Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 7 (2014), 150–159
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Fixed point results for $GP_{(\Lambda,\Theta)}$-contractive mappings

Vahid Parvaneh[a], Peyman Salimi[b,*], Pasquale Vetro[c], Akbar Dehghan Nezhad[d], Stojan Radenović[e]

[a]*Department of Mathematics, College of Science, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran.*
[b]*Young Researchers and Elite Club, Rasht Branch, Islamic Azad University, Rasht, Iran.*
[c]*Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, via Archirafi 34, 90123 Palermo, Italy.*
[d]*Department of Mathematics, Yazd University, Yazd, Iran.*
[e]*Faculty of Mathematics, University of Belgrade, Studentski trg 16, 11000 Beograd, Serbia.*

Communicated by S. M. Vaezpour

Abstract

In this paper, we introduce new notions of GP-metric space and $GP_{(\Lambda,\Theta)}$-contractive mapping and then prove some fixed point theorems for this class of mappings. Our results extend and generalized Banach contraction principle to GP-metric spaces. An example shows the usefulness of our results.

Keywords: GP-metric spaces, $GP_{(\Lambda,\Theta)}$-contractive mappings, O-GP-continuous.
2010 MSC: 47H10, 54H25, 55M20.

1. Introduction and preliminaries

In the fixed point theory of continuous mappings, a well-known theorem of Banach [6] states that if (X,d) is a complete metric space and if f is a self-mapping on X which satisfies the inequality

$$d(fx,fy) \le kd(x,y) \tag{1.1}$$

for some $k \in [0,1)$ and all $x,y \in X$, then f has a unique fixed point z and the sequence of successive approximations $\{f^n x\}$ converges to z for all $x \in X$. On the other hand, the condition $d(fx,fy) < d(x,y)$

*Corresponding author
Email addresses: `vahid.parvaneh@kiau.ac.ir` (Vahid Parvaneh), `salimipeyman@gmail.com` (Peyman Salimi), `vetro@math.unipa.it` (Pasquale Vetro), `anezhad@yazd.ac.ir` (Akbar Dehghan Nezhad), `radens@beotel.net` (Stojan Radenović)

Received 2012-12-30

does not ensures that f has a fixed point. In the last decades, the Banach's theorem [6] has been extensively studied and generalized on many settings, see for example [7, 9, 10, 13, 14, 15, 16, 17, 20, 25, 27, 32, 34, 35].

Partial metric space is a generalized metric space introduced by Matthews [19] in which each object does not necessarily have to have a zero distance from itself. A motivation is to introduce this space to give a modified version of the Banach contraction principle. Subsequently, several authors studied the problem of existence and uniqueness of a fixed point for mappings satisfying different contractive conditions, for example see [2, 5, 8, 11, 18, 28, 29, 30, 33].

On the other hand, in 2006 Mustafa and Sims [21] introduced a new notion of generalized metric spaces called G-metric spaces. Based on the notion of a G-metric space, many fixed point results for different contractive conditions have been presented, for more details see [1, 4, 22, 23, 24, 26, 31]. Recently, based on the two above notions, Zand and Nezhad [36] introduced a new generalized metric space as both a generalization of a partial metric space and a G-metric space. Following this direction of research, Aydi et al. [3] established some fixed point results in GP-metric spaces which were first fixed point results in GP-metric spaces.

In the present work, we introduce new notions of GP-metric space and $GP_{(\Lambda,\Theta)}$-contractive mappings and study some fixed point results for $GP_{(\Lambda,\Theta)}$-contractive mappings in GP-metric spaces. Some fundamental properties of the proposed metric are studied.

A (totally) ordered (abelian) group G is an additive group on which is defined an order relation $<$ such that if $a < b$, then $a + c < b + c$, for all $a, b, c \in G$. We write $\leq$ for $<$ or $=$, and denote by G^+ the set of nonnegative elements of G. In the sequel, the letters $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}^+$ and $\mathbb{N}$ will denote the set of all real numbers, the set of all nonnegative real numbers, the set of all nonnegative integer numbers and the set of all positive integer numbers, respectively.

Definition 1.1 ([12])**.** Let G be an ordered group. An ordered group metric (or OG-metric) on a nonempty set X is a symmetric nonnegative function d_G from $X \times X$ into G such that $d_G(x, y) = 0$ if and only if $x = y$ and such that the triangle inequality is satisfied; the pair (X, d_G) is an ordered group metric space (or OG-metric space).

