_n\|=0. \tag{3.19}$$

Then

$$\begin{aligned}\|x_{n+1}-T_1x_{n+1}\| &\leq \|x_{n+1}-T_1(PT_1)^nx_{n+1}\|+\|T_1(PT_1)^nx_{n+1}-T_1(PT_1)^nx_n\|\\ &\quad +\|T_1(PT_1)^nx_n-T_1x_{n+1}\|\\ &\leq \|x_{n+1}-T_1(PT_1)^nx_{n+1}\|+L\|x_{n+1}-x_n\|\\ &\quad +L\|T_1(PT_1)^{n-1}x_n-x_{n+1}\|\\ &\leq \|x_{n+1}-T_1(PT_1)^nx_{n+1}\|+L\|x_{n+1}-x_n\|\\ &\quad +Lb_n\|T_1(PT_1)^{n-1}x_n-T_2(PT_2)^{n-1}y_n\|\\ &\quad +Lc_n\|T_1(PT_1)^{n-1}x_n-l_n\|\end{aligned}$$

yields

$$\lim_{n\to\infty}\|x_n-T_1x_n\|=0. \tag{3.20}$$

Now

$$\begin{aligned}\|x_n-T_2(PT_2)^{n-1}x_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-T_2(PT_2)^{n-1}y_n\|\\ &\quad +\|T_2(PT_2)^{n-1}y_n-T_2(PT_2)^{n-1}x_n\|\\ &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-T_2(PT_2)^{n-1}y_n\|\\ &\quad +L\|y_n-x_n\| \to 0 \text{ as } n\to\infty. \end{aligned} \tag{3.21}$$

Thus

$$\begin{aligned}\|x_{n+1}-T_2x_{n+1}\| &\leq \|x_{n+1}-T_2(PT_2)^nx_{n+1}\|+\|T_2(PT_2)^nx_{n+1}-T_2x_{n+1}\|\\ &\leq \|x_{n+1}-T_2(PT_2)^nx_{n+1}\|+L\|T_2(PT_2)^{n-1}x_{n+1}-x_{n+1}\|\\ &\leq \|x_{n+1}-T_2(PT_2)^nx_{n+1}\|+L\Big(\|T_2(PT_2)^{n-1}x_{n+1}-T_2(PT_2)^{n-1}y_n\|\\ &\quad +\|T_2(PT_2)^{n-1}y_n-x_{n+1}\|\Big)\\ &\leq \|x_{n+1}-T_2(PT_2)^nx_{n+1}\|+L^2\|x_{n+1}-y_n\|\\ &\quad +L\|T_2(PT_2)^{n-1}y_n-x_{n+1}\|\end{aligned}$$

implies

$$\lim_{n\to\infty} \|x_n - T_2x_n\| = 0. \tag{3.22}$$

This completes the proof. □

Theorem 3.2. *Let E be a real uniformly convex Banach space satisfying Opial's condition and K, T_i $(i = 1,2)$ and $\{x_n\}$ be as in Lemma 3.1. If $F = F(T_1) \cap F(T_2) \neq \phi$, then the sequence $\{x_n\}$ converges weakly to a common fixed point of the mappings T_1 and T_2.*

Proof. Let $p \in F = F(T_1) \cap F(T_2) \neq \phi$. Then, by Lemma 3.1, $\|x_n - p\|$ exists. Assume that $x_{n_i} \to u$ weakly and $x_{n_j} \to v$ weakly as $n \to \infty$. Then $u, v \in F$. We prove that $u = v$. If $u \neq v$, by Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - u\| &= \lim_{i\to\infty} \|x_{n_i} - u\| \\
&< \lim_{i\to\infty} \|x_{n_i} - v\| \\
&= \lim_{n\to\infty} \|x_n - v\| \\
&< \lim_{j\to\infty} \|x_{n_j} - u\| \\
&= \lim_{n\to\infty} \|x_n - u\|
\end{aligned}$$

which is a contradiction. Therefore, we have the conclusion i.e. $u = v$. Thus the sequence $\{x_n\}$ converges weakly to a common fixed point of the mappings T_1 and T_2. This completes the proof. □

Lemma 3.3. *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T_1, T_2\colon K \to E$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive nonself mappings with sequences $\{k_n\}$, $\{h_n\} \subset [1,\infty)$ such that $F = \cap_{i=1}^{2} F(T_i) \neq \phi$. Suppose $N_1 = \lim_n k_n \geq 1$ and $N_2 = \lim_n h_n \geq 1$ such that $\sum_{n=1}^{\infty}(k_nh_n - 1) < \infty$. From arbitrary $x_1 \in K$, the sequence $\{x_n\}$ defined iteratively by (2.5) with the restrictions $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n\bar{c}_n < \infty$. Let $\{a_n\}$ and $\{\bar{a}_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. Then $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all p, $q \in F$ and $t \in [0,1]$.*

