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# Bifurcation in a variational problem on a surface with a distance constraint

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**Abstract**

We describe a variational problem on a surface of a Euclidean space under a distance constraint. We provide sufficient and necessary conditions for the existence of bifurcation points, generalizing Skrypnik's analog described in [P. Vyridis, Int. J. Nonlinear Anal. Appl. **2** (2011), 1–10]. The problem in local coordinates corresponds to an elliptic boundary value problem. ©2014 All rights reserved.

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## 1. Introduction

We consider a bifurcation problem of variational character in the form

$$F'[u] - \lambda\, G'[u] = 0\,, \tag{1.1}$$

where $F$, $G$ are functionals defined on a Hilbert space $X$, with $F'[0] = G'[0] = 0$, and $\lambda$ is a real parameter.

**Definition 1.1.** The number $\lambda_0$ is a bifurcation point for equation (1.1) if and only if in every sufficiently small neighborhood of $(0, \lambda_0)$ there exists a solution $(u, \lambda)$ of (1.1) with $u \neq 0$.

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Suppose that the functionals $F$ and $G$ satisfy the following conditions:
1. The functional $G$ is weakly continuous, differentiable, and its differential is Lipschitz continuous with

$$G'[u] = A\,u + N(u)\,, \tag{1.2}$$

where $A$ is a linear self-adjoint and compact operator. For the nonlinear part $N$ the following estimate holds:

$$\|N(u)\| \leq c\,\|u\|^p\,, \tag{1.3}$$

where $c$ is a positive constant, $p > 1$ and $u \in \mathcal{V}$.
2. The functional $F$ is differentiable with the property:

$$F'[u] = B\,u + L(u)\,, \tag{1.4}$$

where $B$ is a linear, bounded, self-adjoint and positive definite operator. For the nonlinear part $L$ the following estimates hold:

$$\|L(u)\| \leq c\,\|u\|^r\,, \quad \|L(u_1) - L(u_2)\| \leq c\,\left(\|u_1\|^{r-1} + \|u_2\|^{r-1}\right)\,\|u_1 - u_2\|\,, \tag{1.5}$$

where $c$ is a positive constant, $r > 1$, and $u$, $u_1$, $u_2 \in \mathcal{V}$.
Then according to Skrypnik's theory [5], every $\lambda \in \mathbb{R}$, corresponding to a non zero critical point $u$ of the functional

$$I[u, \lambda] = G[u] - \lambda\,F[u],$$

is a bifurcation point for the equation

$$I'[u, \lambda]\,w = G'[\,u\,]\,w - \lambda\,F'[\,u\,]\,w = 0 \tag{1.6}$$

in Hilbert space $X$ if and only if the equation

$$I''[0, \lambda](u, w) = (I''[0, \lambda]\,u,\,w) = 0 \tag{1.7}$$

is satisfied by a non zero solution $u$ for all $w \in X$.
Note that under these assumptions equation (1.7) can be rewritten as

$$A\,u - \lambda\,B\,u = 0.$$

We have developed an analog of Skrypnik's theory [6] on the existence of bifurcation points for the variational problem

$$F'[u] - \lambda\,G'[u] = 0\,, \quad \Phi[u] = 0\,, \quad u \in X \tag{1.8}$$

for constraints of the type $\Phi : X \longrightarrow \mathbb{R}$, where $\Phi$ is a continuous and differentiable mapping with

$$\Phi[\,0\,] = 0\,.$$

In previous applications [6, 7] such constraints have been defined by functionals of integral type. The equation of the constraint

$$\Phi[\,u\,] = 0 \tag{1.9}$$

restricts the domain of (1.1) to a smaller subspace according to Lyapunov - Schmidt decomposition. We consider that the solutions of equation (1.9) for small values of $\|u\|$ are a coset in a neighborhood of $0 \in X$, i.e.

