Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 46–60
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Impulsive first-order functional q_k-integro-difference inclusions with boundary conditions

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Communicated by P. Kumam

Abstract

In this paper, we discuss the existence of solutions for a first order boundary value problem for impulsive functional q_k-integro-difference inclusions. Some new existence results are obtained for convex as well as non-convex multivalued maps with the aid of some classical fixed point theorems. Illustrative examples are also presented.

Keywords: q_k-derivative, q_k-integral, impulsive q_k-difference inclusions, existence, fixed point theorem.
2010 MSC: 34A60, 26A33, 39A13, 34A37.

1. Introduction

In this paper, we investigate the existence of solutions for a boundary value problem of impulsive functional q_k-integro-difference inclusions of the form:

$$\begin{cases} D_{q_k}x(t) \in F(t, x(t), x(\theta(t)), (K_{q_k}x)(t)), \quad t \in J := [0, T],\ t \neq t_k, \\ \Delta x(t_k) = I_k\left(x(t_k)\right), \quad k = 1, 2, \ldots, m, \\ \alpha x(0) = \beta x(T) + \displaystyle\sum_{i=0}^{m} \gamma_i \int_{t_i}^{t_{i+1}} x(s) d_{q_i} s, \end{cases} \tag{1.1}$$

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Received 2015-05-19

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R}^3 \to \mathbb{R}$, $\theta : J \to J$,

$$(K_{q_k}x)(t) = \int_{t_k}^{t} \phi(t,s)x(s)d_{q_k}s, \qquad k = 0,1,2,\dots,m, \tag{1.2}$$

$\phi : J^2 \to [0,\infty)$ is a continuous function, $I_k \in C(\mathbb{R},\mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$ for $k = 1,2,\dots,m$, $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$, $\alpha, \beta, \gamma_i, i = 0,1,\dots,m$ are real constants, $0 < q_k < 1$ for $k = 0,1,2,\dots,m$ and $\phi_0 = \sup_{(t,s)\in J^2} |\phi(t,s)|$.

Fractional differential equations have been shown to be very useful in the study of models of many phenomena in various fields of science and engineering, such as physics, chemistry, biology, signal and image processing, biophysics, blood flow phenomena, control theory, economics, aerodynamics and fitting of experimental data. For examples and recent developments of the topic, see ([1, 4, 6, 8, 9, 11, 12, 23, 27, 28, 29, 30]) and the references cited therein.

The book of Kac and Cheung [22] covers many of the fundamental aspects of quantum calculus. In recent years, the topic of q-calculus has attracted the attention of several researchers and a variety of new results can be found in the papers [2, 3, 5, 7, 10, 18, 34] and the references cited therein. On the other hand, for some monographs on impulsive differential equations we refer the reader to [13, 25, 31].

The notions of q_k-derivative and q_k-integral on finite intervals were introduced recently by the authors in [32]. Their basic properties were studied and, as applications, existence and uniqueness results were proved for initial value problems for first and second order impulsive q_k-difference equations.

Recently, in [33], the problem (1.1) was considered in the case when $F = \{f\}$, that is, when F is single-valued. Existence and uniqueness results were obtained by using Banach's contraction principle, Krasnoselskii's fixed point theorem and Leray-Schauder degree theory. In this paper we continue the study on this new subject to cover the multi-valued boundary value problem (1.1).

We establish some existence results for the problem (1.1), when the right hand side is convex as well as non-convex valued. In the first result we use the nonlinear alternative of Leray-Schauder type, and in the second result we combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued mappings with nonempty closed and decomposable values. The third result relies on the fixed point theorem for contraction multivalued mappings due to Covitz and Nadler. The methods used are well known, however their exposition in the framework of problem (1.1) is new.

2. Preliminaries

In this section we recall the notions of q_k-derivative and q_k-integral on finite intervals. For a fixed $k \in \mathbb{N} \cup \{0\}$ let $J_k := [t_k, t_{k+1}] \subset \mathbb{R}$ be an interval and $0 < q_k < 1$ be a constant. We define the q_k-derivative of a function $f : J_k \to \mathbb{R}$ at a point $t \in J_k$ as follows:

Definition 2.1. Assume $f : J_k \to \mathbb{R}$ is a continuous function and let $t \in J_k$. Then the expression

$$D_{q_k}f(t) = \frac{f(t) - f(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)}, t \neq t_k, \quad D_{q_k}f(t_k) = \lim_{t\to t_k} D_{q_k}f(t), \tag{2.1}$$

is called the q_k-derivative of function f at t.

We say that f is q_k-differentiable on J_k if $D_{q_k}f(t)$ exists for all $t \in J_k$. Note that if $t_k = 0$ and $q_k = q$ in (2.1), then $D_{q_k}f = D_q f$, where D_q is the well-known q-derivative of the function $f(t)$ defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}. \tag{2.2}$$

In addition, we can define the higher order q_k-derivative of functions.

Definition 2.2. Let $f : J_k \to \mathbb{R}$ be a continuous function. If $D_{q_k} f$ is q_k-differentiable on J_k, we define the second-order q_k-derivative of f by $D^2_{q_k} f : J_k \to \mathbb{R}$, $D^2_{q_k} f = D_{q_k}(D_{q_k} f)$. Similarly, we can consider higher order q_k-derivatives $D^n_{q_k} : J_k \to \mathbb{R}$.

The q_k-integral is defined as follows:

Definition 2.3. Assume $f : J_k \to \mathbb{R}$ is a continuous function. Then the q_k-integral is given by

$$\int_{t_k}^{t} f(s)d_{q_k}s = (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1-q_k^n)\, t_k), \tag{2.3}$$

for $t \in J_k$. Moreover, if $a \in (t_k, t)$ then the definite q_k-integral is given by

$$\begin{aligned}\int_a^t f(s)d_{q_k}s &= \int_{t_k}^t f(s)d_{q_k}s - \int_{t_k}^a f(s)d_{q_k}s \\ &= (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1-q_k^n)t_k) - (1-q_k)(a-t_k)\sum_{n=0}^{\infty} q_k^n f(q_k^n a + (1-q_k^n)t_k).\end{aligned}$$

Note that if $t_k = 0$ and $q_k = q$, then (2.3) reduces to the q-integral of a function $f(t)$, defined by

$$\int_0^t f(s)d_q s = (1-q)t\sum_{n=0}^{\infty} q^n f(q^n t) \quad \text{for} \quad t \in [0,\infty).$$

For the basic properties of the q_k-derivative and q_k-integral we refer the reader to [32].

Let $J = [0,T]$, $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k = 1,2,\ldots,m$. Let $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1,2,\ldots,m\}$. $PC(J,\mathbb{R})$ is a Banach space with the norm $\|x\|_{PC} = \sup\{|x(t)|; t \in J\}$.