Proof. By Lemma 3.1, we know that $\{x_n\}$ is bounded. Letting

$$a_n(t) = \|tx_n + (1-t)p - q\|$$

for all $t \in [0,1]$. Then $\lim_{n\to\infty} a_n(0) = \|p - q\|$ and $\lim_{n\to\infty} a_n(1) = \|x_n - q\|$ exists by Lemma 3.1. It, therefore, remains to prove the Lemma 3.3 for $t \in (0,1)$. For all $x \in K$, we define the mapping $W_n\colon K \to K$ by

$$W_n\, x \;=\; P(a_nT_1(PT_1)^{n-1}x + b_nT_2(PT_2)^{n-1}P(\bar{a}_nx + \bar{b}_nT_1(PT_1)^{n-1}x + \bar{c}_nm_n) + c_nl_n).$$

Then

$$\begin{aligned}
\|W_nx - W_ny\| &\leq [1 + (k_n^2h_n^2 - 1)]\|x - y\| \\
&= [1 + \lambda_n]\|x - y\| \\
&= H_n\|x - y\|
\end{aligned} \tag{3.23}$$

for all $x, y \in K$, where $H_n = [1 + \lambda_n]$ and $\lambda_n = (k_n^2h_n^2 - 1)$ with $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $H_n \to 1$ as $n \to \infty$. Setting

$$S_{n,\,m} = W_{n+m-1}W_{n+m-2}\dots W_n,\; m \geq 1 \tag{3.24}$$

and

$$b_{n,m} = \|S_{n,m}(tx_n + (1-t)p) - (tS_{n,m}x_n + (1-t)S_{n,m}q)\|. \tag{3.25}$$

From (3.23) and (3.24), we have

$$\begin{aligned} \|S_{n,m}x - S_{n,m}y\| &\leq H_n\, H_{n+1}\, \ldots\, H_{n+m-1}\|x-y\| \\ &\leq \Big(\prod_{j=n}^{n+m-1} H_j\Big)\|x-y\| \\ &= \sigma_n\|x-y\| \end{aligned} \tag{3.26}$$

for all $x, y \in K$, where $\sigma_n = \prod_{j=n}^{n+m-1} H_j$ and $S_{n,m}x_n = x_{n+m}$, $S_{n,m}p = p$ for all $p \in F$. Thus

$$\begin{aligned} a_{n+m}(t) &= \|tx_{n+m} + (1-t)p - q\| \\ &\leq b_{n,m} + \|S_{n,m}(tx_n + (1-t)p) - q\| \\ &\leq b_{n,m} + \sigma_n a_n(t). \end{aligned} \tag{3.27}$$

It follows from (3.25), (3.26) and Lemma 2.6 that

$$b_{n,m} \leq \sigma_n \phi^{-1}(\|x_n - p\| - \sigma_n^{-1}\|x_{n+m} - p\|).$$

By Lemma 3.1 and $\lim_{n\to\infty}\sigma_n = 1$, we have $\lim_{n,m\to\infty} b_{n,m} = 0$ and so

$$\limsup_{m\to\infty} a_m(t) \leq \lim_{n,m\to\infty} b_{n,m} + \liminf_{n\to\infty} \sigma_n a_n(t) = \liminf_{n\to\infty} a_n(t).$$

This shows that $\lim_{n\to\infty} a_n(t)$ exists, that is,

$$\lim_{n\to\infty}\|tx_n + (1-t)p - q\|$$

exists for all $t \in [0,1]$. This completes the proof. □

Theorem 3.4. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property and K be a nonempty closed convex subset which is also a nonexpansive retract of E. Let $T_1, T_2\colon K \to E$ be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive nonself mappings with sequences $\{k_n\}$, $\{h_n\} \subset [1,\infty)$ such that $F = \cap_{i=1}^2 F(T_i) \neq \phi$. Suppose $N_1 = \lim_n k_n \geq 1$ and $N_2 = \lim_n h_n \geq 1$ such that $\sum_{n=1}^\infty (k_n h_n - 1) < \infty$. From arbitrary $x_1 \in K$, the sequence $\{x_n\}$ defined iteratively by (2.5) with the restrictions $\sum_{n=1}^\infty c_n < \infty$ and $\sum_{n=1}^\infty b_n \bar{c}_n < \infty$. Let $\{a_n\}$ and $\{\bar{a}_n\}$ be sequences in $[\delta, 1-\delta]$ for some $\delta \in (0,1)$. If the mappings $I - T_1$ and $I - T_2$, where I denotes the identity mapping, are demiclosed at zero. Then $\{x_n\}$ converges weakly to a common fixed point of the mappings T_1 and T_2.*

Proof. By Lemma 3.1, we know that $\{x_n\}$ is bounded and since E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $p \in K$. By Lemma 3.1, we have

$$\lim_{n\to\infty}\|x_{n_j} - T_1 x_{n_j}\| = 0, \quad \lim_{n\to\infty}\|x_{n_j} - T_2 x_{n_j}\| = 0.$$