$$X = X_1 \oplus X_2\,,$$

where

$$X_1 = \mathrm{Ker}\Phi'[0] \neq 0\,, \quad X_2 = X_1^{\perp}\,,$$

and there exists a continuous differentiable mapping $h$ from a small neighborhood of $0 \in X_1$ to a small neighborhood of $0 \in X_2$ such that the set of all solutions

$$u = v + w\,,\;\; v \in X_1\,,\;\; w \in X_2$$

is written in the form:

$$u = v + h(v)\,, \quad v \in X_1 \tag{1.10}$$

with

$$h(0) = 0\,, \quad h'(0) = 0\,. \tag{1.11}$$

According to (1.10), we define the functionals:

$$J[\,v] = G[v + h(v)] = G[\,u]\,, \quad v \in X_1\,, \tag{1.12}$$

and

$$Q[\,v] = F[v + h(v)] = F[\,u]\,, \quad v \in X_1\,. \tag{1.13}$$

Then the derivatives

$$\mathcal{D}F[\,u] = Q'[\,v]\,, \quad \mathcal{D}G[\,u] = J'[\,v]$$

have the meaning of differentiation of the functionals $F$ and $G$ along the tangential direction of the manifold $\{v + h(v)\,,\, v \in X_1\}$. Thus the bifurcation problem (1.8) is equivalent to the problem:

$$Q'[\,v\,] - \lambda\, J'[\,v\,] = 0\,, \quad v \in X_1\,, \tag{1.14}$$

or equivalently

$$\mathcal{D}F[\,u] - \lambda\,\mathcal{D}G[\,u] = 0\,, \quad u \in X\,. \tag{1.15}$$

**Definition 1.2.** The number $\lambda_0$ is a bifurcation point for equation (1.15) if in the intersection of any sufficiently small neighborhood of $(0, \lambda_0)$ with the manifold $\{v + h(v)\,,\, v \in X_1\}$ ther exists a solution $(u, \lambda)$ of (1.15) with $u \neq 0$.

It has been proved [6] that the functionals (1.13) and (1.12) satisfy the properties (1.4), (1.5), (1.2), (1.3), and the appropriate conditions of continuity and differentiability, with the additional condition $r \geq 2$ in a small neighborhood of subspace $X_1$. This leads to the following result [6]:

**Theorem 1.3.** *Let $X$ be a Hilbert space and the functionals $G[\,u]$, $F[\,u]$, defined in a neighborhood of $0 \in X$, satisfy properties (1.4), (1.5), (1.2), (1.3) and the appropriate conditions of continuity and differentiability for $r \geq 2$. Let $\Phi : X \longrightarrow \mathbb{R}$ be a continuous differentiable functional, which satisfies the conditions:*

$$\Phi[\,0\,] = 0\,, \quad Ker\,\Phi'[\,0\,] = X_1 \neq 0\,.$$

*Then the number $\lambda \neq 0$ is a bifurcation point for problem (1.15) if and only if the equation*

$$(PAP - \lambda\, PBP)\, u = 0\,, \quad u \in X\,,$$

*where $P : X \longrightarrow X_1$ the orthogonal projector, has a non zero solution.*

It is obvious that bifurcation points exists when $PAP \neq 0$.

In this work, we extend this suggested analog for constraints of a more general type, represented by a differentiable mapping $\Phi : X \longrightarrow Y$ between the Hilbert spaces $X, Y$. Suppose that $\Phi$ is a weakly continuous mapping in $X$. Thus, $\Phi'[0]$ is a compact operator in and invertible in $X_2$. This implies that the identity operator $Id = \Phi'[0]^{-1}\Phi'[0] : Y \longrightarrow Y$ is a compact operator, which is true only in the case that the Hilbert space $Y$ is of finite dimension. Thus the mapping $\Phi$ has to be weakly continuous in a larger space $Y_1$, such that $Y \subset Y_1$.

