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Some common fixed point theorems in dislocated metric spaces

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Abstract

Our purpose in this paper is to establish some new common fixed point theorems for four self-mappings of a dislocated metric space.

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1. Introduction

In 2012, Panthi and Jha [3] have established the following result.

Theorem 1.1. ([3]) *Let A, B, T and S be four continuous self-mappings of a complete d-metric space (X, d) such that*

1. $TX \subset AX$ *and* $SX \subset BX$;
2. *The pairs* (S, A) *and* (T, B) *are weakly compatible;*
3. $d(Sx, Ty) \leq \alpha\,[d(Ax, Ty) + d(By, Sx)] + \beta\,[d(Ax, Sx) + d(By, Ty)] + \gamma d(Ax, By)$
   for all $x, y \in X$ *where* $\alpha, \beta, \gamma \geq 0$ *satisfying* $\alpha + \beta + \gamma < \dfrac{1}{4}$.

Then A, B, T and S have a unique common fixed point in X.

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Our purpose, here, is to prove that this theorem can be improved without any continuity requirement. Further, we will point out that if one supposes that $\gamma > 0$, then one can replace condition $\alpha + \beta + \gamma < \frac{1}{4}$ by $\alpha + \beta + \gamma \leq \frac{1}{4}$. Recall that the notion of dislocated metric, introduced in 2000 by Hitzler and Seda [1], is characterized by the fact that self distance of a point need not be equal to zero and has useful applications in topology, logical programming and in electronics engineering. For further details on dislocated metric spaces, see, for example [2, 4]. We begin by recalling some basic concepts of the theory of dislocated metric spaces.

Definition 1.2. Let X be a non empty set and let $d : X \times X \to [0, \infty)$ be a function satisfying the following conditions

1. $d(x, y) = d(y, x)$
2. $d(x, y) = d(y, x) = 0$ implies $x = y$
3. $d(x, y) = d(x, z) + d(z, y)$ forall $x, y, z \in X$.

Then d is called dislocated metric(or simply d-metric) on X.

Definition 1.3. A sequence (x_n) in a d-metric space (X, d) is called a Cauchy sequence if for given $\epsilon > 0$, there exists $n_0 \in N$ such that for all $m, n \geq n_0$, we have $d(x_m, x_n) < \epsilon$.

Definition 1.4. A sequence in a d-metric space converges if there exists $x \in X$ such that $d(x_n, x) \to 0$.

Definition 1.5. A d-metric space (X, d) is called complete if every Cauchy sequence is convergent.

Remark 1.6. It is easy to verify that in a dislocated metric space, we have the following technical properties:

- A subsequence of a Cauchy sequence in d-metric space is a Cauchy sequence.
- A Cauchy sequence in d-metric space which possesses a convergent subsequence, converges.
- Limits in a d-metric space are unique.

Definition 1.7. Let A and S be two self-mappings of a d-metric space (X, d).
A and S are said to be weakly compatible if they commute at their coincident point; that is, $Ax = Sx$ for some $x \in X$ implies $ASx = SAx$.

2. Main results

Theorem 2.1. *Let A, B, T and S be four self-mappings of a d-metric space (X, d) such that*

1. *$TX \subset AX$ and $SX \subset BX$;*
2. *The pairs (S, A) and (T, B) are weakly compatible;*
3. *For all $x, y \in X$ and $\alpha, \beta, \gamma \geq 0$ satisfying $\alpha + \beta + \gamma < \frac{1}{4}$, we have*

$$d(Sx, Ty) \leq \alpha \left[d(Ax, Ty) + d(By, Sx)\right] + \beta \left[d(Ax, Sx) + d(By, Ty)\right] + \gamma d(Ax, By); \qquad (2.1)$$