We now consider the following linear problem:

$$\begin{cases} D_{q_k} x(t) = y(t), \quad t \in J := [0,T],\ t \neq t_k, \\ \Delta x(t_k) = I_k\left(x(t_k)\right), \quad k = 1,2,\ldots,m, \\ \alpha x(0) = \beta x(T) + \displaystyle\sum_{i=0}^{m} \gamma_i \int_{t_i}^{t_{i+1}} x(s)d_{q_i}s, \end{cases} \tag{2.4}$$

where $y : J \to \mathbb{R}$.

Lemma 2.4 ([33])**.** *Let $\alpha \neq \beta + \sum_{i=0}^{m} \gamma_i(t_{i+1} - t_i)$. The unique solution of problem (2.4) is given by*

$$\begin{aligned} x(t) = \ & \frac{\beta}{\Omega}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k} y(s)d_{q_{k-1}}s + \frac{\beta}{\Omega}\sum_{k=1}^{m} I_k\left(x(t_k)\right) + \sum_{i=0}^{m}\frac{\gamma_i}{\Omega}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r} y(s)d_{q_i}s d_{q_i}r \\ & + \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega}\int_{t_{k-1}}^{t_k} y(s)d_{q_{k-1}}s + \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega} I_k\left(x(t_k)\right) \\ & + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} y(s)d_{q_{k-1}}s + I_k\left(x(t_k)\right)\right) + \int_{t_k}^{t} y(s)d_{q_k}s, \end{aligned} \tag{2.5}$$

with $\sum_{i=a}^{b}(\cdot) = 0$ for $a > b$, where

$$\Omega = \left(\alpha - \beta - \sum_{i=0}^{m}\gamma_i(t_{i+1}-t_i)\right)^{-1}. \tag{2.6}$$

Next, we recall some definitions and notations about multifunctions ([17], [21]).

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed }\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = U_{x\in\mathbb{B}}G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$.) G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there exists $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : J \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable.

3. Main results

Before studying the boundary value problem (1.1), let us begin by defining its solution.

Definition 3.1. A function $x \in PC(J, \mathbb{R})$ is called a solution of problem (1.1) if $\Delta x(t_k) = I_{t_k}(x(t_k))$, $\alpha x(0) = \beta x(T) + \sum_{i=0}^{m} \gamma_i \int_{t_i}^{t_{i+1}} x(s) d_{q_i} s$, $k = 1, 2, \ldots, m$, and there exists a function $f \in L^1(J, \mathbb{R})$ such that $f(t) \in F(t, x(t), x(\theta(t)), (S_{q_k}x)(t))$, a.e. $t \in J$ and

$$\begin{aligned} x(t) = \;& \frac{\beta}{\Omega}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k} f(s)d_{q_{k-1}}s + \frac{\beta}{\Omega}\sum_{k=1}^{m} I_k\left(x(t_k)\right) + \sum_{i=0}^{m}\frac{\gamma_i}{\Omega}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r} f(s)d_{q_i}s d_{q_i}r \\ & + \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega}\int_{t_{k-1}}^{t_k} f(s)d_{q_{k-1}}s + \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega} I_k\left(x(t_k)\right) \\ & + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f(s)d_{q_{k-1}}s + I_k\left(x(t_k)\right)\right) + \int_{t_k}^{t} f(s)d_{q_k}s. \end{aligned}$$

3.1. The Carathéodory case

In this subsection, we consider the case when F has convex values and prove an existence result based on the nonlinear alternative of Leray-Schauder type, assuming that F is Carathéodory.

Definition 3.2. A multi-valued map $F : J \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \longmapsto F(t, x_1, x_2, x_3)$ is measurable for each $(x_1, x_2, x_3) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}$;
(ii) $(x_1, x_2, x_3) \longmapsto F(t, x_1, x_2, x_3)$ is upper semi-continuous for almost all $t \in J$;

Also, a Carathéodory multifunction $F : J \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ is called L^1-Carathéodory if

(iii) for each $\kappa > 0$, there exists $\varphi_\kappa \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x_1, x_2, x_3)\| = \sup_{t\in J}\{|v| : v \in F(t, x_1, x_2, x_3)\} \le \varphi_\kappa(t)$$

for all $|x_1|, |x_2|, |x_3| \le \kappa$ and for almost all $t \in J$.

For each $x \in PC(J, \mathbb{R})$, define the set of selections of F by

$$S_{F,x} = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, x(t), x(\theta(t)), (K_{q_k}x)(t)) \;\text{ for a.e. }\; t \in J\}.$$

We define the graph of a function G to be the set $Gr(G) = \{(x, y) \in X \times Y, y \in G(x)\}$ and recall two results for closed graphs and upper semi-continuity.

Lemma 3.3 ([17], Proposition 1.2)**.** *If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$, i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if $x_n \to x_*, y_n \to y_*$ when $n \to \infty$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Next, we recall a well-known fixed point theorem which is needed in the following.

Lemma 3.4 (Nonlinear alternative for Kakutani maps [20])**.** *Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,cv}(C)$ is a upper semi-continuous compact map. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.

Lemma 3.5 ([26])**.** *Let X be a Banach space. Let $F : J \times \mathbb{R}^3 \to \mathcal{P}_{cp,cv}(X)$ be an L^1- Carathéodory function and let Θ be a linear continuous mapping from $L^1(J,\mathbb{R})$ to $C(J,\mathbb{R})$. Then the operator*

$$\Theta \circ S_F : C(J,\mathbb{R}) \to \mathcal{P}_{cp,cv}(C(J,\mathbb{R})), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C(J,\mathbb{R}) \times C(J,\mathbb{R})$.

Now we are in a position to prove the existence of the solutions for the boundary value problem (1.1) when the right-hand side is convex valued.

Theorem 3.6. *Suppose that:*

(H_1) *$F : J \times \mathbb{R}^3 \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is an L^1-Carathéodory multifunction.*

(H_2) *There exist continuous nondecreasing functions $\psi_j : [0,\infty) \to (0,\infty)$ and functions $p_j, b \in C(J,\mathbb{R}^+)$, $1 \le j \le 2$, such that*

$$\|F(t,x_1,x_2,x_3)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x_1,x_2,x_3)\} \le \sum_{j=1}^{2} p_j(t)\psi_j(|x_j|) + b(t)|x_3|,$$

for each $(t,x_j) \in J \times \mathbb{R}^3$, $1 \le j \le 3$.

