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Application of new iterative transform method and modified fractional homotopy analysis transform method for fractional Fornberg-Whitham equation

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Abstract

The main purpose of this paper is to present a new iterative transform method (NITM) and a modified fractional homotopy analysis transform method (MFHATM) for time-fractional Fornberg-Whitham equation. The numerical results show that the MFHATM and NITM are very efficient and highly accurate for nonlinear fractional differential equations.

Keywords: Elzaki transform, iterative transform, homotopy analysis, fractional Fornberg-Whitham equation.
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1. Introduction

In the past three decades, the fractional differential equations have gained considerable attention of physicists, mathematicians and engineers [9, 23, 24, 25, 27]. With the help of fractional derivatives, the fractional differential equations can be used to model in many fields of engineering and science such as diffusion and reaction processes, control theory of dynamical system, probability and statistics, electrical networks, signal processing, system identification, financial market and quantum mechanics[14, 28]. In general, it is difficult for fractional differential equations to find their exact solutions therefore numerical and approximate techniques have to be used. Many powerful methods have been used to solve linear and nonlinear fractional differential equations. These powerful techniques include the Adomain decomposition

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method (ADM) [2, 30], the homotopy perturbation method (HPM) [8] and the variational iteration method (VIM) [26]. In recent years, many researchers have used various methods to study the solutions of linear and nonlinear fractional differential equations combined with Fourier transform, Laplace transform [6, 11, 12, 13, 15, 16, 17, 22, 31, 32] and Sumudu transform[29].

In 2006, Daftardar-Gejji and Jafari [5] proposed DJM for solving linear and nonlinear differential equations. The DJM is very easy to understand and implement and obtain better numerical results than Adomian decomposition method (ADM) [2, 30] and Variational iterative method (VIM) [26]. The homotopy analysis method (HAM) was first proposed by S. J. Liao [18, 19, 20, 21] for solving linear and nonlinear integral and differential equation. The advantage of HAM over other perturbation methods is that it does not depend on any large and small parameter. The HAM have been applied by many researchers to solve many kinds of nonlinear equations arising in science and engineering. In [1, 7], the authors applied HPM, HAM and ADM respectively in studying the time-fractional Fornberg-Whitham equation with the initial condition which can be written in operator form as follows

$$\begin{cases} u_t^{\alpha} = u_{xxt} - u_x + uu_{xxx} - uu_x + 3u_x u_{xx}, \\ 0 < \alpha \leq 1, t > 0, \\ u(x,0) = e^{\frac{x}{2}}, \end{cases}$$

where $u(x,t)$ is the fluid velocity, t is the time, x is the spatial coordinate and α is constant. When $\alpha = 1$, the fractional Fornberg-Whitham equation was used to study the qualitative behavior of wave breaking.

In this paper, based on DJM and HAM, we establish the new iterative transform method (NITM) and modified fractional homotopy analysis transform method (MFHATM) with the help of the Elzaki transform [3, 4, 10] for obtaining analytical and numerical solution of the time-fractional Fornberg-Whitham equation. The results show the HPM and ADM can be obtained as a special case of the MFHATM for $h = -1$. The numerical results show that the MFHATM and NITM are simpler and more highly accurate than existing methods (HPM, ADM).

2. Basic definitions

In this section, we give some basic definitions of fractional calculus [7, 9, 24] and Elzaki transform [3, 4, 10] which we shall use in this paper.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number p, $(p > \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu$, $m \in N$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f(x) \in C_\mu$, $\mu \geq -1$ is defined as [7, 9, 24]:

$$I^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, x > 0, \\ I^0 f(x) = f(x), \alpha = 0, \end{cases} \tag{2.1}$$

where $\Gamma(.)$ is the well-known Gamma function.

Properties of the operator I^α, which we will use here, are as follows:
For $f \in C_\mu$, $\mu, \gamma \geq -1$, $\alpha, \beta \geq 0$,

$$(1) I^{\alpha} I^{\beta} = I^{\beta} I^{\alpha} f(x) = I^{\alpha+\beta} f(x),$$

$$(2) I^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}.$$

Definition 2.3. The fractional derivative of $f(x)$ in the Caputo sense is defined as [7, 9, 24]:

$$D^{\alpha}f(x) = I^{n-\alpha}D^{n}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}f^{(n)}(t)dt, \tag{2.2}$$

where $n-1<\alpha\leq n$, $n\in N$, $x>0$, $f\in C_{-1}^{n}$.

The following are the basic properties of the operator D^{α}:

$$(1)D^{\alpha}I^{\alpha}f(x) = f(x),$$
$$(2)I^{\alpha}D^{\alpha}f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^{+})\frac{x}{k!}, \quad x>0.$$

Definition 2.4. The Elzaki transform is defined over the set of function

$$A = \{f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k}}, t\in(-1)^j \times [0,\infty)\}$$

by the following formula [3, 4, 10]:

$$T(v) = E[f(t)] = v\int_v^{\infty} e^{-\frac{t}{v}}f(t)dt, \quad v\in[-k_1,k_2].$$

Lemma 2.5. *The Elzaki transform of the Riemann-Liouville fractional integral is given as follows* [3, 4, 10]:

$$E[I^{\alpha}f(t)] = v^{\alpha+1}T(v). \tag{2.3}$$

Lemma 2.6. *The Elzaki transform of the caputo fractional derivative is given as follows* [3, 4, 10]:

$$E[D_x^{n\alpha}u(x,t)] = \frac{T(v)}{v^{n\alpha}} - \sum_{k=0}^{n-1} v^{2-n\alpha+k}u^{(k)}(0,t), n-1 < n\alpha \leq n. \tag{2.4}$$