Definition 1.2 ([36])**.** Let X be a non empty set and G be an ordered group. A function $G_p : X \times X \times X \longrightarrow G^+$ is called an ordered group partial metric (or OGP-metric) if the following conditions are satisfied:

(GP1) $x = y = z$ if $G_p(x, y, z) = G_p(z, z, z) = G_p(y, y, y) = G_p(x, x, x)$;

(GP2) $0 \leq G_p(x, x, x) \leq G_p(x, x, y) \leq G_p(x, y, z)$ for all $x, y, z \in X$;

(GP3) $G_p(x, y, z) = G_p(x, z, y) = G_p(y, z, x) = \cdots$, symmetry in all three variables;

(GP4) $G_p(x, y, z) \leq G_p(x, a, a) + G_p(a, y, z) - G_p(a, a, a)$ for any $x, y, z, a \in X$.

Then the triple (X, G, G_p) is called an OGP-metric space.

For example we can place $G^+ = \mathbb{Z}^+$ or $\mathbb{R}^+$. In the case $G^+ = \mathbb{R}^+$, the triple $(X, \mathbb{R}, G_p)$ will often be denoted by (X, G_p) and is called a GP-metric space. In the sequel, for simplicity we assume that $G^+ = \mathbb{R}^+$.

Example 1.3 ([36])**.** Let $X = \mathbb{R}^+ = G^+$ and define $G_p(x, y, z) = \max\{x, y, z\}$, for all $x, y, z \in X$. Then (X, G_p) is a GP-metric space.

Proposition 1.4 ([36])**.** *Let (X, G_p) be a GP-metric space, then for any x, y, z and $a \in X$ it follows that*

(i) $G_p(x, y, z) \leq G_p(x, x, y) + G_p(x, x, z) - G_p(x, x, x)$;

(ii) $G_p(x, y, y) \leq 2G_p(x, x, y) - G_p(x, x, x)$;

(iii) $G_p(x, y, z) \leq G_p(x, a, a) + G_p(y, a, a) + G_p(z, a, a) - 2G_p(a, a, a)$;

(iv) $G_p(x,y,z) \leq G_p(x,a,z) + G_p(a,y,z) - G_p(a,a,a)$.

Proposition 1.5 ([36])**.** *Every GP-metric space* (X, G_p) *defines a metric space* (X, D_{G_p}) *where*

$$D_{G_p}(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(x,x,x) - G_p(y,y,y) \quad \textit{for all} \quad x,y \in X.$$

Example 1.6. Let $X = \mathbb{R}^+ = G^+$ and define $G_p(x,y,z) = \max\{x,y,z\}$, for all $x,y,z \in X$. Then (X, G_p) is a GP-metric space and $D_{G_P}(x,y) = |x-y|$ for all $x,y \in X$.

Definition 1.7 ([36])**.** Let (X, G_p) be a GP-metric space and let $\{x_n\}$ a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ or $x_n \to x$ if

$$\lim_{m,n\to+\infty} G_p(x, x_m, x_n) = G_p(x,x,x).$$

If $G_p(x,x,x) = 0$ we say that the sequence $\{x_n\}$ is 0-GP-convergent to x.

Proposition 1.8 ([36])**.** *Let* (X, G_p) *be a GP-metric space. Then, for any sequence* $\{x_n\}$ *in* X *and a point* $x \in X$ *the following are equivalent:*

(A) $\{x_n\}$ *is GP-convergent to* x*;*
(B) $G_p(x_n, x_n, x) \to G_p(x,x,x)$ *as* $n \to +\infty$*;*
(C) $G_p(x_n, x, x) \to G_p(x,x,x)$ *as* $n \to +\infty$.

From the definition of D_{G_P}, we deduce the following proposition.

Proposition 1.9. *Let* (X, G_p) *be a GP-metric space. Then, for any sequence* $\{x_n\}$ *in* X *convergent to a point* $x \in X$ *such that* $\lim_{n\to+\infty} G_p(x_n, x_n, x_n) = G_p(x,x,x)$*, then* $D_{G_P}(x_n, x) \to 0$.

Definition 1.10 ([36])**.** Let (X, G_p) be a GP-metric space.

(S1) A sequence $\{x_n\}$ is called a GP-Cauchy sequence if and only if $\lim_{m,n\to+\infty} G_p(x_n, x_m, x_m)$ exists (and is finite).

(S2) A GP-metric space (X, G_p) is said to be GP-complete if and only if every GP-Cauchy sequence in X is GP-convergent to some $x \in X$ such that $G_p(x,x,x) = \lim_{m,n\to+\infty} G_p(x_n, x_m, x_m)$.

Lemma 1.11 ([3])**.** *Let* (X, G_p) *be a GP-metric space. Then*

(A) if $G_p(x,y,z) = 0$*, then* $x = y = z$*;*

(B) if $x \neq y$*, then* $G_p(x,y,y) > 0$.