(H_3) *There exist constants c_k such that $|I_k(y)| \le c_k$, $1 \le k \le m$ for each $y \in \mathbb{R}$.*

(H_4) *There exists a constant $M > 0$ such that*

$$\frac{(1-\Psi\|b\|\phi_0)M}{\Lambda\sum_{j=1}^{2}\|p_j\|\psi_j(M) + \Phi} > 1, \quad \Psi\|b\|\phi_0 < 1,$$

where

$$\Lambda = \frac{|\beta|T}{|\Omega|} + \sum_{i=0}^{m}\frac{|\gamma_i|(t_{i+1}-t_i)^2}{|\Omega|(1+q_i)} + \sum_{i=1}^{m}\frac{|\gamma_i|(t_{i+1}-t_i)t_i}{|\Omega|} + T, \tag{3.1}$$

$$\Psi = \left(\frac{|\beta|+|\Omega|}{|\Omega|}\right)\sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}} + \sum_{i=0}^{m}\frac{|\gamma_i|(t_{i+1}-t_i)^3}{|\Omega|(1+q_i+q_i^2)} + \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{|\gamma_i|(t_{i+1}-t_i)(t_k-t_{k-1})^2}{|\Omega|(1+q_{k-1})}, \tag{3.2}$$

$$\Phi = \left(\frac{|\beta|+|\Omega|}{|\Omega|}\right)\sum_{k=1}^{m}c_k + \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{|\gamma_i|(t_{i+1}-t_i)}{|\Omega|}c_k. \tag{3.3}$$

Then the boundary value problem (1.1) has at least one solution on $J := [0,T]$.

Proof. Define the operator $\mathcal{H} : PC(J,\mathbb{R}) \to \mathcal{P}(PC(J,\mathbb{R}))$ by

$$\mathcal{H}(x) = \left\{ \begin{array}{l} h \in PC(J,\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \dfrac{\beta}{\Omega}\displaystyle\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k} f(s)d_{q_{k-1}}s + \dfrac{\beta}{\Omega}\sum_{k=1}^{m} I_k(x(t_k)) \\ +\displaystyle\sum_{i=0}^{m}\frac{\gamma_i}{\Omega}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r} f(s)d_{q_i}sd_{q_i}r \\ +\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega}\int_{t_{k-1}}^{t_k} f(s)d_{q_{k-1}}s \\ +\displaystyle\sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega} I_k(x(t_k)) \\ +\displaystyle\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f(s)d_{q_{k-1}}s + I_k(x(t_k))\right) + \int_{t_k}^{t} f(s)d_{q_k}s, \end{array}\right. \end{array}\right\}$$

for $f \in S_{F,x}$.

We shall show that $\mathcal{H}$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that $\mathcal{H}$ is convex for each $x \in PC(J,\mathbb{R})$. This step is obvious since $S_{F,x}$ is convex (F has convex values), and therefore we omit the proof.

In the second step, we show that $\mathcal{H}$ maps bounded sets (balls) into bounded sets in $PC(J,\mathbb{R})$. For a positive number ρ, let $B_\rho = \{x \in PC(J,\mathbb{R}) : \|x\| \le \rho\}$ be a bounded ball in $PC(J,\mathbb{R})$. Then, for each $h \in \mathcal{H}(x), x \in B_\rho$, there exists $f \in S_{F,x}$ such that

$$\begin{aligned} h(t) =& \frac{\beta}{\Omega}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k} f(s)d_{q_{k-1}}s + \frac{\beta}{\Omega}\sum_{k=1}^{m} I_k(x(t_k)) + \sum_{i=0}^{m}\frac{\gamma_i}{\Omega}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r} f(s)d_{q_i}sd_{q_i}r \\ &+ \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega}\int_{t_{k-1}}^{t_k} f(s)d_{q_{k-1}}s + \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega} I_k(x(t_k)) \\ &+ \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f(s)d_{q_{k-1}}s + I_k(x(t_k))\right) + \int_{t_k}^{t} f(s)d_{q_k}s, \quad t \in J. \end{aligned}$$

Then we have

$$\begin{aligned} |h(t)| \le& \frac{|\beta|}{|\Omega|}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k} |f(s)|d_{q_{k-1}}s + \frac{|\beta|}{|\Omega|}\sum_{k=1}^{m} |I_k(x(t_k))| + \sum_{i=0}^{m}\frac{|\gamma_i|}{|\Omega|}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r} |f(s)|d_{q_i}sd_{q_i}r \\ &+ \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{|\gamma_i|(t_{i+1}-t_i)}{|\Omega|}\int_{t_{k-1}}^{t_k} |f(s)|d_{q_{k-1}}s + \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{|\gamma_i|(t_{i+1}-t_i)}{|\Omega|}|I_k(x(t_k))| \\ &+ \sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} |f(s)|d_{q_{k-1}}s + \sum_{0<t_k<t}|I_k(x(t_k))| + \int_{t_k}^{t} |f(s)|d_{q_k}s \\ \le& \frac{|\beta|}{|\Omega|}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\Bigg(p_1(s)\psi_1(|x(s)|) + p_2(s)\psi_2(|x(s)|) \\ &+ b(s)\int_{t_{k-1}}^{s}|\phi(s,u)||x(u)|d_{q_{k-1}}u\Bigg)d_{q_{k-1}}s + \sum_{i=0}^{m}\frac{|\gamma_i|}{|\Omega|}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r}\Bigg(p_1(s)\psi_1(|x(s)|) \end{aligned}$$