4. *The range of one of the mappings A, B, S or T is a complete subspace of X.*

Then A, B, T and S have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Choose $x_1 \in X$ such that $Bx_1 = Sx_0$. Choose $x_2 \in X$ such that $Ax_2 = Tx_1$. Continuing in this fashion, choose $x_n \in X$ such that $Sx_{2n} = Bx_{2n+1}$ and $Tx_{2n+1} = Ax_{2n+2}$ for $n = 0, 1, 2, ...$ To simplify, we consider the sequence (y_n) defined by $y_{2n} = Sx_{2n}$ and $y_{2n+1} = Tx_{2n+1}$ for $n = 0, 1, 2, ...$

We claim that (y_n) is a Cauchy sequence. Indeed, by using (2.1) for $n \geq 1$, we have

$$\begin{aligned}
d(y_{2n+1}, y_{2n}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\leq \alpha\,[d(Ax_{2n}, Tx_{2n+1}) + d(Bx_{2n+1}, Sx_{2n})] \\
&\qquad + \beta\,[d(Ax_{2n}, Sx_{2n}) + d(Bx_{2n+1}, Tx_{2n+1})] + \gamma d(Ax_{2n}, Bx_{2n+1}) \\
&\leq \alpha\,[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})] \\
&\qquad + \beta\,[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] + \gamma d(y_{2n-1}, y_{2n}) \\
&\leq \alpha\,[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n})] \\
&\qquad + \beta\,[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] + \gamma d(y_{2n-1}, y_{2n}) \\
&\leq (\alpha + \beta + \gamma) d(y_{2n-1}, y_{2n}) + (3\alpha + \beta) d(y_{2n}, y_{2n+1}).
\end{aligned}$$

Therefore

$$d(y_{2n}, y_{2n+1}) \leq h\; d(y_{2n-1}, y_{2n})$$

where $h = \dfrac{\alpha + \beta + \gamma}{1 - 3\alpha - \beta} \in [0, 1[$. Hence (y_n) is a Cauchy sequence in X and therefore, according to Remarks 1.1, (Sx_{2n}), (Bx_{2n+1}), (Tx_{2n+1}) and (Ax_{2n+2}) are also Cauchy sequence.

Suppose that SX is a complete subspace of X, then the sequence (Sx_{2n}) converges to some Sa such that $a \in X$. According to Remark (1.6), (y_n), (Bx_{2n+1}), (Tx_{2n+1}) and (Ax_{2n+2}) converge to Sa. Since $SX \subset BX$, there exists $u \in X$ such that $Sa = Bu$. We show that $Bu = Tu$. In fact, by using (2.1), we have

$$d(Sx_{2n}, Tu) \leq \alpha\,[d(Sx_{2n}, Tu) + d(Bu, Sx_{2n})] + \beta\,[d(Ax_{2n}, Sx_{2n}) + d(Bu, Tu)] + \gamma d(Ax_{2n}, Bu)$$

and therefore, on letting n to infty, we get

$$\begin{aligned}
d(Bu, Tu) &\leq \alpha\,[d(Bu, Tu) + d(Bu, Bu)] + \beta\,[d(Bu, Bu) + d(Bu, Tu)] + \gamma d(Bu, Bu) \\
&\leq (\alpha + \beta + \gamma)\;d(Bu, Bu) + (\alpha + \beta)\;d(Bu, Tu) \\
&\leq 2\,(\alpha + \beta + \gamma)\;d(Bu, Tu) + (\alpha + \beta)\;d(Bu, Tu) \\
&\leq (3\alpha + 3\beta + 2\gamma)\;d(Bu, Tu)
\end{aligned}$$

which implies that

$$(1 - 3\alpha - 3\beta - 2\gamma)\;d(Bu, Tu) \leq 0$$

and therefore $d(Bu, Tu) = 0$, since $(1 - 3\alpha - 3\beta - 2\gamma) < 0$, which implies that $Tu = Bu$. Since $TX \subset AX$, there exists $v \in X$ such that $Tu = Av$. We show that $Sv = Av$. Indeed, by using (2.1), we have