Lemma 1.12 ([3])**.** *Let* (X, G_p) *be a GP-metric space,* $x, y \in X$ *and* $\{x_n\}$ *be a sequence in* X*. Assume that* $\lim_{n\to+\infty} G_p(x, x_n, x_n) = \lim_{n\to+\infty} G_p(x_n, y, y) = 0$*, then* $x = y$.

Lemma 1.13. *Let* (X, G) *be a GP-metric space and* $\{y_n\} \subset X$ *be a sequence such that*

$$G_p(y_n, y_{n+1}, y_{n+1}) \leq \lambda G_p(y_{n-1}, y_n, y_n) \tag{1.2}$$

for some $\lambda \in [0,1)$ *and each* $n \in \mathbb{N}$*. Then* $\{y_n\}$ *is a GP-Cauchy sequence in* X *such that* $\lim_{m,n\to+\infty} G_p(x_n, x_m, x_m) = 0$.

Proof. For any $m > n$, by (GP4) and (1.2), we get

$$\begin{aligned} G_p(x_n, x_m, x_m) &\leq G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_m, x_m) \\ &\leq G_p(x_n, x_{n+1}, x_{n+1}) + G_p(x_{n+1}, x_{n+2}, x_{n+2}) + G_p(x_{n+2}, x_m, x_m) \\ &\leq G_p(x_n, x_{n+1}, x_{n+1}) + \cdots + G_p(x_{m-1}, x_m, x_m) \\ &\leq \frac{\lambda^n}{1-\lambda} G_p(x_0, fx_0, fx_0). \end{aligned}$$

This implies that $\lim_{m,n\to+\infty} G_p(x_n, x_m, x_m) = 0$, that is, $\{x_n\}$ is a GP-Cauchy sequence. □

Definition 1.14. Let (X, G_p) be a GP-metric space. A mapping $f : X \to X$ is 0-GP-continuous when

$$\lim_{n\to+\infty} G_p(x_n, x_n, x) = 0 \quad \text{implies} \quad \lim_{n\to+\infty} G_p(fx_n, fx_n, fx) = 0.$$

2. Main Results

At first, we define the following notions.

Definition 2.1. Let $f : X \to X$ and $\Theta, \Lambda : X \times X \times X \to \mathbb{R}^+$ be two functions and $\lambda > 0$, $\theta \geq 0$ such that $0 \leq \frac{\theta}{\lambda} < 1$. We say that f is (Λ, Θ)-admissible if

$$x, y, z \in X \quad \text{and} \quad \Lambda(x, y, z) \geq \lambda \quad \text{imply} \quad \Lambda(fx, fy, fz) \geq \lambda$$

and

$$x, y, z \in X \quad \text{and} \quad \Theta(x, y, z) \leq \theta \quad \text{imply} \quad \Theta(fx, fy, fz) \leq \theta.$$

Definition 2.2. Let (X, G_p) be a GP-metric space and $f : X \to X$ be a (Λ, Θ)-admissible mapping. f is a hybrid $GP_{(\Lambda,\Theta)}$-contractive mapping if

$$\frac{1}{3}\Theta(x, y, z) G_p(x, fx, fx) \leq \Lambda(x, y, z) G_p(x, y, z)$$

implies

$$\Lambda(x, y, z) G_p(fx, fy, fz) \leq \Theta(x, y, z) G_p(x, y, z) + LM(x, y, z) \tag{2.1}$$

for all $x, y, z \in X$ where $L \geq 0$ and

$$M(x, y, z) = \min\{\max\{D_{G_p}(fx, y), D_{G_p}(fx, z)\}, \max\{D_{G_p}(fy, y), D_{G_p}(fz, z)\}\}.$$

From the definition of a hybrid $GP_{(\Lambda,\Theta)}$-contractive mapping, we deduce the following lemma.

Lemma 2.3. *Let (X, G_p) be a GP-metric space and $f : X \to X$ a hybrid $GP_{(\Lambda,\Theta)}$-contractive mapping. The following hold:*

(i) if $z \in X$ is a fixed point of the mapping f, then $G_p(z, z, z) = 0$;

(ii) if $z, w \in X$ are fixed points of the mapping f such that $\Theta(z, w, w) \leq \theta < \lambda \leq \Lambda(z, w, w)$, then $z = w$.