$$
\begin{aligned}
&+p_2(s)\psi_2(|x(s)|)+b(s)\int_{t_i}^{s}|\phi(s,u)||x(u)|d_{q_i}u\Bigg)d_{q_i}sd_{q_i}r+\sum_{i=1}^{m}\sum_{k=1}^{i}\frac{|\gamma_i|(t_{i+1}-t_i)}{|\Omega|}\\
&\times\int_{t_{k-1}}^{t_k}\left(p_1(s)\psi_1(|x(s)|)+p_2(s)\psi_2(|x(s)|)+b(s)\int_{t_{k-1}}^{s}|\phi(s,u)||x(u)|d_{q_{k-1}}u\right)d_{q_{k-1}}s\\
&+\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\left(p_1(s)\psi_1(|x(s)|)+p_2(s)\psi_2(|x(s)|)+b(s)\int_{t_{k-1}}^{s}|\phi(s,u)||x(u)|d_{q_{k-1}}u\right)d_{q_{k-1}}s\\
&+\frac{|\beta|}{|\Omega|}\sum_{k=1}^{m}c_k+\sum_{i=1}^{m}\sum_{k=1}^{i}\frac{|\gamma_i|(t_{i+1}-t_i)}{|\Omega|}c_k+\sum_{k=1}^{m}c_k\\
\le&\frac{|\beta|}{|\Omega|}\sum_{j=1}^{2}\|p_j\|\psi_j(\rho)\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}d_{q_{k-1}}s+\|b\|\phi_0\rho\frac{|\beta|}{|\Omega|}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}d_{q_{k-1}}ud_{q_{k-1}}s\\
&+\sum_{j=1}^{2}\|p_j\|\psi_j(\rho)\sum_{i=0}^{m}\frac{|\gamma_i|}{|\Omega|}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r}d_{q_i}sd_{q_i}r+\|b\|\phi_0\rho\sum_{i=0}^{m}\frac{|\gamma_i|}{|\Omega|}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r}\int_{t_i}^{s}d_{q_i}ud_{q_i}sd_{q_i}r\\
&+\sum_{j=1}^{2}\|p_j\|\psi_j(\rho)\sum_{i=1}^{m}\sum_{k=1}^{i}\frac{|\gamma_i|(t_{i+1}-t_i)}{|\Omega|}\int_{t_{k-1}}^{t_k}d_{q_{k-1}}s\\
&+\|b\|\phi_0\rho\sum_{i=1}^{m}\sum_{k=1}^{i}\frac{|\gamma_i|(t_{i+1}-t_i)}{|\Omega|}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}d_{q_{k-1}}ud_{q_{k-1}}s\\
&+\sum_{j=1}^{2}\|p_j\|\psi_j(\rho)\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}d_{q_{k-1}}s+\|b\|\phi_0\rho\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}d_{q_{k-1}}ud_{q_{k-1}}s\\
&+\frac{|\beta|}{|\Omega|}\sum_{k=1}^{m}c_k+\sum_{i=1}^{m}\sum_{k=1}^{i}\frac{|\gamma_i|(t_{i+1}-t_i)}{|\Omega|}c_k+\sum_{k=1}^{m}c_k\\
=&\left(\frac{|\beta|T}{|\Omega|}+\sum_{i=0}^{m}\frac{|\gamma_i|(t_{i+1}-t_i)^2}{|\Omega|(1+q_i)}+\sum_{i=1}^{m}\frac{|\gamma_i|(t_{i+1}-t_i)t_i}{|\Omega|}+T\right)\sum_{j=1}^{2}\|p_j\|\psi_j(\rho)\\
&+\Bigg(\left(\frac{|\beta|+|\Omega|}{|\Omega|}\right)\sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+\sum_{i=0}^{m}\frac{|\gamma_i|(t_{i+1}-t_i)^3}{|\Omega|(1+q_i+q_i^2)}\\
&+\sum_{i=1}^{m}\sum_{k=1}^{i}\frac{|\gamma_i|(t_{i+1}-t_i)(t_k-t_{k-1})^2}{|\Omega|(1+q_{k-1})}\Bigg)\|b\|\phi_0\rho\\
&+\left(\frac{|\beta|+|\Omega|}{|\Omega|}\right)\sum_{k=1}^{m}c_k+\sum_{i=1}^{m}\sum_{k=1}^{i}\frac{|\gamma_i|(t_{i+1}-t_i)}{|\Omega|}c_k\\
=&\Lambda\sum_{j=1}^{2}\|p_j\|\psi_j(\rho)+\Psi\|b\|\phi_0\rho+\Phi,
\end{aligned}
$$

for all $t \in J$.

Thus we get

$$\|h\| \le \Lambda\sum_{j=1}^{2}\|p_j\|\psi_j(\rho)+\Psi\|b\|\phi_0\rho+\Phi,$$

which implies that $\mathcal{H}$ maps bounded sets into bounded sets in $PC(J,\mathbb{R})$.

Now we prove that $\mathcal{H}$ maps bounded sets into equi-continuous subsets of $PC(J,\mathbb{R})$. Suppose that $x \in B_\rho$

and $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$ with $\tau_1 \in J_v$, $\tau_2 \in J_u$, $v \le u$ for some $u, v \in \{0, 1, 2, \ldots, m\}$. Then we obtain

$$\begin{aligned} |h(\tau_2) - h(\tau_1)| &\le \left| \sum_{\tau_1 < t_k < \tau_2} \int_{t_{k-1}}^{t_k} f(s) d_{q_{k-1}} s \right| + \left| \sum_{\tau_1 < t_k < \tau_2} I_k(x(t_k)) \right| + \left| \int_{t_u}^{\tau_2} f(s) d_{q_k} s - \int_{t_v}^{\tau_1} f(s) d_{q_k} s \right| \\ &\le \sum_{\tau_1 < t_k < \tau_2} \int_{t_{k-1}}^{t_k} |f(s)| \, d_{q_{k-1}} s + \sum_{\tau_1 < t_k < \tau_2} |I_k(x(t_k))| + \left| \int_{t_u}^{\tau_2} f(s) d_{q_k} s - \int_{t_v}^{\tau_1} f(s) d_{q_k} s \right|. \end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\rho$ as $\tau_2 - \tau_1 \to 0$. As $\mathcal{H}$ satisfies the above three assumptions, it follows by the Arzelá-Ascoli theorem that $\mathcal{H} : PC(J, \mathbb{R}) \to \mathcal{P}(PC(J, \mathbb{R}))$ is completely continuous.

In the next step, we show that $\mathcal{H}$ is upper semi-continuous. By Lemma 3.3, $\mathcal{H}$ will be upper semi-continuous if we prove that it has a closed graph, since $\mathcal{H}$ was already shown to be completely continuous. Thus we will prove that $\mathcal{H}$ has a closed graph.

Let $x_n \to x_*, h_n \in \mathcal{H}(x_n)$ and $h_n \to h_*$. We need to show that $h_* \in \mathcal{H}(x_*)$. Associated with $h_n \in \mathcal{H}(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in J$,

$$\begin{aligned} h_n(t) =& \frac{\beta}{\Omega} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} f_n(s) d_{q_{k-1}} s + \frac{\beta}{\Omega} \sum_{i=1}^{m} I_k(x_n(t_k)) + \sum_{i=0}^{m} \frac{\gamma_i}{\Omega} \int_{t_i}^{t_{i+1}} \int_{t_i}^{r} f_n(s) d_{q_i} s d_{q_i} r \\ &+ \sum_{i=0}^{m} \sum_{k=1}^{i} \frac{\gamma_i (t_{i+1} - t_i)}{\Omega} \int_{t_{k-1}}^{t_k} f_n(s) d_{q_{k-1}} s + \sum_{i=0}^{m} \sum_{k=1}^{i} \frac{\gamma_i (t_{i+1} - t_i)}{\Omega} I_k(x_n(t_k)) \\ &+ \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} f_n(s) d_{q_{k-1}} s + \sum_{0 < t_k < t} I_k(x_n(t_k)) + \int_{t_k}^{t} f_n(s) d_{q_k} s. \end{aligned}$$