**Proposition 1.4.** *Suppose that the the functional $G$ is weakly continuous and the mapping $\Phi : X \longrightarrow Y_1$ is weakly continuous. Then the functional $J : X_1 \longrightarrow \mathbb{R}$, defined by (1.12), is also weakly continuous.*

*Proof.* Let $v_n \in X_1$ be the sequence with $\|v_n\| < \delta$, for all $n \in \mathbb{N}$ with respect to the norm of $X$, such that $v_n$ converges weakly to $v$. We suppose that $J[v_n]$ does not converge to $J[v]$. Then there exists $\varepsilon > 0$ and a subsequence $v_n$ (we keep the same index) such that

$$|J[v_n] - J[v]| = |G[v_n + h(v_n)] - G[v + h(v)]| \geq \varepsilon\,. \tag{1.16}$$

The sequence $h(v_n)$ is bounded, as the values of the mapping h located in a small neighborhood of $0 \in X_2$, so there exists a subsequence $h(v_k)$, which converges weakly to $w$. The equation of constraint (1.9) also implies that

$$\Phi[v_k + h(v_k)] = 0\,.$$

Since the mapping $\Phi$ is weakly continuous in $Y_1$ we deduce that

$$\lim_{k\to\infty} \Phi[v_k + h(v_k)] = \Phi[v + w] = 0\,. \tag{1.17}$$

The solutions of (1.17) for small values of $\|v\|$ and $\|w\|$ are represented by

$$w = h(v)\,. \tag{1.18}$$

Since the functional $G$ is weakly continuous, equation (1.18) and inequality (1.16) lead to a contradiction for $n = k$. □

Using proposition (1.4), we generalize theorem (1.3):

**Theorem 1.5.** *Let $X$ be a Hilbert space and the functionals $G[u]$, $F[u]$, defined in a neighborhood of $0 \in X$, satisfy the properties (1.4), (1.5), (1.2), (1.3), and the appropriate conditions of continuity and differentiability for $r \geq 2$. Let $\Phi : X \longrightarrow Y$ be a continuous differentiable mapping, which satisfies the conditions*

$$\Phi[0] = 0\,, \quad Ker\,\Phi'[0] = X_1 \neq 0\,,$$

*and there exists a Hilbert space $Y_1$ with $Y \subset Y_1$ such that the mapping $\Phi : X \longrightarrow Y_1$ is weakly continuous. Then the number $\lambda \neq 0$ is a bifurcation point for problem (1.15) if and only if the equation*

$$(PAP - \lambda\, PBP)\, u = 0\,, \quad u \in X\,,$$

*where $P : X \longrightarrow X_1$ is the orthogonal projector, has a non zero solution.*

## 2. Description of the constraint

Let $M$ be a smooth and connected surface in $\mathbb{R}^3$ and $S \subset M$ an open region in $M$ with a smooth boundary $\partial S$. Consider the closed curve $\partial S$ as a one - dimensional compact submanifold in $M$, the tangent space $T_x\partial S$ of $\partial S$ as a linear subspace of the tangent space $T_xM$ of $M$ at $x \in \partial S$, as well as the continuously differentiable vector field $\vec{\nu}(x)$, $x \in \mathbb{R}^3$ identified to the normal vector field of the curve $\partial S$ at $x \in \partial S$, located in the tangent space $T_xM \subseteq \mathbb{R}^3$ and is vertical to the tangent space $T_x\partial S$. Since the boundary $\partial S$ is a compact submanifold in $M$, for fixed $x \in \partial S$ the ball

$$B = \{\vec{\nu}(x) \in T_xM : \quad |\vec{\nu}(x)| < \varepsilon\}$$

is diffeomorphical to a neighborhood $U$ of $x \in \partial S$ in $M$. Thus, starting from $y \in U$, there exists a localy unique geodesic $\gamma$ to $\partial S$, which realizes the distance from $y \in U$ to the boundary $\partial S$, denoted by