Lemma 2.7. *Let $f(t)$ and $g(t)$ be defined in A having ELzaki transform $M(v)$ and $N(v)$, then the ELzaki transform of convolution of f and g is given as:*

$$E[(f*g)(t)] = \frac{1}{v}M(v)N(v). \tag{2.5}$$

3. Modified fractional homotopy analysis transform method(MFHATM)

To illustrate the basic idea of the MFHATM for the fractional nonlinear partial differential equation, we consider the following equation with the initial condition as:

$$\begin{cases} D_t^{n\alpha}U(x,t) + LU(x,t) + RU(x,t) = g(x,t), \\ n-1 < n\alpha \leq n, \\ U(x,0) = h(x), \end{cases} \tag{3.1}$$

where $D_t^{n\alpha}$ is the Caputo fractional derivative operator, $D_t^{n\alpha} = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$, L is a linear operator, R is general nonlinear operator, $g(x,t)$ is a continuous function.
Applying Elzaki transform on both sides of Eq. (3.1), we can get:

$$E[D_t^{n\alpha}U(x,t)] + E[LU(x,t) + RU(x,t) - g(x,t)] = 0. \tag{3.2}$$

Using the property of Elzaki transform, we have the following form:

$$E[U(x,t)] - v^{n\alpha}\sum_{k=0}^{n-1} v^{2-n\alpha+k}U^{(k)}(x,0) + v^{n\alpha}E[LU(x,t) + RU(x,t) - g(x,t)] = 0. \tag{3.3}$$

Define the nonlinear operator:

$$\begin{aligned} N[\Phi(x,t;p)] =& E[\Phi(x,t;p)] - v^{n\alpha}\sum_{k=0}^{n-1} v^{2-n\alpha+k}h^{(k)}(x) \\ &+ v^{n\alpha}E[L\Phi(x,t;p) + R\Phi(x,t;p) - g(x,t;p)]. \end{aligned} \tag{3.4}$$

By means of homotopy analysis method [18], we construct the so-called the zero-order deformation equation:

$$(1-p)E[\Phi(x,t;p) - \Phi(x,t;0)] = phH(x,t)N[\Phi(x,t;p)], \tag{3.5}$$

where p is an embedding parameter and $p \in [0,1]$, $H(x,t) \neq 0$ is an auxiliary function, $h \neq 0$ is an auxiliary parameter, E is an auxiliary linear Elzaki operator. When $p = 0$ and $p = 1$, we have:

$$\begin{cases} \Phi(x,t;0) = U_0(x,t), \\ \Phi(x,t;1) = U(x,t). \end{cases} \tag{3.6}$$

When p increasing from 0 to 1, the $\Phi(x,t;p)$ various from $U_0(x,t)$ to $U(x,t)$. Expanding $\Phi(x,t;p)$ in Taylor series with respect to the p, we have:

$$\Phi(x,t;p) = U_0(x,t) + \sum_{m=1}^{\infty} U_m(x,t)p^m, \tag{3.7}$$

where

$$U_m(x,t) = \frac{1}{m!}\frac{\partial^m \Phi(x,t;p)}{\partial p^m}|_{p=0}. \tag{3.8}$$

When $p = 1$, the (3.7) becomes:

$$U(x,t) = U_0(x,t) + \sum_{m=1}^{\infty} U_m(x,t). \tag{3.9}$$

Define the vectors:

$$\overrightarrow{U}_n = \{U_0(x,t), U_1(x,t), U_2(x,t)...U_n(x,t)\}. \tag{3.10}$$

Differentiating (3.5) m-times with respect to p and then setting $p = 0$ and finally dividing them by $m!$, we obtain the so-called mth order deformation equation:

$$E[U_m(x,t) - \kappa_m U_{m-1}(x,t)] = hpH(x,t)R_m(\overrightarrow{U}_{m-1}(x,t)), \tag{3.11}$$

where

$$R_m(\overrightarrow{U}_{m-1}(x,t)) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\Phi(x,t;p)}{\partial p^{m-1}}|_{p=0}, \tag{3.12}$$

and

$$\kappa_m = \begin{cases} 0, m \leq 1 \\ 1, m > 1. \end{cases}$$

Applying the inverse Elzaki transform on both sides of Eq. (3.11), we can obtain:

$$U_m(x,t) = \kappa_m U_{m-1}(x,t) + E^{-1}[hpH(x,t)R_m(\overrightarrow{U}_{m-1}(x,t))]. \tag{3.13}$$

The mth deformation equation (3.13) is a linear which can be easily solved. So, the solution of Eq. (3.1) can be written into the following form:

$$U(x,t) = \sum_{m=0}^{N} U_m(x,t), \tag{3.14}$$

when $N \to \infty$, we can obtain an accurate approximation solution of Eq. (3.1).

Similarly, the proof of the convergence of the modified fractional homotopy analysis transform method (MFHATM) is the same as [20].

4. The new iterative transform method (NITM)

To illustrate the basic idea of the NITM for the fractional nonlinear partial differential equation, applying Elzaki transform on both sides of (3.1), we get:

$$E[U(x,t)] = v^{n\alpha}\sum_{k=0}^{n-1} v^{2-n\alpha+k}h^{(k)}(x) - v^{n\alpha}E[LU(x,t)+RU(x,t)-g(x,t)]. \tag{4.1}$$

Operating the inverse Elzaki transform on both sides of (4.1), we can obtain:

$$U(x,t) = E^{-1}[v^{n\alpha}\sum_{k=0}^{n-1} v^{2-n\alpha+k}h^{(k)}(x)] - E^{-1}[v^{n\alpha}E[LU(x,t)+RU(x,t)-g(x,t)]]. \tag{4.2}$$

Let

$$\begin{cases} f(x,t) = E^{-1}[v^{n\alpha}\sum_{k=0}^{n-1} v^{2-n\alpha+k}h^{(k)}(x) + v^{n\alpha}E[g(x,t)]], \\ N(U(x,t)) = -E^{-1}[v^{n\alpha}E[RU(x,t)]], \\ K(U(x,t)) = -E^{-1}[v^{n\alpha}E[LU(x,t)]]. \end{cases}$$