Theorem 2.4. *Let (X, G_p) be a GP-metric space such that (X, G_p) is GP-complete and f is a 0-GP-continuous hybrid $GP_{(\Lambda,\Theta)}$-contractive mapping. If there exists $x_0 \in X$ such that $\Lambda(x_0, fx_0, fx_0) \geq \lambda$ and $\Theta(x_0, fx_0, fx_0) \leq \theta$. Then f has a fixed point in X.*

Proof. Let $x_0 \in X$ such that $\Lambda(x_0, fx_0, fx_0) \geq \lambda$ and $\Theta(x_0, fx_0, fx_0) \leq \theta$. Define a sequence $\{x_n\}$ in X by $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. Since f is a (Λ, Θ)-admissible mapping and $\Lambda(x_0, x_1, x_1) = \Lambda(x_0, fx_0, fx_0) \geq \lambda$, we deduce that $\Lambda(x_1, x_2, x_2) = \Lambda(fx_0, fx_1, fx_1) \geq \lambda$. By continuing this process, we get $\Lambda(x_n, x_{n+1}, x_{n+1}) \geq \lambda$ for all $n \in \mathbb{N} \cup \{0\}$. Similarly, $\Theta(x_n, x_{n+1}, x_{n+1}) \leq \theta$ for all $n \in \mathbb{N} \cup \{0\}$. Also, if $x_{n-1} = x_n$ then x_n is a fixed point for f and we have nothing to prove and so we assume that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. That is $G_p(x_n, x_{n+1}, x_{n+1}) > 0$ for all $n \in \mathbb{N} \cup \{0\}$ and hence

$$\begin{aligned} \frac{1}{3}\Theta(x_{n-1}, x_n, x_n) G_p(x_{n-1}, x_n, x_n) &\leq \frac{\theta\lambda}{3\lambda} G_p(x_{n-1}, x_n, x_n) \\ &\leq \Lambda(x_{n-1}, x_n, x_n) G_p(x_{n-1}, x_n, x_n). \end{aligned}$$

Now, using (2.1) with $x = x_{n-1}$ and $y = z = x_n$, we get

$$\begin{aligned} \lambda G_p(x_n, x_{n+1}, x_{n+1}) &\leq \Lambda(x_{n-1}, x_n, x_n) G_p(x_n, x_{n+1}, x_{n+1}) \\ &\leq \Theta(x_{n-1}, x_n, x_n) G_p(x_{n-1}, x_n, x_n) + L\, M(x_{n-1}, x_n, x_n) \\ &\leq \theta G_p(x_{n-1}, x_n, x_n) \end{aligned}$$

and hence

$$G_p(x_n, x_{n+1}, x_{n+1}) \leq \frac{\theta}{\lambda} G_p(x_{n-1}, x_n, x_n), \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

Since, $0 \leq \dfrac{\theta}{\lambda} < 1$, by Lemma 1.13, we deduce that $\{x_n\}$ is a GP-Cauchy sequence such that $\lim_{m,n \to +\infty} G_p(x_n, x_m, x_m) = 0$. Since X is GP-complete, then $\{x_n\}$ GP-converges to $z \in X$ such that $G_p(z, z, z) = \lim\limits_{m,n \to +\infty} G_p(x_n, x_m, x_m) = 0$. That is, the sequence $\{x_n\}$ is 0-GP-convergent to z. Now, using the 0-GP-continuity of the mapping f and Proposition 1.4 (ii), we get

$$\begin{aligned} \lim_{n \to +\infty} G_p(fz, fz, x_{n+1}) &\leq \lim_{n \to +\infty} 2G_p(fz, x_{n+1}, x_{n+1}) - \lim_{n \to +\infty} G_p(x_{n+1}, x_{n+1}, x_{n+1}) \\ &\leq \lim_{n \to +\infty} 2G_p(fz, fx_n, fx_n) = 0. \end{aligned}$$

Consequently,

$$\lim_{n \to +\infty} G_p(x_n, fz, fz) = 0.$$

As

$$\lim_{n \to +\infty} G_p(x_n, x_n, z) = 0,$$

by Lemma 1.12, this yields $z = fz$. □

For hybrid $GP_{(\Lambda,\Theta)}$-contractive mappings that are not 0-GP-continuous we have the following result.

Theorem 2.5. *Let (X, G_p) be a GP-metric space such that (X, G_p) is GP-complete and f is a hybrid $GP_{(\Lambda,\Theta)}$-contractive mapping. Assume that the following conditions hold:*

(i) there exists $x_0 \in X$ such that $\Lambda(x_0, fx_0, fx_0) \geq \lambda$ and $\Theta(x_0, fx_0, fx_0) \leq \theta$;

(ii) if $\{x_n\}$ is a sequence in X such that $\Lambda(x_n, x_{n+1}, x_{n+1}) \geq \lambda$ and $\Theta(x_n, x_{n+1}, x_{n+1}) \leq \theta$ for all $n \in \mathbb{N}$ and $x_n \to z \in X$, then $\Lambda(x_n, z, z) \geq \lambda$ and $\Theta(x_n, z, z) \leq \theta$ for all $n \in \mathbb{N} \cup \{0\}$.