$$\rho(y) = \operatorname{dist}(y, \partial S)\,. \tag{2.1}$$

The function $\rho$ depends smoothly on the points $y \in U$ [2]. Let $y \in M$, $y \notin \partial S$. Since the curve $\partial S$ is a smooth submanifold of $M$, there exists $\varepsilon > 0$ such that the set

$$\Gamma_\varepsilon = \{y \in M : \quad \rho(y) = \varepsilon\}$$

is a smooth curve on $M$ and such that the distance $\rho = \mathrm{dist}(\cdot, \partial S)$ is a smooth function on the set

$$U_\varepsilon = \{y \in M : \quad 0 < \rho(y) < 2\,\varepsilon\}\,.$$

For such $\varepsilon > 0$ the geodesic

$$\gamma : [0,1] \longrightarrow U_\varepsilon \subset M\,, \quad \gamma(0) \in \partial S\,, \quad \gamma(\varepsilon) = y$$

realizes the distance from $y$ to $\partial S$. Let $z$ be the point on $\partial S$ such that

$$\rho(y) = \mathrm{dist}(y, \partial S) = \mathrm{dist}(y, z)\,, \qquad \gamma(0) = z\,.$$

Since $M$ is a connected surface, we have that

$$\rho(y) = \rho(z) + \int_\gamma \mathrm{grad}\rho(y)\, ds = \int_\gamma \mathrm{grad}\rho(y)\, ds = \int_0^\varepsilon \mathrm{grad}\rho(\gamma(t))\, \dot{\gamma}(t)\, dt\,.$$

On the other hand, the length of the curve $\gamma$ is given by

$$L(\gamma) = \int_\gamma ds = \int_0^\varepsilon \sqrt{g_{ij}(\gamma(t))\, \dot{\gamma}^i(t)\, \dot{\gamma}^j(t)}\, dt = \int_0^\varepsilon \frac{g_{ij}(\gamma(t))\, \dot{\gamma}^i(t)\, \dot{\gamma}^j(t)}{\sqrt{g_{ij}(\gamma(t))\, \dot{\gamma}^i(t)\, \dot{\gamma}^j(t)}}\, dt\,,$$

where $g_{ij}(y)$ are the components of metric tensor $g(y)$ at $y \in M$. Since the curve $\gamma$ is the geodesic that connects the point $z$ and $y$, we obtain

$$\rho(y) = L(\gamma)\,,$$

or equivalently

$$\mathrm{grad}\,\rho(\gamma(t)) = \frac{g(\gamma(t))\, \dot{\gamma}(t)}{|\dot{\gamma}(t)|} = g(\gamma(t))\, \vec{\nu}(\gamma(t))\,,$$

where $\vec{\nu}(\gamma(t))$ is the unit normal vector field of the curve $\Gamma_t$ at point $\gamma(t)$ for $t \le \varepsilon$. Evaluating at $t = 0$, and considering a local coordinate system at $z \in \partial S$, we obtain

$$\frac{\partial \rho(z)}{\partial z^i} = g_{ij}(z)\, \nu^j(z)\,. \tag{2.2}$$

We consider now the mapping

$$y : \partial S \longrightarrow U_\varepsilon \subset M\,, \qquad y(x) = x + \vec{u}(x)\,, \tag{2.3}$$

where $\vec{u} \in W_2^2(\partial S, T_x M)$ for small values of $\|\vec{u}\|$. The mapping (2.3) leaves invariant the boundary $\partial S$ if and only if

$$y(\partial S) \subset \partial S\,. \tag{2.4}$$

We define the mapping

$$\Phi : W_2^2(\partial S, T_x M) \longrightarrow W_2^2(\partial S)\,, \qquad \Phi[\,\vec{u}\,] = \rho(y) = \rho(x + \vec{u}(x)) \tag{2.5}$$