$$\begin{aligned}
d(Sv, Av) &= d(Sv, Tu) \\
&\leq \alpha\,[d(Av, Tu) + d(Bu, Sv)] + \beta\,[d(Av, Sv) + d(Bu, Tu)] + \gamma d(Av, Bu) \\
&\leq \alpha\,[d(Av, Av) + d(Av, Sv)] + \beta\,[d(Av, Sv) + d(Av, Av)] + \gamma d(Av, Av) \\
&\leq \alpha\,[d(Av, Sv) + d(Sv, Av) + d(Av, Sv)] + \beta\,[d(Av, Sv) + d(Av, Sv) + d(Sv, Av)] \\
&\qquad + \gamma\,[d(Av, Sv) + d(Sv, Av)] \\
&\leq (3\alpha + 3\beta + 2\gamma)\;d(Av, Sv)
\end{aligned}$$

which implies that

$$(1 - 3\alpha - 3\beta - 2\gamma)\;d(Av, Sv) \leq 0$$

and therefore $d(Av, Sv) = 0$, since $1 - 3\alpha - 3\beta - 2\gamma < 0$, which implies that $Av = Sv$. Hence $Bu = Tu = Av = Sv$.

The weak compatibility of S and A implies that $ASv = SAv$, from which it follows that $AAv = ASv =$

$SAv = SSv$.
The weak compatibility of B and T implies that $BTu = TBu$, from which it follows that $BBu = BTu = TBu = TTu$.
Let us show that Bu is a fixed point of T. Indeed, from (2.1), we get

$$\begin{aligned} d(Bu, TBu) &= d(Sv, TBu) \\ &\leq \alpha\left[d(Av, TBu) + d(BBu, Sv)\right] + \beta\left[d(Av, Sv) + d(BBu, TBu)\right] + \gamma d(Av, BBu) \\ &\leq \alpha\left[d(Bu, TBu) + d(TBu, Bu)\right] + \beta\left[d(Bu, Bu) + d(TBu, TBu)\right] + \gamma d(Bu, TBu) \\ &\leq 2\alpha d(Bu, TBu) + \beta\left[d(Bu, TBu) + d(TBu, Bu) + d(TBu, Bu) + d(Bu, TBu)\right] \\ &\quad + \gamma d(Bu, TBu) \\ &\leq (2\alpha + 4\beta + \gamma)\ d(Bu, TBu) \end{aligned}$$

and therefore $d(Bu, TBu) = 0$, since $1 - 2\alpha - 4\beta - \gamma < 0$, which implies that $TBu = Bu$. Hence Bu is a fixed point of T. It follows that $BBu = TBu = Bu$, which implies that Bu is a fixed point of B.
On the other hand, in view of (2.1), we have

$$\begin{aligned} d(SBu, Bu) &= d(SBu, TBu) \\ &\leq \alpha\left[d(ABu, TBu) + d(BBu, SBu)\right] + \beta\left[d(ABu, SBu) + d(BBu, TBu)\right] + \gamma d(ABu, BBu) \\ &\leq \alpha\left[d(SBu, Bu) + d(Bu, SBu)\right] + \beta\left[d(Bu, Bu) + d(Bu, Bu)\right] + \gamma d(Bu, Bu) \\ &\leq 2\alpha d(Bu, SBu) + \beta\left[d(Bu, SBu) + d(SBu, Bu) + d(Bu, SBu) + d(SBu, Bu)\right] \\ &\quad + \gamma\left[d(Bu, SBu) + d(SBu, Bu)\right] \\ &\leq (2\alpha + 4\beta + 2\gamma)\ d(Bu, SBu) \end{aligned}$$

and therefore $d(Bu, SBu) = 0$, since $1 - 2\alpha - 4\beta - 2\gamma < 0$, which implies that $SBu = Bu$. Hence Bu is a fixed point of S. It follows that $ABu = SBu = Bu$, which implies that Bu is also a fixed point of S. Thus Bu is a common fixed point of S, T, A and B.
Finally to prove uniqueness, suppose that there exists $u, v \in X$ such that $Su = Tu = Au = Bu$ and $Su = Tu = Au = Bv$. If $d(u, v) \neq 0$, then, by using (2.1), we get