Thus, (4.2) can be written in the following form:

$$U(x,t) = f(x,t) + K(U(x,t)) + N(U(x,t)), \tag{4.3}$$

where f is a known function, K and N are linear and nonlinear operator of u. The solution of Eq. (4.3) can be written in the following series form:

$$U(x,t) = \sum_{i=0}^{\infty} U_i(x,t). \tag{4.4}$$

We have

$$K(\sum_{i=0}^{\infty} U_i) = \sum_{i=0}^{\infty} K(U_i). \tag{4.5}$$

The nonlinear operator N is decomposed as (see [5]):

$$N(\sum_{i=0}^{\infty} U_i) = N(U_0) + \sum_{i=0}^{\infty}\{N(\sum_{j=0}^{i} U_j) - N(\sum_{j=0}^{i-1} U_j)\}. \tag{4.6}$$

Therefore, Eq. (4.3) can be represented as the following form:

$$\sum_{i=1}^{\infty} U_i = f + \sum_{i=0}^{\infty} K(U_i) + N(U_0) + \sum_{i=0}^{\infty}\{N(\sum_{j=0}^{i} U_j) - N(\sum_{j=0}^{i-1} U_j)\}. \tag{4.7}$$

Defining the recurrence relation:

$$\begin{cases} U_0 = f, \\ U_1 = K(u_0) + N(u_0), \\ U_{m+1} = K(U_m) + N(U_0+U_1+...+U_m) - N(U_0+U_1+...+U_{m-1}), \end{cases}$$

we have:

$$(U_1+...+U_{m+1}) = K(U_0+...+U_m) + N(U_0+...+U_m), \tag{4.8}$$

namely

$$\sum_{i=0}^{\infty} U_i = f + K(\sum_{i=0}^{\infty} U_i) + N(\sum_{i=0}^{\infty} U_i). \tag{4.9}$$

The m-term approximate solution of (4.3) is given by:

$$U = U_0 + U_1 + U_2 + U_3 + ... + U_{m-1}. \tag{4.10}$$

Similarly, the convergence of the NITM, we refer the paper [5].

5. Illustrative examples

In this section, we apply MFHATM and NITM to solve the time-fractional Fornberg-Whitham equation.

Consider the time-fractional Fornberg-Whitham equation with the initial condition that can be written in operator form as[7]:

$$\begin{cases} u_t^{\alpha} = u_{xxt} - u_x + uu_{xxx} - uu_x + 3u_x u_{xx}, \\ 0 < \alpha \leq 1, t > 0, \\ u(x,0) = e^{\frac{x}{2}}. \end{cases} \tag{5.1}$$

5.1. Applying the MFHATM

Applying the Elzaki transform and the differentiation property of Elzaki transform on both sides of Eq. (5.1), we get:

$$E[u] - v^2 u(x,0) = v^{\alpha} E[u_{xxt} - u_x + uu_{xxx} - uu_x + 3u_x u_{xx}], \tag{5.2}$$

on simplifying (5.2), we have:

$$E[u] - v^2 e^{\frac{x}{2}} - v^{\alpha} E[u_{xxt} - u_x + uu_{xxx} - uu_x + 3u_x u_{xx}] = 0. \tag{5.3}$$

We define the nonlinear operator as:

$$N[\phi(x,t;p)] = E[\phi] - v^2 e^{\frac{x}{2}} - v^{\alpha} E[\phi_{xxt} - \phi_x + \phi\phi_{xxx} - \phi\phi_x + 3\phi_x\phi_{xx}]. \tag{5.4}$$

Constructing zeroth order deformation equation with assumption $H(x,t) = 1$, we have:

$$(1-p)E[\phi(x,t;p) - \phi(x,t;0)] = phN[\phi(x,t;p)]. \tag{5.5}$$

When $p = 0$ and $p = 1$, we can obtain:

$$\begin{cases} \phi(x,t;0) = u_0(x,t), \\ \phi(x,t;1) = u(x,t). \end{cases} \tag{5.6}$$

Therefore, we have the mth order deformation equation:

$$E[u_m(x,t) - \kappa_m u_{m-1}(x,t)] = hp[R_m(\overrightarrow{u}_{m-1}(x,t))]. \tag{5.7}$$

Operating the inverse Elzaki operator on both sides of Eq. (5.7), we get the result as follows:

$$u_m(x,t) = \kappa_m u_{m-1}(x,t) + hpE^{-1}[R_m(\overrightarrow{u}_{m-1}(x,t))], \tag{5.8}$$

where

$$\begin{aligned} R_m(\overrightarrow{u}_{m-1}) =& E[u_{m-1}] - (1-\kappa_m)v^2 e^{\frac{x}{2}} - v^{\alpha} E[(u_{m-1})_{xxt} - (u_{m-1})_x \\ &+ \sum_{k=0}^{m-1} [u_k (u_{m-1-k})_{xxx} - u_k (u_{m-1-k})_x + 3(u_k)_x (u_{m-1-k})_{xx}]]. \end{aligned} \tag{5.9}$$

According to (5.8), (5.9), we obtain:

$$\begin{aligned} u_m =&(h+\kappa_m)u_{m-1} - h(1-\kappa_m)e^{\frac{x}{2}} - hE^{-1}[v^\alpha E[(u_{m-1})_{xxt} - (u_{m-1})_x \\ &+ \sum_{k=0}^{m-1}[u_k(u_{m-1-k})_{xxx} - u_k(u_{m-1-k})_x + 3(u_k)_x(u_{m-1-k})_{xx}]]]. \end{aligned} \tag{5.10}$$