in a small neighborhood of $\vec{0} \in W_2^2(\partial S, T_x M)$. Thus, the constraint (2.4) holds if and only if

$$\Phi[\,\vec{u}\,] = 0\,. \tag{2.6}$$

The mapping $\Phi$ is differentiable, due to the differentiability of the distance function. For a vector field $\vec{v} \in W_2^2(\partial S, T_x M)$ we consider the following representation:

$$\vec{v}(x) = \varphi(x)\, \vec{\tau}(x) + \psi(x)\, \vec{\nu}(x)\,, \quad x \in \partial S\,, \quad \varphi, \psi \in W_2^2(\partial S)\,, \tag{2.7}$$

where $\vec{\tau}(x) \in T_x \partial S$ is the tangent unit vector of the curve $\partial S$, and $\vec{\nu}(x) \in T_x M$ is the normal unit vector of the curve $\partial S$ vertical to $T_x \partial S$ at point $x \in \partial S$.

**Proposition 2.1.** *There exists a decomposition of the space $W_2^2(\partial S, T_xM)$ in a direct sum*

$$W_2^2(\partial S,\, T_xM) = X_1 \oplus X_2\,, \tag{2.8}$$

*where*

$$\begin{aligned} X_1 = &\ \left\{\vec{v} \in W_2^2(\partial S,\, T_xM),\quad \vec{v}(x) = \varphi(x)\,\vec{\tau}(x),\quad \varphi \in W_2^2(\partial S)\right\}\,, \\ X_2 = &\ \left\{\vec{v} \in W_2^2(\partial S,\, T_xM),\quad \vec{v}(x) = \psi(x)\,\vec{\nu}(x),\quad \psi \in W_2^2(\partial S)\right\}\,, \end{aligned}$$

*and a differentiable mapping $h$ from a neighborhood of $X_1$ to a neighborhood of $X_2$, such that the solutions of equation (2.6) can be expressed as*

$$\vec{u} = \vec{v} + h[\,\vec{v}\,]\,,\quad \vec{v} \in X_1\,, \tag{2.9}$$

*with*

$$h[\,\vec{0}\,] = \vec{0}\,,\quad h'[\,\vec{0}\,] = 0\,. \tag{2.10}$$

*Furthermore, $\Phi$ is weakly continuous as a mapping from $W_2^2(\partial S,\, T_xM)$ into the space $C(\partial S)$.*

*Proof.* First, we observe that for $\vec{u}, \vec{v} \in W_2^2(\partial S, T_xM)$ the variation of the mapping $\Phi$ is

$$\Phi[\,\vec{u} + \vec{v}\,] - \Phi[\,\vec{u}\,] = \Phi'[\,\vec{u}\,]\,\vec{v} + B[\,\vec{u}\,](\vec{v},\vec{v})\,,$$

where

$$\Phi'[\,\vec{u}\,]\,\vec{v} = \frac{\partial}{\partial x^i}\,\rho(x + \vec{u}(x))\,v^i(x)\,,$$

and

$$B[\,\vec{u}\,](\vec{v},\vec{v}) = \int_0^1 (1-t)\,\frac{\partial^2}{\partial x^i \partial x^j}\,\rho(x + \vec{u}(x) + t\,\vec{v}(x))\,v^i(x)\,v^j(x)\,dt\,.$$

Following the methods described in [4], by the boudness of the embedding of $W_2^2(\partial S)$ into spaces $C(\partial S)$ and $C^1(\partial S)$ we obtain the estimates:

$$\begin{aligned} \|\Phi'[\,\vec{u}\,]\,\vec{v}\|_{W_2^2(\partial S)} &\le C\big[\,\|\vec{v}\|_{W_2^2} + \|\vec{u}\|_{C^1}\|\vec{v}\|_{C^1} + \|\vec{u}\|_{C^1}^2\|\vec{v}\|_C + \|\vec{u}\|_{W_2^2}\|\vec{v}\|_C\,\big] \le \\ &\le C'\,(1 + \|\vec{u}\|_{W_2^2})\,\|\vec{v}\|_{W_2^2} \end{aligned}$$