Since the mappings $I - T_1$ and $I - T_2$ are demiclosed at zero, therefore $T_1 p = p$ and $T_2 p = p$, which means $p \in F$. Now, we show that $\{x_n\}$ converges weakly to p. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converges weakly to some $q \in K$. By the same method as above, we have $q \in F$ and $p, q \in w_w(x_n)$. By Lemma 3.3, the limit

$$\lim_{n\to\infty}\|tx_n + (1-t)p - q\|$$

exists for all $t \in [0,1]$ and so $p = q$ by Lemma 2.5. Thus, the sequence $\{x_n\}$ converges weakly to $p \in F$. This completes the proof. □

Remark 3.5. If we put $c_n = \bar{c}_n = 0$, $T_1 = I$ and $T_2 = T$ then Theorem 3.2 extends Theorem 2.1 of Schu [21] to the case of more general class of non-self maps considered in this paper.

Remark 3.6. Theorem 3.4 extends Theorem 3.10 of Chidume et al. [5] to the case of modified Ishikawa type iteration process with errors in the sense of Xu [28] for two mappings considered in this paper.

References

[1] S.C. Bose, *Weak convergence to a fixed point of an asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **68** (1978), 305–308. 1

[2] S.S. Chang, Y.J. Cho and H. Zhou, *Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings*, J. Korean Math. Soc. **38:6** (2001), 1245–1260. 1, 2.6

[3] C.E. Chidume, *On the approximation of fixed points of nonexpansive mappings*, Houston J. Math. **7** (1981), 345–355.

[4] C.E. Chidume, *Nonexpansive mappings, generalizations and iterative algorithms*. In: Agarwal R.P., O'Reagan D.eds. Nonlinear Analysis and Application. To V. Lakshmikantam on his 80th Birthday (Research Monograph), Dordrecht: Kluwer Academic Publishers, pp. 383-430. 1

[5] C.E. Chidume, E.U. Ofoedu and H. Zegeye, *Strong and weak convergence theorems for asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **280** (2003), 364–374. 2.1, 3.6

[6] C.E. Chidume, N. Shahzad and H. Zegeye, *Convergence theorems for mappings which are asymptotically nonexpansive in the intermediate sense*, Numer. Funct. Anal. Optimiz. **25:3-4** (2004), 239–257. 2

[7] C.E. Chidume, N. Shahzad and H. Zegeye, *Strong convergence theorems for nonexpansive mappings in arbitrary Banach spaces*, Nonlinear Anal. (Submitted). 1

[8] K. Goebel and W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171–174. 1

[9] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), 147–150. 1

[10] S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc. **59** (1976), 65–71. 1

[11] W. Kaczor, *Weak convergence of almost orbits of asymptotically nonexpansive commutative semigroups*, J. Mathematical Analysis and Applications **272:2** (2002), 565–574. 2.5

[12] S.H. Khan and W. Takahashi, *Approximating common fixed points of two asymptotically nonexpansive mappings*, Sci. Math. Jpn. **53** (2001), 143–148. 1

[13] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510. 1

[14] Z. Opial, *Weak convergence of the sequence of successive approximatins for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597. 1, 2

[15] M.O. Osilike and S.C. Aniagbosor, *Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings*, Math. Comput. Modelling **32** (2000), 1181–1191. 1

[16] G.B. Passty, *Construction of fixed points for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **84** (1982), 212–216. 1

[17] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 274–276.

[18] B.E. Rhoades, *Fixed point iteration for certain nonlinear mappings*, J. Math. Anal. Appl. **183** (1994), 118–120. 1

[19] G.S. Saluja, *Convergence of fixed point of asymptotically quasi-nonexpansive type mappings in convex metric spaces*, J. Nonlinear Sci. Appl. **1:3** (2008), 132–144. 1

[20] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **158** (1991), 407–413. 1

[21] J. Schu, *Weak and strong convergence theorems to fixed points of asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. **43** (1991), 153–159. 1, 2.4, 3.5

[22] K.K. Tan and H.K. Xu, *A nonlinear ergodic theorem for asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. **45** (1992), 25–36. 1

[23] K.K. Tan and H.K. Xu, *The nonlinear ergodic theorem for asymptotically nonexpansive mapping in Banach spaces*, Proc. Amer. Math. Soc. **114** (1992), 399–404. 1

[24] K.K. Tan and H.K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **178** (1993), 301–308. 2.3

[25] K.K. Tan and H.K. Xu, *Fixed point iteration processes for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **122** (1994), 733–739. 1

[26] T. Thianwan, *Convergence criteria of modified Noor iterations with errors for three asymptotically nonexpansive non-self mappings*, J. Nonlinear Sci. Appl. **6:3** (2013), 181–197. 1

[27] H.K. Xu, *Existence and convergence for fixed points of mappings of asymptotically nonexpansive type*, Nonlinear Analysis, **16** (1991), 1139–1146. 1

[28] Y. Xu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl. **224** (1998), 91–101. 2, 2, 3.6