Using the initial condition $u_0(x,t) = e^{\frac{x}{2}}$, we get the following results as:

$$\begin{aligned} u_1 =& -hE^{-1}[v^\alpha E[-\frac{1}{2}e^{\frac{x}{2}}]] = \frac{he^{\frac{x}{2}}t^\alpha}{2\Gamma(\alpha+1)}, \\ u_2 =& \frac{(1+h)he^{\frac{x}{2}}t^\alpha}{2\Gamma(\alpha+1)} - \frac{h^2e^{\frac{x}{2}}t^{2\alpha-1}}{8\Gamma(2\alpha)} + \frac{h^2e^{\frac{x}{2}}t^{2\alpha}}{4\Gamma(2\alpha+1)}, \\ u_3 =& \frac{(1+h)^2he^{\frac{x}{2}}t^\alpha}{2\Gamma(\alpha+1)} - \frac{(1+h)h^3e^{\frac{x}{2}}t^{2\alpha-1}}{8\Gamma(2\alpha)} + \frac{(1+h)h^3e^{\frac{x}{2}}t^{2\alpha}}{4\Gamma(2\alpha+1)} - \frac{(1+h)h^2e^{\frac{x}{2}}t^{2\alpha-1}}{8\Gamma(2\alpha)} \\ &+ \frac{h^3e^{\frac{x}{2}}t^{3\alpha-2}}{32\Gamma(3\alpha-1)} - \frac{h^3e^{\frac{x}{2}}t^{3\alpha-1}}{8\Gamma(3\alpha)} + \frac{h^3e^{\frac{x}{2}}t^{3\alpha}}{8\Gamma(3\alpha+1)} + \frac{(1+h)h^2e^{\frac{x}{2}}t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_4 =& \frac{(1+h)^3he^{\frac{x}{2}}t^\alpha}{2\Gamma(\alpha+1)} - \frac{(1+h)^2h^3e^{\frac{x}{2}}t^{2\alpha-1}}{8\Gamma(2\alpha)} + \frac{(1+h)^2h^3e^{\frac{x}{2}}t^{2\alpha}}{4\Gamma(2\alpha+1)} - \frac{(1+h)^2h^2e^{\frac{x}{2}}t^{2\alpha-1}}{8\Gamma(2\alpha)} \\ &+ \frac{(1+h)h^3e^{\frac{x}{2}}t^{3\alpha-2}}{32\Gamma(3\alpha-1)} - \frac{(1+h)h^3e^{\frac{x}{2}}t^{3\alpha-1}}{8\Gamma(3\alpha)} + \frac{(1+h)h^3e^{\frac{x}{2}}t^{3\alpha}}{8\Gamma(3\alpha+1)} + \frac{(1+h)^2h^2e^{\frac{x}{2}}t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &- \frac{(1+h)^2h^2e^{\frac{x}{2}}t^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{(1+h)h^3e^{\frac{x}{2}}t^{3\alpha-1}}{4\Gamma(3\alpha)} + \frac{(1+h)h^3e^{\frac{x}{2}}t^{3\alpha-2}}{32\Gamma(3\alpha-1)} - \frac{(1+h)h^3e^{\frac{x}{2}}t^{3\alpha-1}}{16\Gamma(3\alpha)} \\ &+ \frac{(1+h)h^3e^{\frac{x}{2}}t^{3\alpha-2}}{32\Gamma(3\alpha-1)} - \frac{h^4e^{\frac{x}{2}}t^{4\alpha-3}}{128\Gamma(4\alpha-2)} + \frac{h^4e^{\frac{x}{2}}t^{4\alpha-2}}{32\Gamma(4\alpha-1)} - \frac{h^4e^{\frac{x}{2}}t^{4\alpha-1}}{32\Gamma(4\alpha)} \\ &+ \frac{(1+h)^2h^2e^{\frac{x}{2}}t^{2\alpha}}{4\Gamma(2\alpha+1)} - \frac{(1+h)h^4e^{\frac{x}{2}}t^{3\alpha-1}}{16\Gamma(3\alpha)} + \frac{(1+h)h^4e^{\frac{x}{2}}t^{3\alpha}}{8\Gamma(3\alpha+1)} - \frac{(1+h)h^3e^{\frac{x}{2}}t^{3\alpha-1}}{16\Gamma(3\alpha)} \\ &+ \frac{h^4e^{\frac{x}{2}}t^{4\alpha-2}}{64\Gamma(4\alpha-1)} - \frac{h^4e^{\frac{x}{2}}t^{4\alpha-1}}{16\Gamma(4\alpha)} + \frac{h^4e^{\frac{x}{2}}t^{4\alpha}}{32\Gamma(4\alpha+1)} + \frac{h^3(1+h)e^{\frac{x}{2}}t^{3\alpha}}{2\Gamma(3\alpha+1)}, \\ \vdots \end{aligned}$$

Thus, we use five terms in evaluating the approximate solution:

$$u(x,t) = \sum_{m=0}^{4} u_m(x,t). \tag{5.11}$$

Remark 5.1. When $h=-1$, the 5-order approximate solution of Eq. (5.1) is given by:

$$\begin{aligned} u(x,t) =& \sum_{m=0}^{4} u_m(x,t) \\ =& e^{\frac{x}{2}}[1 - \frac{t^\alpha}{2\Gamma(\alpha+1)} - \frac{t^{2\alpha-1}}{8\Gamma(2\alpha)} + \frac{t^{2\alpha}}{4\Gamma(2\alpha+1)} - \frac{t^{3\alpha-1}}{32\Gamma(3\alpha-1)} + \frac{t^{3\alpha-1}}{8\Gamma(3\alpha)} - \frac{t^{3\alpha}}{8\Gamma(3\alpha+1)} \\ &- \frac{t^{3\alpha-3}}{128\Gamma(4\alpha-2)} + \frac{t^{4\alpha-2}}{32\Gamma(4\alpha-1)} - \frac{t^{4\alpha-1}}{32\Gamma(4\alpha)} + \frac{t^{4\alpha-2}}{64\Gamma(4\alpha-1)} - \frac{t^{4\alpha-1}}{16\Gamma(4\alpha)} + \frac{t^{4\alpha}}{32\Gamma(4\alpha+1)}]. \end{aligned} \tag{5.12}$$

Remark 5.2. The exact solution of Eq. (5.1) for $\alpha = 1$ is given as the following form [1]:

$$u(x,t) = e^{\frac{x}{2} - \frac{2t}{3}}. \tag{5.13}$$

Remark 5.3. In this paper, we apply modified fractional homotopy analysis transform method and obtain the same result as HPM [7] for $h = -1$. When $h = -1$, $\alpha = 1$, the result is complete agreement with HAM and ADM by F. Abidi and K. Omrani [1]. Therefore, the MFHATM is rather general and contains the HPM, HAM and ADM.