Thus it suffices to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in J$,

$$\begin{aligned} h_*(t) =& \frac{\beta}{\Omega} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} f_*(s) d_{q_{k-1}} s + \frac{\beta}{\Omega} \sum_{i=1}^{m} I_k(x_*(t_k)) + \sum_{i=0}^{m} \frac{\gamma_i}{\Omega} \int_{t_i}^{t_{i+1}} \int_{t_i}^{r} f_*(s) d_{q_i} s d_{q_i} r \\ &+ \sum_{i=0}^{m} \sum_{k=1}^{i} \frac{\gamma_i (t_{i+1} - t_i)}{\Omega} \int_{t_{k-1}}^{t_k} f_*(s) d_{q_{k-1}} s + \sum_{i=0}^{m} \sum_{k=1}^{i} \frac{\gamma_i (t_{i+1} - t_i)}{\Omega} I_k(x_*(t_k)) \\ &+ \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} f_*(s) d_{q_{k-1}} s + \sum_{0 < t_k < t} I_k(x_*(t_k)) + \int_{t_k}^{t} f_*(s) d_{q_k} s. \end{aligned}$$

Let us consider the linear operator $\Theta : L^1(J, \mathbb{R}) \to PC(J, \mathbb{R})$ given by

$$\begin{aligned} f \mapsto \Theta(f)(t) =& \frac{\beta}{\Omega} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} f(s) d_{q_{k-1}} s + \frac{\beta}{\Omega} \sum_{i=1}^{m} I_k(x(t_k)) + \sum_{i=0}^{m} \frac{\gamma_i}{\Omega} \int_{t_i}^{t_{i+1}} \int_{t_i}^{r} f(s) d_{q_i} s d_{q_i} r \\ &+ \sum_{i=0}^{m} \sum_{k=1}^{i} \frac{\gamma_i (t_{i+1} - t_i)}{\Omega} \int_{t_{k-1}}^{t_k} f(s) d_{q_{k-1}} s + \sum_{i=0}^{m} \sum_{k=1}^{i} \frac{\gamma_i (t_{i+1} - t_i)}{\Omega} I_k(x(t_k)) \\ &+ \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} f(s) d_{q_{k-1}} s + \sum_{0 < t_k < t} I_k(x(t_k)) + \int_{t_k}^{t} f(s) d_{q_k} s. \end{aligned}$$

Note that

$$\|h_n(t) - h_*(t)\| = \Bigg\| \frac{\beta}{\Omega} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} (f_n(s) - f_*(s)) d_{q_{k-1}} s + \frac{\beta}{\Omega} \sum_{i=1}^{m} (I_k(x_n(t_k)) - I_k(x_*(t_k)))$$

$$
\begin{aligned}
&+\sum_{i=0}^{m}\frac{\gamma_i}{\Omega}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r}(f_n(s)-f_*(s))d_{q_i}sd_{q_i}r\\
&+\sum_{i=0}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega}\int_{t_{k-1}}^{t_k}(f_n(s)-f_*(s))d_{q_{k-1}}s\\
&+\sum_{i=0}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega}(I_k(x_n(t_k))-I_k(x_*(t_k)))\\
&+\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(f_n(s)-f_*(s))d_{q_{k-1}}s+\sum_{0<t_k<t}(I_k(x_n(t_k))-I_k(x_*(t_k)))\\
&+\int_{t_k}^{t}(f_n(s)-f_*(s))d_{q_k}s\Bigg\|\to 0,\quad \text{as}\quad n\to\infty.
\end{aligned}
$$

Thus, it follows by Lemma 3.5 that $\Theta\circ S_F$ is a closed graph operator. Furthermore, we have $h_n(t)\in\Theta(S_{F,x_n})$. Since $x_n\to x_*$, therefore, we have

$$
\begin{aligned}
h_*(t)=&\frac{\beta}{\Omega}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}f_*(s)d_{q_{k-1}}s+\frac{\beta}{\Omega}\sum_{i=1}^{m}I_k(x_*(t_k))+\sum_{i=0}^{m}\frac{\gamma_i}{\Omega}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r}f_*(s)d_{q_i}sd_{q_i}r\\
&+\sum_{i=0}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega}\int_{t_{k-1}}^{t_k}f_*(s)d_{q_{k-1}}s+\sum_{i=0}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega}I_k(x_*(t_k))\\
&+\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}f_*(s)d_{q_{k-1}}s+\sum_{0<t_k<t}I_k(x_*(t_k))+\int_{t_k}^{t}f_*(s)d_{q_k}s
\end{aligned}
$$

for some $f_*\in S_{F,x_*}$.

Finally, we show there exists an open set $U\subseteq PC(J,\mathbb{R})$ with $x\notin\mathcal{H}(x)$ for any $\lambda\in(0,1)$ and all $x\in\partial U$. Let $\lambda\in(0,1)$ and $x\in\lambda\mathcal{H}(x)$. Then there exists $f\in L^1(J,\mathbb{R})$ with $f\in S_{F,x}$ such that, for $t\in J$, we have

$$
\begin{aligned}
x(t)=&\frac{\lambda\beta}{\Omega}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}f(s)d_{q_{k-1}}s+\frac{\lambda\beta}{\Omega}\sum_{i=1}^{m}I_k(x(t_k))+\sum_{i=0}^{m}\frac{\lambda\gamma_i}{\Omega}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r}f(s)d_{q_i}sd_{q_i}r\\
&+\sum_{i=0}^{m}\sum_{k=1}^{i}\frac{\lambda\gamma_i(t_{i+1}-t_i)}{\Omega}\int_{t_{k-1}}^{t_k}f(s)d_{q_{k-1}}s+\sum_{i=0}^{m}\sum_{k=1}^{i}\frac{\lambda\gamma_i(t_{i+1}-t_i)}{\Omega}I_k(x(t_k))\\
&+\lambda\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}f(s)d_{q_{k-1}}s+\lambda\sum_{0<t_k<t}I_k(x(t_k))+\lambda\int_{t_k}^{t}f(s)d_{q_k}s.
\end{aligned}
$$

Using the computations of the second step above we have

$$
\|x\|\leq\Lambda\sum_{j=1}^{2}\|p_j\|\psi_j(\|x\|)+\Psi\|b\|\phi_0\|x\|+\Phi,
$$

which implies that

$$
\frac{(1-\Psi\|b\|\phi_0)\|x\|}{\Lambda\sum\limits_{j=1}^{2}\|p_j\|\psi_j(\|x\|)+\Phi}\leq 1.
$$

In view of (H_4), there exists M such that $\|x\|\neq M$. Let us set

$$
U=\{x\in PC(J,\mathbb{R}):\|x\|<M\}.
$$

Note that the operator $\mathcal{H} : \overline{U} \to \mathcal{P}(PC(J,\mathbb{R}))$ is upper semi-continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x \in \lambda\mathcal{H}(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.4), we deduce that $\mathcal{H}$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

3.2. The lower semi-continuous case

The next result concerns the case when F is not necessarily convex valued. Our strategy to deal with this problem is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo [14] for lower semi-continuous maps with decomposable values.