Then f has a fixed point.

Proof. Let $x_0 \in X$ such that $\Lambda(x_0, fx_0, fx_0) \geq \lambda$ and $\Theta(x_0, fx_0, fx_0) \leq \theta$. Define a sequence $\{x_n\}$ in X by $x_n = f^n x_0 = fx_{n-1}$ for all $n \in \mathbb{N}$. Following the proof of the Theorem 2.4, we can say that $\{x_n\}$ is a GP-Cauchy sequence such that $\Lambda(x_n, x_{n+1}, x_{n+1}) \geq \lambda$ and $\Theta(x_n, x_{n+1}, x_{n+1}) \leq \theta$ for all $n \in \mathbb{N} \cup \{0\}$. Since X is GP-complete, then there is $z \in X$ such that the sequence $\{x_n\}$ 0-GP-converges to z. Then by (ii), we get $\Lambda(x_n, z, z) \geq \lambda$ and $\Theta(x_n, z, z) \leq \theta$ for all $n \in \mathbb{N} \cup \{0\}$.

Now, we suppose that there exists $n_0 \in \mathbb{N}$ such the following inequalities hold:

$$\frac{1}{3}\Theta(x_{2n_0}, z, z)G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}) > \Lambda(x_{2n_0}, z, z)G(x_{2n_0}, z, z)$$

and

$$\frac{1}{3}\Theta(x_{2n_0+1}, z, z)G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2}) > \Lambda(x_{2n_0+1}, z, z)G(x_{2n_0+1}, z, z).$$

These relations imply

$$\frac{1}{3}G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}) > G(x_{2n_0}, z, z) \quad \text{and} \quad \frac{1}{3}G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2}) > G(x_{2n_0+1}, z, z).$$

Then, by Proposition 1.4 (iii) and (2.2), we have

$$\begin{aligned} G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}) &\leq G(x_{2n_0}, z, z) + 2G(x_{2n_0+1}, z, z) \\ &< \frac{1}{3}G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}) + \frac{2}{3}G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2}) \\ &< \frac{1}{3}G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}) + \frac{2}{3}G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}) \\ &= G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}) \end{aligned}$$

which is a contradiction. Thus, for all $n \in \mathbb{N}$, either

$$\frac{1}{3}\Theta(x_{2n_0}, z, z)G(x_{2n_0}, x_{2n_0+1}, x_{2n_0+1}) \leq \Lambda(x_{2n_0}, z, z)G(x_{2n_0}, z, z)$$

or

$$\frac{1}{3}\Theta(x_{2n_0+1}, z, z)G(x_{2n_0+1}, x_{2n_0+2}, x_{2n_0+2}) \leq \Lambda(x_{2n_0+1}, z, z)G(x_{2n_0+1}, z, z).$$

holds for every $n \in \mathbb{N}$. Assume that the first of the previous inequalities holds for all $n \in J \subset \mathbb{N}$. If J is an infinite set, then using the contractive condition (2.1) and condition (ii), we deduce that

$$\begin{aligned} G_p(z, fz, fz) &\leq G_p(z, x_{2n+1}, x_{2n+1}) + G_p(x_{2n+1}, fz, fz) \\ &\leq G_p(z, x_{2n+1}, x_{2n+1}) + \frac{1}{\lambda}\Lambda(x_{2n}, z, z)G_p(fx_{2n}, fz, fz) \\ &\leq G_p(z, x_{2n+1}, x_{2n+1}) + \frac{1}{\lambda}\Theta(x_{2n}, z, z)\, G_p(x_{2n}, z, z) + \frac{L}{\lambda}M(x_{2n}, z, z) \\ &\leq G_p(z, x_{n+1}, x_{n+1}) + \frac{\theta}{\lambda}\, G_p(x_n, z, z) + \frac{L}{\lambda}M(x_{2n}, z, z) \end{aligned}$$

holds for all $n \in J$.

Since the sequence $\{x_n\}$ 0-GP-converges to z, we deduce

$$M(x_{2n}, z, z) = \min\{D_{G_P}(x_{2n+1}, z), D_{G_P}(fz, z)\} \to 0 \quad \text{as} \quad n \to +\infty.$$

Letting $n \to +\infty$ with $n \in J$, in the previous inequality, we obtain that $G_p(z, fz, fz) \leq 0$, that is, $z = fz$. Hence, f has a fixed point. If J is a finite set, we obtain the same result by considering the second inequality. □

Example 2.6. Define a GP-metric G_p on $X = \mathbb{R}^+$ by $G_p(x, y, z) = \max\{x, y, z\}$. Let $f : X \to X$ be defined by

$$f(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } \ x \in [0, 1] \\ 3\ln x + \frac{1}{2} & \text{If } \ x \in \mathbb{R}^+ \setminus [0, 1] \end{cases}$$

and $\Lambda, \Theta : X \times X \times X \to \mathbb{R}^+$ be defined by

$$\Lambda(x,y,z) = \begin{cases} \frac{2}{3} & \text{if } x,y,z \in [0,1] \\ \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad \Theta(x,y,z) = \frac{1}{3}.$$

Now, we prove that all the hypotheses of Theorem 2.5 are satisfied and hence f has a fixed point; but Banach contraction principle, with respect to the metric D_{G_P}, cannot be applied to f.