$$\begin{aligned} d(u, v) &= d(Su, Tv) \\ &\leq \alpha\left[d(Au, Tv) + d(Bv, Su)\right] + \beta\left[d(Au, Su) + d(Bv, Tv)\right] + \gamma d(Au, Bv) \\ &\leq \alpha\left[d(u, v) + d(u, v)\right] + \beta\left[d(u, u) + d(v, v)\right] + \gamma d(u, v) \\ &\leq (2\alpha + 4\beta + \gamma) d(u, v) \end{aligned}$$

from which it follows that $(1 - 2\alpha - 4\beta - \gamma)\, d(u, v) \leq 0$ which is a contradiction since $1 - 2\alpha - 4\beta - \gamma < 0$.
Hence $d(u, v) = 0$ and therefore $u = v$.
The proof is similar when TX or AX or Bx is a complete subspace of X. This completes the proof. □

For $A = B$ and $S = T$ in (2.1), we have the following result.

Corollary 2.2. *Let (X, d) be a d-metric space. Let A and T be two self-mappings of X such that*

1. $TX \subset AX$
2. *The pair (T, A) is weakly compatible and*
3. $d(Tx, Ty) \leq \alpha\left[d(Ax, Ty) + d(Ay, Tx)\right] + \beta\left[d(Ax, Tx) + d(Ay, Ty)\right] + \gamma d(Ax, Ay)$ *for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ satisfying $\alpha + \beta + \gamma < \frac{1}{4}$*
4. *TX or AX is a complete subspace of X.*

Then A and T have a unique common fixed point in X.

For $A = B = Id_X$ in (2.1), we get the following corollary.

Corollary 2.3. *Let (X, d) be a d-metric space. Let T and S be two self-mappings of X such that*

1. $d(Sx,Ty) \leq \alpha\,[d(x,Ty)+d(y,Sx)] + \beta\,[d(x,Sx)+d(y,Ty)] + \gamma d(x,y)$
   for all $x,y \in X$ where $\alpha,\beta,\gamma \geq 0$ satisfying $\alpha+\beta+\gamma < \frac{1}{4}$
2. *TX or SX is a complete subspace of X.*

Then T and S have a unique common fixed point in X.

For $S = T = Id_X$ in (2.1), we have the following result.

Corollary 2.4. *Let (X,d) be a complete d-metric space. Let A and B be two surjective self-mappings of X such that*

$$d(x,y) \leq \alpha\,[d(Ax,y)+d(By,x)] + \beta\,[d(Ax,x)+d(By,y)] + \gamma d(Ax,By)$$

for all $x,y \in X$ where $\alpha,\beta,\gamma \geq 0$ satisfying $\alpha+\beta+\gamma < \frac{1}{4}$. Then A and B have a unique common fixed point in X.

Remark 2.5. Following the procedure used in the proof of Theorem in 2.1, we have the next new result in which we replace the condition $\alpha+\beta+\gamma < \frac{1}{4}$ by $\alpha+\beta+\gamma \leq \frac{1}{4}$

Theorem 2.6. *Let A, B, T and S be four self-mappings of a d-metric space (X,d) such that*

1. *$TX \subset AX$ and $SX \subset BX$*
2. *The pairs (S,A) and (T,B) are weakly compatible and*
3. *For all $x,y \in X$, $\alpha,\beta \geq 0$ and $\gamma > 0$ satisfying $\alpha+\beta+\gamma \geq \frac{1}{4}$, we have*

$$d(Sx,Ty) \leq \alpha\,[d(Ax,Ty)+d(By,Sx)] + \beta\,[d(Ax,Sx)+d(By,Ty)] + \gamma d(Ax,By) \tag{2.2}$$

4. *The range of one of the mappings A, B, S or T is a complete subspace of X.*

Then A, B, T and S have a unique common fixed point in X.