Let X be a nonempty closed subset of a Banach space E and $G : X \to \mathcal{P}(E)$ be a multi-valued operator with nonempty closed values. G is lower semi-continuous (l.s.c.) if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E.

Let A be a subset of $[0,T] \times \mathbb{R}$. A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the $\sigma-$algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and $\mathcal{D}$ is Borel measurable in $\mathbb{R}$. A subset $\mathcal{A}$ of $L^1([0,T],\mathbb{R})$ is decomposable if for all $x, y \in \mathcal{A}$ and measurable $\mathcal{J} \subset [0,T] = J$, the function $x\chi_{\mathcal{J}} + y\chi_{J-\mathcal{J}} \in \mathcal{A}$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 3.7. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1(J,\mathbb{R}))$ be a multivalued operator. We say N has a property (BC) if N is lower semi-continuous (l.s.c.) and has nonempty closed and decomposable values.

Let $F : J \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ be a multivalued map with nonempty compact values. Define a multi-valued operator $\mathcal{F} : PC(J \times \mathbb{R}) \to \mathcal{P}(L^1(J,\mathbb{R}))$ associated with F as

$$\mathcal{F}(x) = \{w \in L^1(J,\mathbb{R}) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in J\},$$

which is called the Nemytskii operator associated with F.

Definition 3.8. Let $F : J \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ be a multi-valued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Nemytskii operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.

Lemma 3.9 ([19])**.** *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1(J,\mathbb{R}))$ be a multi-valued operator satisfying the property (BC). Then N has a continuous selection, that is, there exists a continuous (single-valued) function $g : Y \to L^1(J,\mathbb{R})$ such that $g(x) \in N(x)$ for every $x \in Y$.*

Theorem 3.10. *Assume that (H_2)-(H_4) and the following condition hold:*

(H_5) *$F : J \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ is a nonempty compact-valued map such that*

- (a) $(t, x_1, x_2, x_3) \longmapsto F(t, x_1, x_2, x_3)$ *is $\mathcal{L} \otimes \mathcal{B}$ measurable,*
- (b) $(x_1, x_2, x_3) \longmapsto F(t, x_1, x_2, x_3)$ *is lower semi-continuous for each $t \in [0,T]$;*

Then the boundary value problem (1.1) *has at least one solution on $J := [0,T]$.*

Proof. It follows from (H_2) and (H_5) that F is of l.s.c. type. Then from Lemma 3.9, there exists a continuous function $f : PC(J,\mathbb{R}) \to L^1(J,\mathbb{R})$ such that $f(x) \in F(x)$ for all $x \in PC(J,\mathbb{R})$. Consider the problem

$$\begin{cases} D_{q_k} x(t) = f(x(t)), \quad t \in J := [0,T],\ t \neq t_k, \\ \Delta x(t_k) = I_k\left(x(t_k)\right), \quad k = 1, 2, \ldots, m, \\ \alpha x(0) = \beta x(T) + \displaystyle\sum_{i=0}^{m} \gamma_i \int_{t_i}^{t_{i+1}} x(s) d_{q_i} s. \end{cases} \tag{3.4}$$

Note that if $x \in PC(J, \mathbb{R})$ is a solution of (3.4), then x is a solution to the problem (1.1). In order to transform the problem (3.4) into a fixed point problem, we define the operator $\overline{\mathcal{H}}$ as

$$\begin{aligned}\overline{\mathcal{H}}x(t) =& \frac{\beta}{\Omega}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k} f(x(s))d_{q_{k-1}}s + \frac{\beta}{\Omega}\sum_{k=1}^{m} I_k\left(x(t_k)\right) + \sum_{i=0}^{m}\frac{\gamma_i}{\Omega}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r} f(x(s))d_{q_i}s d_{q_i}r \\ &+ \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega}\int_{t_{k-1}}^{t_k} f(x(s))d_{q_{k-1}}s + \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega} I_k\left(x(t_k)\right) \\ &+ \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f(x(s))d_{q_{k-1}}s + I_k\left(x(t_k)\right)\right) + \int_{t_k}^{t} f(x(s))d_{q_k}s.\end{aligned}$$

It can easily be shown that $\overline{\mathcal{H}}$ is continuous and completely continuous. The remaining part of the proof is similar to that of Theorem 3.6, so we omit it. □

3.3. The Lipschitz case

Now we prove the existence of solutions for problem (1.1) with a not necessarily nonconvex valued right hand side, by applying a fixed point theorem for multivalued mappings due to Covitz and Nadler [16].

Let (X, d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider the Pompeiu-Hausdoff metric $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$

where $d(A, b) = \inf_{a\in A} d(a; b)$ and $d(a, B) = \inf_{b\in B} d(a; b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [24]).

Definition 3.11. A multi-valued operator $\mathcal{H} : X \to \mathcal{P}_{cl}(X)$ is called:

(a) $\kappa-$Lipschitz if and only if there exists $\kappa > 0$ such that

$$H_d(\mathcal{H}(x), \mathcal{H}(y)) \le \kappa d(x, y) \quad \text{for each} \quad x, y \in X;$$

(b) a contraction if and only if it if $\kappa-$Lipschitz with $\kappa < 1$.

Lemma 3.12 ([16]). *Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Theorem 3.13. *Assume that:*

(L_1) *$F : J \times \mathbb{R}^3 \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, x_1, x_2, x_3) : J \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x_j \in \mathbb{R}$, $1 \le j \le 3$.*

(L_2) *For almost all $t \in J$ and $x_j, y_j \in \mathbb{R}, 1 \le j \le 3$ we have*

$$H_d(F(t, x_1, x_2, x_3), F(t, y_1, y_2, y_3)) \le p(t)\sum_{j=1}^{3}|x_j - y_j|,$$

with $p \in C(J, \mathbb{R}^+)$ and $d(0, F(t, 0, 0, 0)) \le p(t)$ for almost all $t \in J$.