Remark 5.4. In the MFHATM, the auxiliary parameter h can be apply to adjust and control the convergence region and rate of the analytical approximate solutions. Fig.1–Fig.3 respectively show the so-call h-curve [18] of the 5th-order MFHATM approximate solution for different values of α, it is very easy to see the valid region of h which corresponds to the line segment nearly parallel to the horizontal axis. So, the series is convergent when $-5 < h < 5$.

5.2. Applying the NITM

Applying the Elzaki transform and the differentiation property of Elzaki transform on both sides of Eq. (5.1), we get:

$$E[u] = v^2 e^{\frac{x}{2}} + v^\alpha E[u_{xxt} - u_x + uu_{xxx} - uu_x + 3u_x u_{xx}] = 0. \tag{5.14}$$

Operating the inverse Elzaki transform on both sides of Eq. (5.14), we have:

$$u = e^{\frac{x}{2}} + E^{-1}[v^\alpha E[u_{xxt} - u_x]] + E^{-1}[v^\alpha E[uu_{xxx} - uu_x + 3u_x u_{xx}]]. \tag{5.15}$$

Applying the NITM, we can obtain:

$$\begin{cases} u_0 = e^{\frac{x}{2}}, \\ K[u(x,t)] = E^{-1}[v^\alpha E[u_{xxt} - u_x]], \\ N[u(x,t)] = E^{-1}[v^\alpha E[uu_{xxx} - uu_x + 3u_x u_{xx}]]. \end{cases}$$

By iteration, the following results is obtained:

$$\begin{aligned}
u_0 &= e^{\frac{x}{2}}, \\
u_1 &= -\frac{e^{\frac{x}{2}} t^\alpha}{2\Gamma(\alpha+1)}, \\
u_2 &= -\frac{\alpha e^{\frac{x}{2}} \Gamma(\alpha) t^{2\alpha-1}}{8\Gamma(\alpha+1)\Gamma(2\alpha)} + \frac{e^{\frac{x}{2}} t^{2\alpha}}{4\Gamma(2\alpha+1)}, \\
u_3 &= -\frac{\alpha(2\alpha-1) e^{\frac{x}{2}} \Gamma(\alpha)\Gamma(2\alpha-1) t^{3\alpha-2}}{32\Gamma(\alpha+1)\Gamma(2\alpha)\Gamma(3\alpha-1)} + \frac{\alpha e^{\frac{x}{2}} \Gamma(2\alpha) t^{3\alpha-1}}{8\Gamma(2\alpha+1)\Gamma(3\alpha)} \\
&\quad + \frac{\alpha \Gamma(\alpha) e^{\frac{x}{2}} t^{3\alpha-1}}{16\Gamma(\alpha+1)\Gamma(3\alpha)} - \frac{e^{\frac{x}{2}} t^{3\alpha}}{8\Gamma(3\alpha+1)}, \\
u_4 &= \frac{-\alpha(2\alpha-1)(3\alpha-2) e^{\frac{x}{2}} \Gamma(\alpha)\Gamma(2\alpha-1)\Gamma(3\alpha-2) t^{4\alpha-3}}{128\Gamma(\alpha+1)\Gamma(2\alpha)\Gamma(3\alpha-1)\Gamma(4\alpha-2)} \\
&\quad + \frac{\alpha(3\alpha-1) e^{\frac{x}{2}} \Gamma(2\alpha)\Gamma(3\alpha-1) t^{4\alpha-2}}{32\Gamma(2\alpha+1)\Gamma(3\alpha)\Gamma(4\alpha-1)} + \frac{\alpha(3\alpha-1) e^{\frac{x}{2}} \Gamma(\alpha)\Gamma(3\alpha-1) t^{4\alpha-2}}{64\Gamma(\alpha+1)\Gamma(3\alpha)\Gamma(4\alpha-1)} \\
&\quad + \frac{\alpha(2\alpha-1) e^{\frac{x}{2}} \Gamma(\alpha)\Gamma(2\alpha-1) t^{4\alpha-2}}{64\Gamma(\alpha+1)\Gamma(2\alpha)\Gamma(4\alpha-1)} - \frac{3\alpha e^{\frac{x}{2}} \Gamma(3\alpha) t^{4\alpha-1}}{32\Gamma(3\alpha+1)\Gamma(4\alpha)} \\
&\quad - \frac{\alpha e^{\frac{x}{2}} \Gamma(2\alpha) t^{4\alpha-1}}{16\Gamma(2\alpha+1)\Gamma(4\alpha)} - \frac{\alpha e^{\frac{x}{2}} \Gamma(\alpha) t^{4\alpha-1}}{32\Gamma(\alpha+1)\Gamma(4\alpha)} + \frac{e^{\frac{x}{2}} t^{4\alpha}}{16\Gamma(4\alpha+1)}, \\
&\ \vdots
\end{aligned}$$