Proof. For $\alpha,\beta \geq 0$ and $\gamma > 0$ satisfying $\alpha+\beta+\gamma < \frac{1}{4}$, we apply Theorem 2.1. For $\alpha,\beta \geq 0$ and $\gamma > 0$ satisfying $\alpha+\beta+\gamma = \frac{1}{4}$, we consider, as in Theorem 2.1, an arbitrary point x_0 in X and the sequence (x_n) defined in X by $Sx_{2n} = Bx_{2n+1}$ and $Tx_{2n+1} = Ax_{2n+2}$ for $n = 0,1,2,....$ To simplify, we consider the sequence (y_n) defined by $y_{2n} = Sx_{2n}$ and $y_{2n+1} = Tx_{2n+1}$ for $n = 0,1,2,....$
We claim that (y_n) is a Cauchy sequence. Indeed, by using (2.2) for $n \geq 1$, we have

$$\begin{aligned}
d(y_{2n+1},y_{2n}) &= d(Sx_{2n},Tx_{2n+1})\\
&\leq \alpha\,[d(Ax_{2n},Tx_{2n+1})+d(Bx_{2n+1},Sx_{2n})]\\
&\qquad + \beta\,[d(Ax_{2n},Sx_{2n})+d(Bx_{2n+1},Tx_{2n+1})] + \gamma d(Ax_{2n},Bx_{2n+1})\\
&\leq \alpha\,[d(y_{2n-1},y_{2n+1})+d(y_{2n},y_{2n})]\\
&\qquad + \beta\,[d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})] + \gamma d(y_{2n-1},y_{2n})\\
&\leq \alpha\,[d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})+d(y_{2n},y_{2n+1})+d(y_{2n+1},y_{2n})]\\
&\qquad + \beta\,[d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})] + \gamma d(y_{2n-1},y_{2n})\\
&\leq (\alpha+\beta+\gamma)d(y_{2n-1},y_{2n}) + (3\alpha+\beta)d(y_{2n},y_{2n+1})\\
&\leq \frac{1}{4}d(y_{2n-1},y_{2n}) + (3\alpha+\beta)d(y_{2n},y_{2n+1}).
\end{aligned}$$

Therefore $d(y_{2n},y_{2n+1}) \leq h\ d(y_{2n-1},y_{2n})$, where $h = \dfrac{1}{4(1-3\alpha-\beta)} \in [0,1[$. Hence (y_n) is a Cauchy sequence in X and therefore, according to Remarks 1.1, (Sx_{2n}), (Bx_{2n+1}), (Tx_{2n+1}) and (Ax_{2n+2}) are also Cauchy sequence.
Suppose that SX is a complete subspace of X, then the sequence (Sx_{2n}) converges to some Sa such

that $a \in X$. According to Remark (1.6), (y_n), (Bx_{2n+1}), (Tx_{2n+1}) and (Ax_{2n+2}) converge to Sa. Since $SX \subset BX$, there exists $u \in X$ such that $Sa = Bu$. We show that $Bu = Tu$. In fact, in view of by using (2.1), we have

$$d(Sx_{2n}, Tu) \leq \alpha\left[d(Sx_{2n}, Tu) + d(Bu, Sx_{2n})\right] + \beta\left[d(Ax_{2n}, Sx_{2n}) + d(Bu, Tu)\right] + \gamma d(Ax_{2n}, Bu)$$

and therefore, on letting n to infty, we get

$$\begin{aligned} d(Bu, Tu) &\leq \alpha\left[d(Bu, Tu) + d(Bu, Bu)\right] + \beta\left[d(Bu, Bu) + d(Bu, Tu)\right] + \gamma d(Bu, Bu) \\ &\leq (\alpha + \beta + \gamma)\ d(Bu, Bu) + (\alpha + \beta)\ d(Bu, Tu) \\ &\leq 2(\alpha + \beta + \gamma)\ d(Bu, Tu) + (\alpha + \beta)\ d(Bu, Tu) \\ &\leq (3\alpha + 3\beta + 2\gamma)\ d(Bu, Tu) \\ &\leq \left(\frac{3}{4} - \gamma\right)\ d(Bu, Tu) \end{aligned}$$