Then the boundary value problem (1.1) has at least one solution on J if

$$\|p\|\Lambda(2 + \phi_0 T) < 1.$$

Proof. Consider the operator $\mathcal{H} : PC(J,\mathbb{R}) \to \mathcal{P}(PC(J,\mathbb{R}))$ defined in the beginning of the proof of Theorem 3.6. Note that the set $S_{F,x}$ is nonempty for each $x \in PC(J,\mathbb{R})$ by the assumption (L_1), so F has a measurable selection (see Theorem III.6 [15]). Now we show that the operator $\mathcal{H}$ satisfies the assumptions of Lemma 3.12. To show that $\mathcal{H}(x) \in \mathcal{P}_{cl}(PC(J,\mathbb{R}))$ for each $x \in PC(J,\mathbb{R})$, let $\{x_n\}_{n\geq 0} \in \mathcal{H}(x)$ be such that $x_n \to x$ $(n \to \infty)$ in $PC(J,\mathbb{R})$. Then $x \in PC(J,\mathbb{R})$ and there exists $f_n \in S_{F,x_n}$ such that, for each $t \in J$,

$$\begin{aligned} x_n(t) =&\frac{\beta}{\Omega}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k} f_n(s)d_{q_{k-1}}s + \frac{\beta}{\Omega}\sum_{k=1}^{m} I_k(x_n(t_k)) + \sum_{i=0}^{m}\frac{\gamma_i}{\Omega}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r} f_n(s)d_{q_i}sd_{q_i}r \\ &+ \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega}\int_{t_{k-1}}^{t_k} f_n(s)d_{q_{k-1}}s + \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega} I_k(x_n(t_k)) \\ &+ \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f_n(s)d_{q_{k-1}}s + I_k(x_n(t_k))\right) + \int_{t_k}^{t} f_n(s)d_{q_k}s. \end{aligned}$$

As F has compact values, we pass to a subsequence (if necessary) to obtain that f_n converges to f in $L^1(J,\mathbb{R})$. Thus, $f \in S_{F,x}$ and for each $t \in J$, we have

$$\begin{aligned} f_n(t) \to f(t) =&\frac{\beta}{\Omega}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k} f(s)d_{q_{k-1}}s + \frac{\beta}{\Omega}\sum_{k=1}^{m} I_k(x(t_k)) + \sum_{i=0}^{m}\frac{\gamma_i}{\Omega}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r} f(s)d_{q_i}sd_{q_i}r \\ &+ \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega}\int_{t_{k-1}}^{t_k} f(s)d_{q_{k-1}}s + \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega} I_k(x(t_k)) \\ &+ \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f(s)d_{q_{k-1}}s + I_k(x(t_k))\right) + \int_{t_k}^{t} f(s)d_{q_k}s. \end{aligned}$$

Hence, $x \in \mathcal{H}(x)$.

Next we show that $\mathcal{H}$ is a contractive multifunction with constant $\kappa = \|p\|\Lambda(2+\phi_0 T) < 1$. Let $x, y \in PC(J,\mathbb{R})$ and $h_1 \in \mathcal{H}(x)$. Then there exists $f_1 \in S_{F,x}$ such that, for each $t \in J$,

$$\begin{aligned} h_1(t) =&\frac{\beta}{\Omega}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k} f_1(s)d_{q_{k-1}}s + \frac{\beta}{\Omega}\sum_{k=1}^{m} I_k\left(x(t_k)\right) + \sum_{i=0}^{m}\frac{\gamma_i}{\Omega}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r} f_1(s)d_{q_i}sd_{q_i}r \\ &+ \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega}\int_{t_{k-1}}^{t_k} f_1(s)d_{q_{k-1}}s + \sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega} I_k\left(x(t_k)\right) \\ &+ \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f_1(s)d_{q_{k-1}}s + I_k\left(x(t_k)\right)\right) + \int_{t_k}^{t} f_1(s)d_{q_k}s. \end{aligned}$$

By (L_2) we get

$$\begin{aligned} H_d(F(t,x(t),x(\theta(t)),(K_{q_k}x)(t),F(t,y(t),y(\theta(t)),(K_{q_k}y)(t)) \leq\; & p(t)\left(|x(t)-y(t)| + |x(\theta(t)) - y(\theta(t))|\right. \\ & \left.+ |(K_{q_k}x)(t) - (K_{q_k}y)(t)|\right), \end{aligned}$$

so, there exists $z \in F(t,x(t),x(\theta(t)),K_{q_k}x)(t)$ such that

$$|f_1(t) - z| \leq p(t)\left(|x(t)-y(t)| + |x(\theta(t)) - y(\theta(t))| + |(K_{q_k}x)(t) - (K_{q_k}y)(t)|\right),$$

for almost all $t \in J$. Define the multifunction $\mathcal{T} : J \to \mathcal{P}(\mathbb{R})$ by

$$\mathcal{T}(t) =\{z \in \mathbb{R} : |f_1(t) - z| \leq p(t)(|x(t)-y(t)| + |x(\theta(t)) - y(\theta(t))|$$

$$+\left|(K_{q_k}x)(t)-(K_{q_k}y)(t)\right|), \quad \text{for almost all} \quad t\in J\}.$$

It is easy to check that the multifunction $\mathcal{T}(\cdot)\cap F(\cdot,x(\cdot),x(\theta(\cdot)),(K_{q_k}x)(\cdot))$ is measurable. Hence, we choose $f_2\in S_{F,x}$ such that

$$|f_1(t)-f_2(t)|\le p(t)\left(|x(t)-y(t)|+|x(\theta(t))-y(\theta(t))|+|(K_{q_k}x)(t)-(K_{q_k}y)(t)|\right),$$

for almost all $t\in J$. Consider $h_2\in\mathcal{H}(x)$ which is defined by

$$\begin{aligned}
h_2(t)=&\frac{\beta}{\Omega}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}f_2(s)d_{q_{k-1}}s+\frac{\beta}{\Omega}\sum_{k=1}^{m}I_k\left(x(t_k)\right)+\sum_{i=0}^{m}\frac{\gamma_i}{\Omega}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r}f_2(s)d_{q_i}sd_{q_i}r\\
&+\sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega}\int_{t_{k-1}}^{t_k}f_2(s)d_{q_{k-1}}s+\sum_{i=1}^{m}\sum_{k=1}^{i}\frac{\gamma_i(t_{i+1}-t_i)}{\Omega}I_k\left(x(t_k)\right)\\
&+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}f_2(s)d_{q_{k-1}}s+I_k\left(x(t_k)\right)\right)+\int_{t_k}^{t}f_2(s)d_{q_k}s.
\end{aligned}$$