and

$$\begin{aligned} \|B[\,\vec{u}\,](\vec{v},\vec{v})\|_{W_2^2(\partial S)} &\le C\big[\,\|\vec{v}\|_C\|\vec{v}\|_{W_2^2} + \|\vec{v}\|_{C^1}^2 + (\|\vec{u}\|_C + \|\vec{v}\|_C + \|\vec{u}\|_{W_2^2} + \|\vec{v}\|_{W_2^2})\|\vec{v}\|_{W_2^2}\big] \\ &\le C'\big(1 + \|\vec{u}\|_{W_2^2} + \|\vec{v}\|_{W_2^2}\big)\|\vec{v}\|_{W_2^2}^2\,, \end{aligned}$$

where $C$ and $C'$ are various constants. This means that the mapping $\Phi$ is continiously differentiable. In the same manner, we can verify that $\Phi'[\,\vec{u}\,]$ depends continiously on $\vec{u}$.
Now the conclusion comes from the Lyapunov - Schmidt reduction, and the implicit function theorem [4]. By the definition (2.5) of the mapping $\Phi$, it is obvious that

$$\Phi[\,\vec{0}\,] = 0\,,$$

and by (2.2), and the reprsentation (2.7) of a vector field $\vec{v} \in W_2^2(\partial S, T_xM)$, we obtain that

$$\Phi'[\,\vec{0}\,]\,\vec{v} = g_{ij}(x)\nu^i(x)v^j(x) = \psi(x)\,,\quad \psi \in W_2^2(\partial S)\,.$$

Thus we set

$$X_1 = \operatorname{Ker}\Phi'[\,\vec{0}\,] \ne \{\,\vec{0}\,\}\,,\quad X_2 = X_1^{\perp}\,.$$

Finally, $\Phi$ is weakly continuous as a maping from $W_2^2(\partial S,\, T_xM)$ into $C(\partial S)$, due to the compactness of the embedding of $W_2^2(\partial S)$ into $C(\partial S)$. □

## 3. The constrained variational problem

Let $M$ be a smooth surface in $\mathbb{R}^3$, $\vec{\eta}(x)\,,\ x \in \mathbb{R}^3$ a continuously differentiable vector field identified to the normal vector field for every $x \in M$ and $S$ an open and connected set in $M$, with boundary $\partial S$ consisting of two non-intersecting sufficiently smooth components $\Gamma$ and $\Gamma_1$. We assume that the mean curvature $H$ of surface $M$ does not vanish [7].

Let a vector field $\vec{u} \in H_0(S,\, T_xM)$, where

$$H_0(S,\, T_xM) = \left\{ \vec{u} \in W_2^1(S,\, T_xM)\,,\ \vec{u}|_\Gamma \in W_2^2(\Gamma,\, T_xM)\,,\ \vec{u}|_{\Gamma_1} = \vec{0} \right\}\,.$$

We denote by $W_2^1(S, T_xM)$ and $W_2^2(\Gamma, T_xM)$ the Sobolev spaces of functions defined on $S$ and $\Gamma$ with values in $T_xM \subset \mathbb{R}^3$, respectively. We introduce the following functionals

$$F[\,\vec{u}\,] = \frac{1}{2}\int_S a_{ijkl}(x)\,\xi_{ij}(\vec{u})\,\xi_{kl}(\vec{u})\,dS + \frac{1}{2}\int_\Gamma |\delta_i\delta_i\,\vec{u}|^2 ds\,, \tag{3.1}$$

$$G[\,\vec{u}\,] = \int_\Gamma q(\vec{u}, x)\,ds\,, \tag{3.2}$$

$$I[\,\vec{u}, \lambda\,] = F[\,\vec{u}\,] - \lambda\, G[\,\vec{u}\,]\,, \quad \lambda \in \mathbb{R}. \tag{3.3}$$