We apply five terms in evaluating the approximate solution, the solution of the Eq. (5.1) is given by:

$$\begin{aligned}
u(x,t) =& u_0+u_1+u_2+u_3+u_4\\
=& e^{\frac{x}{2}} - \frac{e^{\frac{x}{2}}t^{\alpha}}{2\Gamma(\alpha+1)} - \frac{\alpha e^{\frac{x}{2}}\Gamma(\alpha)t^{2\alpha-1}}{8\Gamma(\alpha+1)\Gamma(2\alpha)} + \frac{e^{\frac{x}{2}}t^{2\alpha}}{4\Gamma(2\alpha+1)}\\
&- \frac{\alpha(2\alpha-1)e^{\frac{x}{2}}\Gamma(\alpha)\Gamma(2\alpha-1)t^{3\alpha-2}}{32\Gamma(\alpha+1)\Gamma(2\alpha)\Gamma(3\alpha-1)} + \frac{\alpha e^{\frac{x}{2}}\Gamma(2\alpha)t^{3\alpha-1}}{8\Gamma(2\alpha+1)\Gamma(3\alpha)} + \frac{\alpha\Gamma(\alpha)e^{\frac{x}{2}}t^{3\alpha-1}}{16\Gamma(\alpha+1)\Gamma(3\alpha)}\\
&- \frac{e^{\frac{x}{2}}t^{3\alpha}}{8\Gamma(3\alpha+1)} - \frac{\alpha(2\alpha-1)(3\alpha-2)e^{\frac{x}{2}}\Gamma(\alpha)\Gamma(2\alpha-1)\Gamma(3\alpha-2)t^{4\alpha-3}}{128\Gamma(\alpha+1)\Gamma(2\alpha)\Gamma(3\alpha-1)\Gamma(4\alpha-2)}\\
&+ \frac{\alpha(3\alpha-1)e^{\frac{x}{2}}\Gamma(2\alpha)\Gamma(3\alpha-1)t^{4\alpha-2}}{32\Gamma(2\alpha+1)\Gamma(3\alpha)\Gamma(4\alpha-1)} + \frac{\alpha(3\alpha-1)e^{\frac{x}{2}}\Gamma(\alpha)\Gamma(3\alpha-1)t^{4\alpha-2}}{64\Gamma(\alpha+1)\Gamma(3\alpha)\Gamma(4\alpha-1)}\\
&+ \frac{\alpha(2\alpha-1)e^{\frac{x}{2}}\Gamma(\alpha)\Gamma(2\alpha-1)t^{4\alpha-2}}{64\Gamma(\alpha+1)\Gamma(2\alpha)\Gamma(4\alpha-1)} - \frac{3\alpha e^{\frac{x}{2}}\Gamma(3\alpha)t^{4\alpha-1}}{32\Gamma(3\alpha+1)\Gamma(4\alpha)}\\
&- \frac{\alpha e^{\frac{x}{2}}\Gamma(2\alpha)t^{4\alpha-1}}{16\Gamma(2\alpha+1)\Gamma(4\alpha)} - \frac{\alpha e^{\frac{x}{2}}\Gamma(\alpha)t^{4\alpha-1}}{32\Gamma(\alpha+1)\Gamma(4\alpha)} + \frac{e^{\frac{x}{2}}t^{4\alpha}}{16\Gamma(4\alpha+1)}.
\end{aligned}$$

Remark 5.5. By using the new iterative transform method, we can directly find the approximate solution without applying any prior knowledge as perturbation, polynomials, auxiliary parameter and so on. Therefore, the NITM is very easy to understand and implement. In Table.1 and Table.2, we compare the exact solution with the 5th-order approximate solutions by MFHATM and NITM at some points for $\alpha = 1$. Fig.4 and Fig.5 show the absolute error between the exact solution and the 5-order approximate solutions by MFHATM and NITM for $\alpha = 1$. The numerical results show that the MFHATM and NITM are highly accurate.

Remark 5.6. In Table.3 and Table.4, we respectively compute exact solution and the 5th-order approximate solutions by MFHATM and NITM for different values of α. Fig.6–Fig.11 show the 5th-order approximate solutions by MFHATM ($h = -1$) and NITM for $\alpha = 0.6$, $\alpha = 0.8$, $\alpha = 1$, respectively. Fig.12 we draw the exact solution of Eq. (5.1) for $\alpha = 1$. By comparison, it is easy to find that the approximate solutions continuously depend on the values of time-fractional derivative α.

Remark 5.7. In Table.5, we compare the exact solution with the 5th-order approximate solutions by NITM, HPM [7] and ADM [1] at some points for $\alpha = 1$. The numerical results show the NITM is more highly accurate than HPM and ADM.

Remark 5.8. In this paper, we only apply five terms to approximate the solution of Eq. (5.1), if we apply more terms of the approximate solution, the accuracy of the approximate solution will be greatly improved.

6. Conclusion

In this paper, the modified fractional homotopy analysis transform method and new iterative transform method have been successfully applied for finding the approximate solution of the nonlinear time-fractional Fornberg-Whitham equation. The numerical results show that the new iterative transform method and the modified fractional homotopy analysis transform method are simpler and more highly accurate than existing methods (HPM, ADM). Therefore, it is obvious that the NITM and MFHATM are very powerful, efficient and easy mathematical methods for solving the nonlinear fractional differential equations in science and engineering.

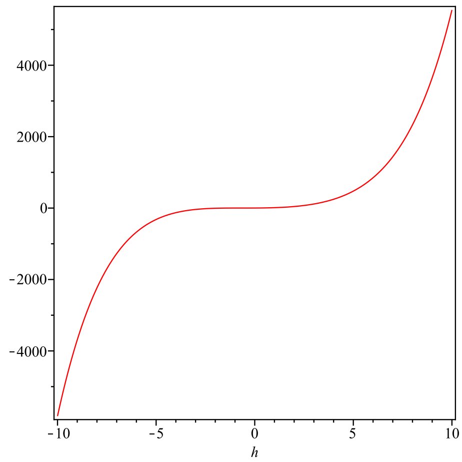

Figure 1: The h-curve of $u(1, 0.5)$ given by the 5th-order MFHATM approximate solution for $\alpha = 0.4$.

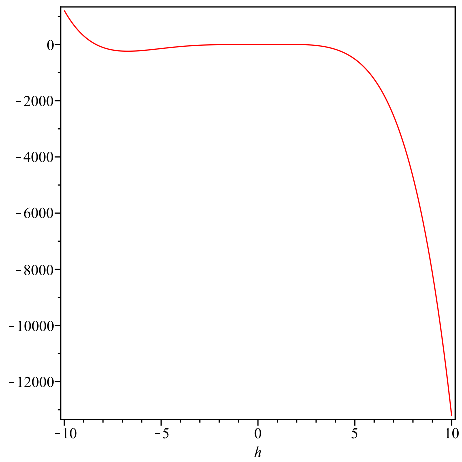

Figure 2: The h-curve of $u(1, 0.5)$ given by the 5th-order MFHATM approximate solution for $\alpha = 0.8$.