Thus,

$$\begin{aligned}
|h_1(t)-h_2(t)|\le&\frac{|\beta|}{|\Omega|}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}|f_1(s)-f_2(s)|d_{q_{k-1}}s+\sum_{i=0}^{m}\frac{|\gamma_i|}{|\Omega|}\int_{t_i}^{t_{i+1}}\int_{t_i}^{r}|f_1(s)-f_2(s)|d_{q_i}sd_{q_i}r\\
&+\sum_{i=1}^{m}\sum_{k=1}^{i}\frac{|\gamma_i|(t_{i+1}-t_i)}{|\Omega|}\int_{t_{k-1}}^{t_k}|f_1(s)-f_2(s)|d_{q_{k-1}}s+\sum_{i=1}^{m+1}\int_{t_{k-1}}^{t_k}|f_1(s)-f_2(s)|d_{q_{k-1}}s\\
\le&\|p\|\left(\frac{|\beta|T}{|\Omega|}+\sum_{i=0}^{m}\frac{|\gamma_i|(t_{i+1}-t_i)^2}{|\Omega|(1+q_i)}+\sum_{i=1}^{m}\frac{|\gamma_i|(t_{i+1}-t_i)t_i}{|\Omega|}+T\right)(2+\phi_0T)\|x-y\|\\
=&\|p\|\Lambda(2+\phi_0T)\|x-y\|.
\end{aligned}$$

Hence,

$$\|h_1-h_2\|\le\|p\|\Lambda(2+\phi_0T)\|x-y\|.$$

Analogously, interchanging the roles of x and y, we obtain

$$H_d(\mathcal{H}_F(x),\mathcal{H}_F(y))\le\|p\|\Lambda(2+\phi_0T)\|x-y\|.$$

Since $\kappa=\|p\|\Lambda(2+\phi_0T)<1$, $\mathcal{H}$ is a contraction, and it follows by Lemma 3.12 that $\mathcal{H}$ has a fixed point x which is a solution of (1.1). This completes the proof. $\square$

4. Examples

In this section, we will illustrate our main results with the help of some examples. Let us consider the following boundary value problem for impulsive first-order functional q_k-integro-difference inclusions

$$\begin{cases}
D_{\frac{k+1}{k+3}}x(t)\in F(t,x(t),x(\theta(t)),(K_{\frac{k+1}{k+3}}x)(t)), \quad t\in[0,2], \ t\ne t_k\\
\Delta x(t_k)=\dfrac{(k+1)|x(t_k)|}{(k+2)(|x(t_k)|+1)}, \quad t_k=\dfrac{k}{2}, \quad k=1,2,3,\\
\dfrac{1}{2}x(0)=\dfrac{2}{3}x(2)+\displaystyle\sum_{i=0}^{3}\left(\frac{i+2}{i+3}\right)\int_{t_i}^{t_{i+1}}x(s)d_{\frac{i+1}{i+3}}s.
\end{cases} \tag{4.1}$$

Here we have $q_k = (k+1)/(k+3)$, $\gamma_k = (k+2)/(k+3)$, $k = 0,1,2,3$, $m = 3$, $T = 2$, $\alpha = 1/2$, $\beta = 2/3$. By using the Maple program, we can find $|\Omega| = 0.5911330049$, $\Lambda = 7.152534722$, $\Psi = 2.511800263$. Since $I_k(x) = ((k+1)|x|)/((k+2)(|x|+1))$, we have $|I_k(x)| \le (k+1)/(k+2) = c_k$, which implies $\Phi = 7.660543981$.

(a) Consider the multi-valued map $F : [0,2] \to \mathcal{P}(\mathbb{R})$ given by

$$\begin{aligned} x &\to F(t, x(t), x(2t/3), K_{\frac{k+1}{k+3}} x(t)) \\ &= \left[0, \frac{x^2(t) + |x(t)| + 1}{6(t+8)(|x(t)|+1)} + \frac{e^{-t}(x(2t/3)+2)}{4(2t+5)} + \frac{t+2}{8}\int_{t_k}^{t} \frac{x(s)\cos^2 st}{10} d_{\frac{k+1}{k+3}} s\right]. \end{aligned} \tag{4.2}$$

Then, we have

$$\sup\{|x| : x \in F(t, x_1, x_3, x_3)\} \le \frac{1}{6(t+8)}(|x_1|+1) + \frac{1}{4(2t+5)}(|x_2|+1) + \frac{t+2}{8}|x_3|,$$

and $\phi_0 = 1/10$. Choosing $p_1(t) = 1/(6(t+8))$, $\psi_1(x) = x+1$, $p_2(t) = 1/(4(2t+5))$, $\psi_2(x) = x+1$, $b(t) = (t+2)/8$ we have $\|p_1\| = 1/48$, $\|p_2\| = 1/20$, $\|b\| = 1/2$. From all information, there exists a positive constant $M > 22.20718108$ which satisfies the condition (H_4).

Thus all the conditions of Theorem 3.6 are satisfied. Therefore, by the conclusion of Theorem 3.6, the problem (4.1) with the $F(t, x_1, x_2, x_3)$ given by (4.2) has at least one solution on $[0,2]$.

(b) Let $F : [0,2] \times \mathbb{R}^3 \to \mathcal{P}(\mathbb{R})$ be a multi-valued map given by

$$\begin{aligned} x &\to F(t, x(t), x(t/2), K_{\frac{k+1}{k+3}} x(t)) \\ &= \left[0, \frac{e^{-t^2}}{8(t+6)}\left(\frac{x^2(t) + 2|x(t)|}{|x(t)|+1}\right) + \frac{|\cos \pi t| x(t/2)}{2(t^4+12)} + \frac{1}{3(4t+8)}\int_{t_k}^{t} \frac{e^{s-t}}{2} x(s) d_{\frac{k+1}{k+3}} s + \frac{1}{48}\right]. \end{aligned} \tag{4.3}$$

It is easy to see that $F(\cdot, x_1, x_2, x_3)$ is measurable for each $x_i \in \mathbb{R}$, $i = 1,2,3$. For $x_i, y_i \in \mathbb{R}$, $i = 1,2,3$, we have

$$H_d(F(t, x_1, x_2, x_3), F(t, y_1, y_2, y_3)) \le \frac{1}{3(4t+8)} \sum_{i=1}^{3} |x_i - y_i|,$$

with $p(t) = 1/(3(4t+8))$. Note that $d(0, F(t,0,0,0)) = 1/48 \le p(t)$ for $t \in [0,2]$ and $\phi_0 = 1/2$. We can find that

$$\|p\|\Lambda(2 + \phi_0 T) = 0.8940668403 < 1.$$

Therefore, all the conditions of Theorem 3.13 are satisfied, so the problem (4.1) with $F(t, x_1, x_2, x_3)$ given by (4.3) has at least one solution on $[0,2]$.

Acknowledgement:

This research was funded by King Mongkut's University of Technology North Bangkok. Contract no. KMUTNB-GOV-58-10.

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