Proof. Let $x, y, z \in X$, if $\Lambda(x,y,z) \geq \frac{2}{3}$ then $x, y, z \in [0,1]$. On the other hand, for all $w \in [0,1]$, we have $fw \leq 1$. Hence $\Lambda(fx, fy, fz) \geq \frac{2}{3}$. Similarly, if $\Theta(x,y,z) \leq 1/3$, then $\Theta(fx, fy, fz) \leq 1/3$. This implies that f is (Λ, Θ)-admissible. Clearly, $\Lambda(0, f0, f0) \geq 2/3$ and $\Theta(0, f0, f0) \leq 1/3$.

Now, if $\{x_n\}$ is a sequence in X such that $\Lambda(x_n, x_{n+1}, x_{n+1}) \geq 2/3$ and $\Theta(x_n, x_{n+1}, x_{n+1}) \leq 1/3$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to z$ as $n \to +\infty$. Then $\{x_n\} \subseteq [0,1]$ and hence $z \in [0,1]$. This implies that $\Lambda(x_n, z, z) \geq 2/3$ and $\Theta(x_n, z, z) \leq 1/3$ for all $n \in \mathbb{N} \cup \{0\}$.

Let $x, y, z \in [0,1]$, then

$$\begin{aligned} \Lambda(x,y,z)G_p(fx,fy,fz) &= \frac{2}{3}\max\{fx,fy,fz\} \\ &= \frac{1}{3}\max\{x^2,y^2,z^2\} \leq \frac{1}{3}\max\{x,y,z\} = \Theta(x,y,z)G_p(x,y,z). \end{aligned}$$

Otherwise, $\Lambda(x,y,z) = 0$ and so

$$0 = \Lambda(x,y,z)G_p(fx,fy,fz) \leq \Theta(x,y,z)G_p(x,y,z).$$

Then, f is a hybrid $GP_{(\Lambda,\Theta)}$-contractive mapping that satisfies all the conditions of Theorem 2.5 and hence f has a fixed point.

Now, let $d = D_{G_P}$. By Example 1.6, we have $d(x,y) = |x-y|$ for all $x, y \in X$. For $x = e$ and $y = 0$, we deduce

$$d(fe, f0) = \frac{7}{2} > k\,e = k\,d(e,0)$$

for all $k \in [0,1)$ and so Banach contraction Principle cannot be applied to f. □

Adding to Theorem 2.5 some hypotheses we can obtain the uniqueness of the fixed point.

Theorem 2.7. *Let all the conditions of Theorem 2.4 (or Theorem 2.5) be satisfied. If the following condition holds:*

(j) for all $z, w \in X$ such that $z = fz$ and $w = fw$, we have $\Lambda(z,w,w) \geq \lambda$ and $\Theta(z,w,w) \leq \theta$,

then f has a unique fixed point.

Proof. Follows by Theorem 2.4 (or Theorem 2.5) and Lemma 2.3. □

If in Theorem 2.4 and Theorem 2.5 we take $\Lambda(x,y,z) = 1$, $\Theta(x,y,z) = \theta$ where $\theta \in [0,1)$ and $L = 0$, then we deduce the Banach contraction principle in the setting of GP-metric spaces.

Corollary 2.8. *Let (X, G_p) be a GP-complete GP-metric space and f be a self-mapping on X. Assume that there exists $\theta \in [0,1)$, such that*

$$G_p(fx,fy,fz) \leq \theta G_p(x,y,z)$$

for all $x, y, z \in X$. Then f has a unique fixed point.

3. Fixed point in ordered GP-metric spaces

Let X be a nonempty set. If (X, G_p) is a GP-metric space and $(X, \preceq)$ is partially ordered, then $(X, G_p, \preceq)$ is called an ordered GP-metric space. The points $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. The mapping $f : X \to X$ is called non-decreasing if $x \preceq y$ implies $fx \preceq fy$ for all $x, y \in X$. In this section, we will show that many fixed point results in ordered GP-metric spaces can be deduced easily from our presented theorems.