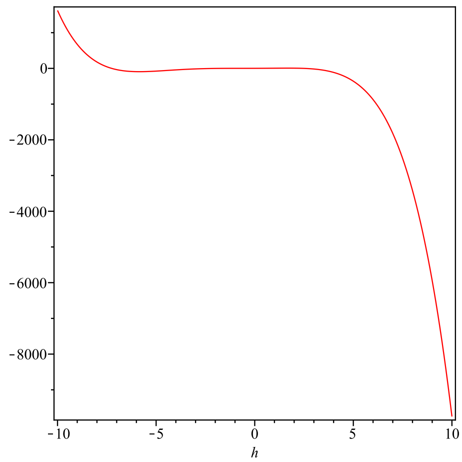

Figure 3: The h-curve of $u(1, 0.5)$ given by the 5th-order MFHATM approximate solution for $\alpha = 1$.

			$\alpha = 1, h = -1$	
x	t	u_{exa}	u_{MFHATM}	$\lvert u_{exa} - u_{MFHATM}\rvert$
0.2	0.3	0.9048374180	0.9087039069	0.0038664889
0.3	0.4	0.8898817710	0.8978703440	0.0079885730
0.6	0.7	0.8464817249	0.8683671592	0.0218854343
0.7	0.3	1.1618342431	1.1667989130	0.0049646700

Table 1: The different values for the 5th-order approximate solution by MFHATM and the exact solution for $\alpha = 1$.

			$\alpha = 1$	
x	t	u_{exa}	u_{NITM}	$\lvert u_{exa} - u_{NITM}\rvert$
0.2	0.3	0.9048374180	0.9049165380	0.0000791200
0.3	0.4	0.8898817710	0.8898004371	0.0000813339
0.6	0.7	0.8464817249	0.8458696883	0.0006120366
0.7	0.3	1.1618342431	1.1619358351	0.0001015920

Table 2: The different values for the 5th-order approximate solution by NITM and the exact solution for $\alpha = 1$.

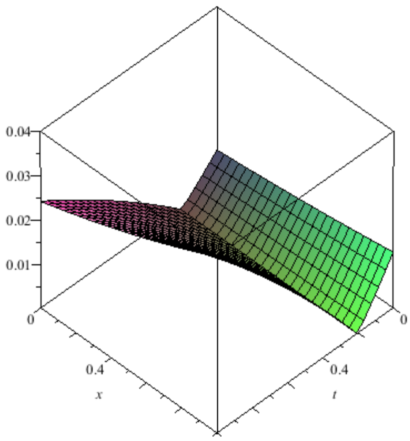

Figure 4: The absolute error $\lvert u_{exa} - u_{MFHATM}\rvert$ for $\alpha = 1$.

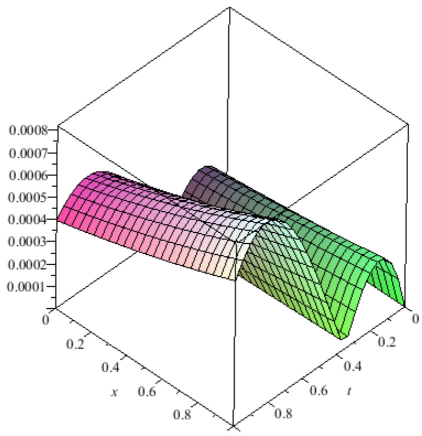

Figure 5: The absolute error $\lvert u_{exa} - u_{NITM}\rvert$ for $\alpha = 1$.

				$h = -1$		
x	t	0.3	0.6	0.9	1	$u_{exa}(\alpha = 1)$
0.2	0.3	0.6999363827	0.7897930519	0.8724356962	0.9087039069	0.9048374180
0.3	0.4	0.7061268867	0.7979109655	0.8632765540	0.8978703440	0.8898817710
0.6	0.7	0.7581237974	0.8315693348	0.8484162865	0.8683671592	0.8464817249
0.7	0.3	0.8987361058	1.0141143530	1.1202296090	1.1667989130	1.1618342431

Table 3: Comparison between the exact solution and the 5th-order approximate solution by MFHATM for different values of α.

x	t	0.3	0.6	0.9	1	u_{exa}
0.2	0.3	0.7542664469	0.7690502464	0.8663502605	0.9049165380	0.9048374180
0.3	0.4	0.7651789940	0.7774053929	0.8516805973	0.8898004371	0.8898817710
0.6	0.7	0.8339806567	0.8189897704	0.8222444558	0.8458696883	0.8464817249
0.7	0.3	0.9684972892	0.9874800634	1.1124157550	1.1619358351	1.1618342431

Table 4: Comparison between the exact solution and the 5th-order approximate solution by NITM for different values of α.

		$\alpha = 1$			
x	t	u_{exa}	u_{HPM}	u_{ADM}	u_{NITM}
0.2	0.3	0.9048374180	0.9087039069	0.9087039069	0.9049165380
0.3	0.4	0.8898817710	0.8978703440	0.8978703440	0.8898004371
0.6	0.7	0.8464817249	0.8683671592	0.8683671592	0.8458696883
0.7	0.3	1.1618342430	1.1667989130	1.1667989130	1.1619358350

Table 5: Comparison the 5th-order approximate solution by NITM, HPM, ADM and the exact solution of Eq.(5.1) for $\alpha = 1$.