which implies that $(\frac{1}{4} + \gamma)\ d(Bu, Tu) \leq 0$. Therefore $d(Bu, Tu) = 0$, which implies that $Tu = Bu$. Since $TX \subset AX$, there exists $v \in X$ such that $Tu = Av$. We show that $Sv = Av$. In fact, by using (2.2), we have

$$\begin{aligned} d(Sv, Av) &= d(Sv, Tu) \\ &\leq \alpha\left[d(Av, Tu) + d(Bu, Sv)\right] + \beta\left[d(Av, Sv) + d(Bu, Tu)\right] + \gamma d(Av, Bu) \\ &\leq \alpha\left[d(Av, Av) + d(Av, Sv)\right] + \beta\left[d(Av, Sv) + d(Av, Av)\right] + \gamma d(Av, Av) \\ &\leq \alpha\left[d(Av, Sv) + d(Sv, Av) + d(Av, Sv)\right] + \beta\left[d(Av, Sv) + d(Av, Sv) + d(Sv, Av)\right] \\ &\qquad + \gamma\left[d(Av, Sv) + d(Sv, Av)\right] \\ &\leq (3\alpha + 3\beta + 2\gamma)\ d(Av, Sv) \\ &\leq \left(\frac{3}{4} - \gamma\right)\ d(Av, Sv) \end{aligned}$$

which implies that $(\frac{1}{4} + \gamma)\ d(Av, Sv) \leq 0$. Therefore $d(Av, Sv) = 0$, which implies that $Av = Sv$. Hence $Bu = Tu = Av = Sv$.

The weak compatibility of S and A implies that $ASv = SAv$, from which it follows that $AAv = ASv = SAv = SSv$.

The weak compatibility of B and T implies that $BTu = TBu$, from which it follows that $BBu = BTu = TBu = TTu$.

Let us show that Bu is a fixed point of T. Indeed, by using (2.2), we have

$$\begin{aligned} d(Bu, TBu) &= d(Sv, TBu) \\ &\leq \alpha\left[d(Av, TBu) + d(BBu, Sv)\right] + \beta\left[d(Av, Sv) + d(BBu, TBu)\right] + \gamma d(Av, BBu) \\ &\leq \alpha\left[d(Bu, TBu) + d(TBu, Bu)\right] + \beta\left[d(Bu, Bu) + d(TBu, TBu)\right] + \gamma d(Bu, TBu) \\ &\leq 2\alpha d(Bu, TBu) + \beta\left[d(Bu, TBu) + d(TBu, Bu) + d(TBu, Bu) + d(Bu, TBu)\right] \\ &\qquad + \gamma d(Bu, TBu) \\ &\leq (2\alpha + 4\beta + \gamma)\ d(Bu, TBu) \end{aligned}$$

and therefore $d(Bu, TBu) = 0$, since $1 - 2\alpha - 4\beta - \gamma \geq 1 - 4(\frac{1}{4} - \gamma) - \gamma = 3\gamma > 0$, which implies that $TBu = Bu$. Hence Bu is a fixed point of T. It follows that $BBu = TBu = Bu$, which implies that Bu is a fixed point of B.