The coefficients $a_{ijkl} \in L_\infty(S)$ satisfy the symmetry properties $a_{ijkl}(x) = a_{klij}(x)$, and are positive definite, i.e.

$$a_{ijkl}(x)\,\xi^{ij}\xi^{kl} \geq \Lambda\,\xi^{ij}\,\xi^{ij}\,, \qquad \Lambda > 0\,. \tag{3.4}$$

The tensor $\xi_{ij}(\vec{u})$ is defined as

$$\xi_{ij}(\vec{u}) = \frac{1}{2}\,(\nabla_i u^j + \nabla_j u^i)\,, \tag{3.5}$$

where $\nabla_i$ is the $i$-th component of the tangent differentiation with respect to the surface $M$ [3]:

$$\nabla_i = \frac{\partial}{\partial x^i} - \eta^i(x)\,\eta^j(x)\,\frac{\partial}{\partial x^j}\,, \quad i = 1, 2, 3\,, \quad x \in M\,, \tag{3.6}$$

and $\delta_i$ is the $i$-th component of the tangent directional differentiation along the curve $\partial S$:

$$\delta_i = \tau^i(x)\,\frac{d}{ds} = \tau^i(x)\,\tau^j(x)\,\frac{\partial}{\partial x^j}\,, \quad i = 1, 2, 3\,, \quad x \in \partial S\,. \tag{3.7}$$

Finally, we assume that function $q$ is three times differentiable with the following properties

$$q(\vec{0}, x) = 0\,, \quad q_{u^i}(\vec{0}, x) = 0\,, \quad x \in \Gamma\,, \quad i = 1, 2, 3. \tag{3.8}$$

Now a critical point for the functional (3.3) under the constraint (2.6), for a given $\lambda \in \mathbb{R}$, is the vector field $\vec{u} \in X$, which satisfies the relation

$$I'[\,\vec{u}, \lambda\,]\,\vec{r} = 0\,, \tag{3.9}$$

or equivalently

$$\int_S a_{ijkl}(x)\,\xi_{ij}(\vec{u})\,\xi_{kl}(\vec{r})\,dS + \int_\Gamma \delta_i\delta_i\,\vec{u}\,\delta_j\delta_j\,\vec{r}\,ds - \lambda\int_\Gamma q_{u^i}(\vec{u}, x)\,r^i ds = 0 \tag{3.10}$$

for all $\vec{r} \in X$, where the vector fields $\vec{u}$ and $\vec{r}$ have the representation (1.10):

$$\vec{u} = \vec{v} + h(\vec{v})\,, \quad \vec{r} = \vec{w} + h(\vec{w})\,, \quad \vec{v}\,, \vec{w} \in X_1\,,$$

and the space $X_1$ is defined in proposition (2.1). The linearised equation (1.7), which corresponds to (3.9), is

$$\int_S a_{ijkl}(x)\,\xi_{ij}(\vec{v})\,\xi_{kl}(\vec{w})\,dS + \int_\Gamma \delta_i\delta_i\vec{v}\,\delta_j\delta_j\vec{w}\,ds - \lambda\int_\Gamma q_{u_iu_j}(\vec{0}, x)\,v^i w^j\,ds = 0\,. \tag{3.11}$$