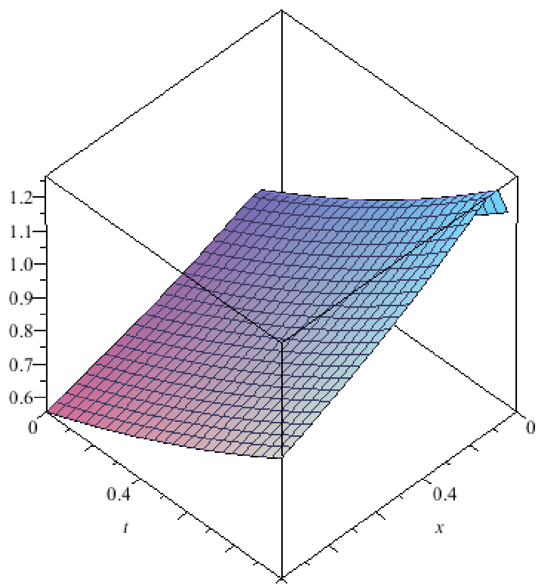

Figure 6: The 5th-order approximate solution by MFHATM for $\alpha = 0.6$, $h = -1$.

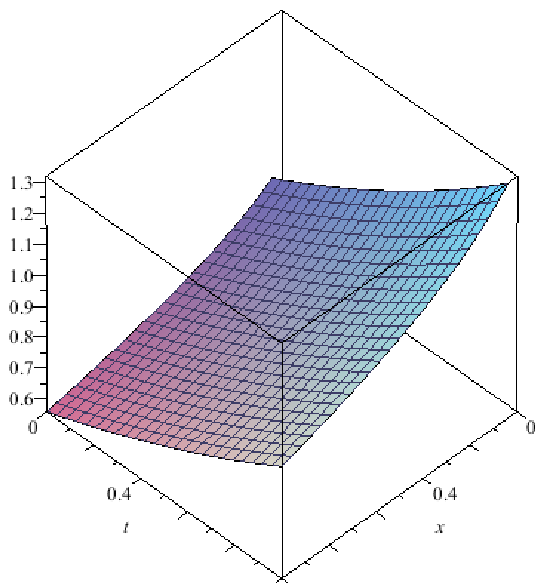

Figure 7: The 5th-order approximate solution by NITM for $\alpha = 0.6$.

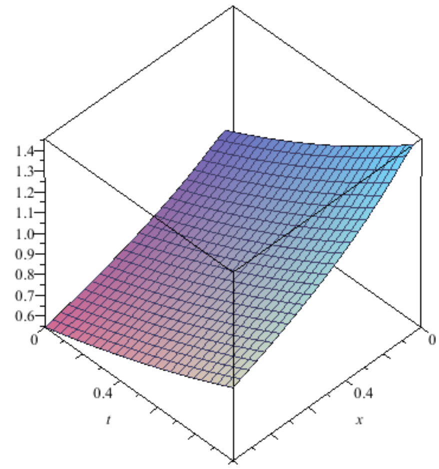

Figure 8: The 5th-order approximate solution by MFHATM for $\alpha = 0.8$, $h = -1$.

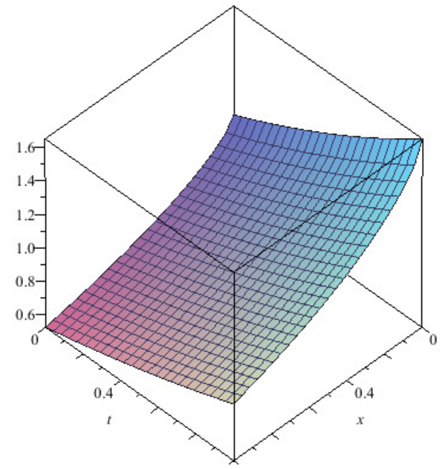

Figure 9: The 5th-order approximate solution by NITM for $\alpha = 0.8$.

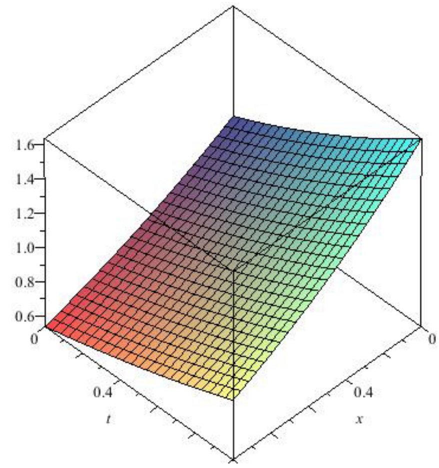

Figure 10: The 5th-order approximate solution by MFHATM for $\alpha = 1$, $h = -1$.

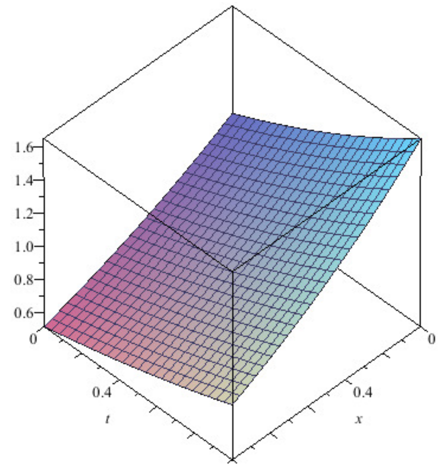

Figure 11: The 5th-order approximate solution by NITM for $\alpha = 1$.

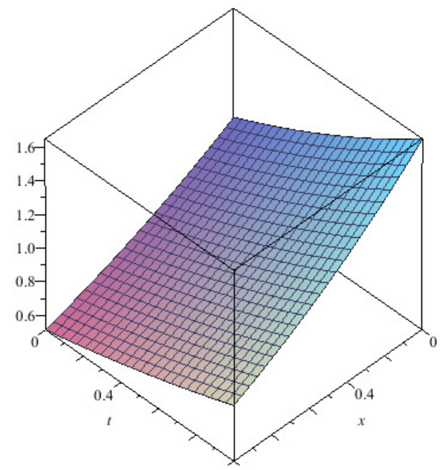

Figure 12: The exact solution for $\alpha = 1$.

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