Theorem 3.1. *Let $(X, G_p, \preceq)$ be an ordered GP-metric space such that (X, G_p) is GP-complete and $f : X \to X$ be a 0-GP-continuous and non-decreasing mapping. Assume that the following assertions hold:*

(i) $\frac{1}{3}G_p(x, fx, fx) \leq G_p(x, y, z)$ *implies*

$$G_p(fx, fy, fz) \leq \lambda G_p(x, y, z) + L\, M(x, y, z)$$

for all $x, y, z \in X$ with $x \preceq y \preceq z$ or $x \succeq y \succeq z$, where $\lambda \in [0, 1)$ and $L \geq 0$;

(ii) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$.

Then f has a fixed point in X.

Proof. Define the mappings $\Lambda : X \times X \times X \to \mathbb{R}^+$ and $\Theta : X \times X \times X \to \mathbb{R}^+$ by

$$\Lambda(x, y, z) = \begin{cases} 1 & \text{if } x \preceq y \preceq z \text{ or } x \succeq y \succeq z \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \Theta(x, y, z) = \lambda.$$

Clearly, the mapping f satisfies the contractive condition (2.1). Now, let $x, y, z \in X$ such that $\Lambda(x, y, z) \geq 1$. By the definition of the function Λ, this implies that $x \preceq y \preceq z$ or $x \succeq y \succeq z$. As the mapping f is non-decreasing, we deduce that $fx \preceq fy \preceq fz$ or $fx \succeq fy \succeq fz$ and hence $\Lambda(fx, fy, fz) \geq 1$. Consequently, f is a hybrid $GP_{(\Lambda,\Theta)}$-contractive mapping. The condition (ii) ensures that there exists $x_0 \in X$ such that $x_0 \preceq fx_0$. This implies that $\Lambda(x_0, fx_0, fx_0) \geq 1$. Therefore, all the hypotheses of Theorem 2.4 are satisfied and hence f has a fixed point. □

For self-mappings that are not 0-GP-continuous we have the following result.

Theorem 3.2. *Let $(X, G_p, \preceq)$ be an ordered GP-metric space such that (X, G_p) is GP-complete and $f : X \to X$ be a non-decreasing mapping. Assume that the following assertions hold:*

(i) there exists $\lambda \in [0, 1)$ and $L \geq 0$ such that $\frac{1}{3}G_p(x, fx, fx) \leq G_p(x, y, z)$ *implies*

$$G_p(fx, fy, fz) \leq \lambda G_p(x, y, z) + L\, M(x, y, z)$$

for all $x, y, z \in X$ with $x \preceq y \preceq z$ or $x \succeq y \succeq z$;

(ii) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$,

(iii) if $\{x_n\}$ is a non-decreasing sequence in X and $x_n \to z \in X$, then $x_n \preceq z$ for all $n \in \mathbb{N} \cup \{0\}$.

Then f has a fixed point in X.

Acknowledgements:

The third author is supported by Università degli Studi di Palermo, Local University Project R. S. ex 60%.

References

[1] M. Abbas, T. Nazir and P. Vetro, *Common fixed point results for three maps in G-metric spaces*, Filomat, **25**:4 (2011), 1–17. 1

[2] H. Aydi, M. Abbas and C. Vetro, *Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces*, Topology Appl., **159** (2012), 3234–3242. 1

[3] H. Aydi, E. Karapinar, P. Salimi, *Some fixed point results in GP-metric spaces*, J. Appl. Math., **2012** (2012), Article ID 891713. 1, 1.11, 1.12

[4] H. Aydi, W. Shatanawi and C. Vetro, *On generalized weak G-contraction mapping in G-metric spaces*, Comput. Math. Appl., **62** (2011), 4223–4229. 1

[5] H. Aydi, C. Vetro, W. Sintunavarat and P. Kumam, *Coincidence and fixed points for contractions and cyclical contractions in partial metric spaces*, Fixed Point Theory Appl., **2012**: 124 (2012). 1

[6] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181. 1, 1

[7] V. Berinde and F. Vetro, *Common fixed points of mappings satisfying implicit contractive conditions*, Fixed Point Theory Appl., **2012**:105 (2012). 1

[8] S.K. Chatterjea, *Fixed point theorem*, Comte Rend. Acad. Bulgare Sc., **25** (1972), 727–730. 1

[9] S. Chauhan and B.D. Pant, *Fixed point theorems for compatible and subsequentially continuous mappings in Menger spaces*, J. Nonlinear Sci. Appl. **7** (2) (2014), 78–89. 1

[10] Lj.B. Ćirić, *A Generalization of Banach's contraction principle*, Proc. Amer. Math. Soc., **45** (1974), 267–273. 1