Under the additional assumptions of smoothness

$$\partial S \in C^\infty\,, \quad a_{ijkl} \in C^\infty(\overline{S})\,, \quad q \in C^\infty(T_xM \times \partial S),$$

using the methods described in [6, 7] and proposition (2.1), the integral equation (3.11) in local coordinates reduces to the equivalent boundary value problem:

$$H\eta^l b_{ijkl}(x)\,\xi_{ij}(\vec{v}) + \nabla_l\,[\,b_{ijkl}(x)\,\xi_{ij}(\vec{v})] = 0, \quad x \in S$$

$$b_{ijkl}(x)\,\xi_{ij}(\vec{v})\nu^k\tau^l + (K^2 + R^2 - K - R)Dv^l\tau^l + D^2v^l\tau^l - \lambda q_{v^kv^l}(\vec{0}, x)v^k\tau^l = 0\,, \quad x \in \Gamma \quad (3.12)$$

$$\vec{v} = \vec{0}, \quad x \in \Gamma_1\,,$$

where $H$ is the mean curvature of surface $M$ [3], $K$ is the geodesic curvature, $R$ is the normal curvature of curve $\partial S$, located in the surface $M$ [1], $D = \delta_i\delta_i$, and $b_{ijkl} = a_{ijkl} - a_{ijlk}$.

We formulate the result:

**Theorem 3.1.** *Consider the functional (3.3), subjected to the constraint (2.6). Then the number $\lambda_0$ is a bifurcation point for equation (3.10) if and only if there exists a nonzero solution $\vec{v}_0 \in X_1$ of equation (3.11) for all $\vec{w} \in X_1$.*

*Proof.* The linearized equation (3.11) can be written in the following equivalent form

$$(\vec{v}, \vec{w}) - \lambda\,(A\vec{v}, \vec{w})\,, \quad \vec{v}, \vec{w} \in X_1\,, \tag{3.13}$$

such as the expression

$$\|u\| = \left[\int_S a_{ijkl}(x)\,\xi_{ij}(\vec{u})\,\xi_{kl}(\vec{u})\,dS + \int_\Gamma \delta_i\delta_i\,\vec{u}\,\delta_j\delta_j\,\vec{u}\,ds\right]^{1/2}$$

defines a norm in the space $H_0(S, T_xM)$, equivalent to the standard one [7], while the operator $A$ defined by,

$$(A\vec{u}, \vec{v})_{H_0} = \int_{\partial S} q_{u^iu^j}(\vec{0}, x)\,u^iv^j ds\,,$$

is linear, compact and symmetric [6, 7]. This implies that

$$\vec{v} - \lambda\,A\,\vec{v} = \vec{0}\,, \quad \vec{v} \in X_1\,,$$

or equivalently

$$(P\,Id\,P - \lambda\,PAP)\,\vec{u} = \vec{0}\,, \quad \vec{u} \in H_0(S, T_xM)\,,$$

where $Id$ is the identity operator of $H_0(S, T_xM)$, and $P$ is the orthogonal projector of $H_0(S, T_xM)$ in $X_1$, considering that $\vec{u}$ has the representation (1.10). Now the conclusion is obvious by the proposition (2.1) and the theorem (1.5). We can observe that bifurcation points exist when $q_{u^iu^j}(\vec{0}, x) \neq 0$. □

## References


[1] H. Cartan, *Differential Calculus - Differential Forms*, Herman Paris (1971). 3

[2] B. A. Dubrovin, A. T. Fomenko, S. P. Novikov, *Modern Geometry - Methods and Applications, Part II*, Springer-Verlag New York Inc. (1990). 2

[3] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Monographs in Mathematics, Vol. **80** Birkhäuser, Boston-Basel-Stuttgart (1984). 3, 3

[4] V. G. Osmolovskii, *Linear and nonlinear perturbations of operator* div, Translations of Mathematical Monographs, Vol. **160** (1997). 2

[5] I. V. Skrypnik, *Nonlinear Partial Differential Equations of Higher Order*, Kiev (1973). 1

[6] P. Vyridis, *Bifurcation in a Variational Problem on a Surface with a Constraint*, Int. J. Nonlinear Anal. Appl. **2** (2011), 1–10. 1, 1, 1, 3, 3

[7] P. Vyridis, *Free and Constrained Equilibrium States in a Variational Problem on a Surface* (to appear) 1, 3, 3, 3