On the other hand, by using (2.2), we have

$$\begin{aligned} d(SBu, Bu) &= d(SBu, TBu) \\ &\leq \alpha\left[d(ABu, TBu) + d(BBu, SBu)\right] + \beta\left[d(ABu, SBu) + d(BBu, TBu)\right] + \gamma d(ABu, BBu) \\ &\leq \alpha\left[d(SBu, Bu) + d(Bu, SBu)\right] + \beta\left[d(Bu, Bu) + d(Bu, Bu)\right] + \gamma d(Bu, Bu) \\ &\leq 2\alpha d(Bu, SBu) + \beta\left[d(Bu, SBu) + d(SBu, Bu) + d(Bu, SBu) + d(SBu, Bu)\right] \\ &\qquad + \gamma\left[d(Bu, SBu) + d(SBu, Bu)\right] \\ &\leq (2\alpha + 4\beta + 2\gamma)\ d(Bu, SBu) \end{aligned}$$

and therefore $d(Bu, SBu) = 0$, since $1 - 2\alpha - 4\beta - 2\gamma \geq 1 - 4(\frac{1}{4} - \gamma) - 2\gamma = 2\gamma > 0$, which implies that $SBu = Bu$. Hence Bu is a fixed point of S. It follows that $ABu = SBu = Bu$, which implies that Bu is also a fixed point of S. Thus Bu is a common fixed point of S, T, A and B.
Finally to prove uniqueness, suppose that there exists $u, v \in X$ such that $Su = Tu = Au = Bu$ and $Su = Tu = Au = Bv$. If $d(u, v) \neq 0$, then by using (2.2), we get

$$\begin{aligned} d(u,v) &= d(Su,Tv) \\ &\leq \alpha\,[d(Au,Tv) + d(Bv,Su)] + \beta\,[d(Au,Su) + d(Bv,Tv)] + \gamma d(Au,Bv) \\ &\leq \alpha\,[d(u,v) + d(u,v)] + \beta\,[d(u,u) + d(v,v)] + \gamma d(u,v) \\ &\leq (2\alpha + 4\beta + \gamma) d(u,v) \end{aligned}$$

from which it follows that $(1 - 2\alpha - 4\beta - \gamma)\, d(u, v) \leq 0$ which is a contradiction since $1 - 2\alpha - 4\beta - \gamma < 0$. Hence $d(u, v) = 0$ and therefore $u = v$.
The proof is similar when TX or AX or Bx is a complete subspace of X. This completes the proof. □

For $A = B$ and $S = T$ in (2.6), we have the following result.

Corollary 2.7. *Let (X, d) be a d-metric space. Let A and T be two self-mappings of X such that*

1. $TX \subset AX$
2. *The pair (T, A) is weakly compatible and*
3. $d(Tx, Ty) \leq \alpha\,[d(Ax, Ty) + d(Ay, Tx)] + \beta\,[d(Ax, Tx) + d(Ay, Ty)] + \gamma d(Ax, Ay)$
   for all $x, y \in X$ where $\alpha, \beta \geq 0$ and $\gamma > 0$ satisfying $\alpha + \beta + \gamma \leq \frac{1}{4}$
4. *TX or AX is a complete subspace of X.*

Then A and T have a unique common fixed point in X.

For $A = B = Id_X$ in (2.6), we get the following corollary.

Corollary 2.8. *Let (X, d) be a d-metric space. Let T and S be two self-mappings of X such that*

1. $d(Sx, Ty) \leq \alpha\,[d(x, Ty) + d(y, Sx)] + \beta\,[d(x, Sx) + d(y, Ty)] + \gamma d(x, y)$
   for all $x, y \in X$ where $\alpha, \beta \geq 0$ and $\gamma > 0$ satisfying $\alpha + \beta + \gamma \leq \frac{1}{4}$
2. *TX or SX is a complete subspace of X.*

Then T and S have a unique common fixed point in X.

For $S = T = Id_X$ in (2.6), we have the following result.

Corollary 2.9. *Let (X, d) be a complete d-metric space. Let A and B be two surjective self-mappings of X such that*

$$d(x, y) \leq \alpha\,[d(Ax, y) + d(By, x)] + \beta\,[d(Ax, x) + d(By, y)] + \gamma d(Ax, By)$$

for all $x, y \in X$ where $\alpha, \beta \geq 0$ and $\gamma > 0$ satisfying $\alpha + \beta + \gamma \leq \frac{1}{4}$. Then A and B have a unique common fixed point in X.

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