[11] Lj.B. Ćirić, B. Samet, H. Aydi and C. Vetro, Common fixed points of generalized contractions on partial metric spaces and an application, Appl. Math. Comput. 218 (2011), 2398–2406. 1

[12] L. W. Cohen and C. Goffman, *The topology of ordered Abelian groups*, Trans. Amer. Math. Soc., **67** (1949), 310–319. 1.1

[13] B. Damjanovic, B. Samet and C. Vetro, *Common fixed point theorems for multi-valued maps*, Acta Math. Sci. Ser. B Engl. Ed. **32** (2012), 818–824. 1

[14] C. Di Bari and P. Vetro, *Fixed points for weak ϕ-contractions on partial metric spaces*, Int. J. of Engineering, Contemporary Mathematics and Sciences, **1** (2011), 5–13. 1

[15] C. Di Bari, M. Milojević, S, Radenović and P. Vetro, *Common fixed points for self-mappings on partial metric spaces*, Fixed Point Theory Appl., **2012**:140 (2012). 1

[16] C. Di Bari, Z. Kadelburg, H. Nashine and S. Radenović, *Common fixed points of g-quasicontractions and related mappings in 0-complete partial metric spaces*, Fixed Point Theory Appl., **2012**:113 (2012). 1

[17] R. Kannan, *Some results on fixed points*, Bull. Cal. Math. Soc., **60** (1968), 71–76. 1

[18] V. La Rosa and P. Vetro, *Fixed points for Geraghty-contractions in partial metric spaces*, J. Nonlinear Sci. Appl. **7** (1) (2014), 1–10. 1

[19] S.G. Matthews, *Partial metric topology*, in: Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci., **728** (1994), 183–197. 1

[20] D. Miheţ, *Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces*, J. Nonlinear Sci. Appl. **6** (1) (2013), 35–40. 1

[21] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7** (2) (2006), 289–297. 1

[22] Z. Mustafa and H. Obiedat, *A fixed point theorem of Reich in G-metric spaces*, CUBO, **12** (1) (2010), 83–93. 1

[23] Z. Mustafa, W. Shatanawi and M. Bataineh, *Existence of fixed point results in G-metric spaces*, Int. J. Math. Math. Sci., **2009** (2009) Article ID 283028, 10 pages. 1

[24] Z. Mustafa and B. Sims, *Fixed point theorems for contractive mappings in complete G-metric spaces*, Fixed Point Theory Appl., **2009** (2009), Article ID 917175, 10 pages. 1

[25] D. Paesano and P. Vetro, *Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces*, Topology Appl., **159** (2012), 911–920. 1

[26] S. Radenović, P. Salimi, S. Pantelic and J. Vujaković, *A note on some tripled coincidence point results in G-metric spaces*, 2012, Int. J. Math. Sci and Engg. Appls., **6** No. VI (November 2012), 23–38. 1

[27] S. Reich, *Kannan's fixed point theorem*, Boll. Un. Mat. Ital. **4** (1971), 1–11. 1

[28] I.A. Rus, *Fixed point theory in partial metric spaces*, Anal. Univ. de Vest, Timisoara, Seria Matematică-Informatică, **46** (2008), 141–160. 1

[29] B. Samet, M. Rajović, R. Lazović and R. Stoiljković, *Common fixed point results for nonlinear contractions in ordered partial metric spaces*, Fixed Point Theory Appl., **2011**:71 (2011). 1

[30] B. Samet, C. Vetro and P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Anal., **75** (2012), 2154–2165. 1

[31] R. Saadati, S.M. Vaezpour, P. Vetro and B.E. Rhoades *Fixed point theorems in generalized partially ordered G-metric spaces.* Mathematical and Computer Modelling., **52** (2010), 797–801. 1

[32] T. Suzuki, *A generalized Banach contraction principle that characterizes metric completeness*, Proc. Amer. Math. Soc., **136** (2008), 1861–1869. 1

[33] C. Vetro and F. Vetro, *Common fixed points of mappings satisfying implicit relations in partial metric spaces*, J. Nonlinear Sci. Appl. **6** (3) (2013), 152–161. 1

[34] F. Vetro, *On approximating curves associated with nonexpansive mappings*, Carpathian J. Math., **27** (2011), 142–147. 1

[35] F. Vetro and S. Radenović, *Nonlinear ψ-quasi-contractions of Ćirić-type in partial metric spaces*, Appl. Math. Comput., **219** (4) (2012), 1594–1600. 1

[36] M.R.A. Zand and A.D. Nezhad, *A generalization of partial metric spaces*, Journal of Contemporary Applied Mathematics. **24** (2011), 86–93. 1, 1.2, 1.3, 1.4, 1.5, 1.7, 1.8, 1.10