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Fixed point results and an application to homotopy in modular metric spaces

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Abstract

The purpose of this paper is to define new concepts, such as T-orbitally w-completeness, orbitally w-continuity and almost weakly w-contractive mapping in the modular metric spaces. We prove some fixed point theorems for these related concepts and mappings in this space. Further, we give an application using the technique in [Lj. B. Ćirić, B. Samet, H. Aydi, C. Vetro, Appl. Math. Comput., **218** (2011), 2398–2406] and show that our results can be applied to homotopy.

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1. Introduction

Fixed point theory is an active field of research with wide range of applications in a variety of areas such as nonlinear analysis, functional analysis, differential equations, operator theory, engineering, game theory, etc. It is a very powerful and significant tool in solving existence and uniqueness problems.

Fixed point theorems are concerned with the results which state that under certain conditions a self map f on a set X allow one or more fixed point. Fixed point theory started after the classical analysis began rapidly. Afterwards, it was used mainly to prove existence theorems for differential equations.

The Polish mathematician Banach [5] formulated and proved a theorem which related to under suitable conditions the existence and uniqueness of a fixed point in a complete metric space. This result is well-known as Banach's fixed point theorem or the Banach contraction principle. Since it has a useful structure, many mathematicians have drawn attention to the contraction principle. One of them is Ćirić [14] and gave a well-known generalization of it.

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Nakano is the first researcher who introduced modular spaces [22]. Then Chistyakov presented the modular metric space [8] and got some results in [9, 10]. Mongkolkeha et al. [21] gave some theorems of fixed points for contraction mappings in modular metric spaces. Dehghan et al. [16] gave an example related to results in [21]. Azadifar *et. al.*, [4] proved the existence and uniqueness of a common fixed point of compatible mappings of integral type in this space. Kilinc and Alaca [19] defined (ϵ, k)-uniformly locally contractive mappings and η-chainable concept and proved a fixed point theorem for these concepts in a complete modular metric spaces. Further, different fixed point results in this space were proved in [2, 3, 7, 15, 17, 18] and [20].

In the present paper, as a new perspective in modular metric spaces we introduce T-orbitally w-completeness, orbitally w-continuity and almost weakly w-contractive mapping in the modular metric spaces. We prove some fixed point theorems for these related concepts and mappings satisfying the condition

$$\omega_\lambda(Tx, Ty) \leq \phi[\omega_{3\lambda}(x, y)]$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a real function, upper semicontinuous from the right such that $\phi(t) < t$ for $t > 0$. Our results are the modular metric version of Ćirić [12, 13] and Boyd and Wong [6]. An application for our main result to homotopy is given at the end of the paper.

2. Preliminaries

Definition 2.1 ([23]). A modular on a real linear space X is a functional $\rho : X \longrightarrow [0, \infty]$ satisfying the followings:

(A1) $\rho(0) = 0$;
(A2) If $x \in X$ and $\rho(\alpha x) = 0$ for all numbers $\alpha > 0$, then $x = 0$;
(A3) $\rho(-x) = \rho(x)$ for all $x \in X$;
(A4) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $x, y \in X$.

Let X be a non-empty set and $\lambda \in (0, \infty)$. We remark that the function $\omega : (0, \infty) \times X \times X \longrightarrow [0, \infty]$ is denoted by $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$.

Definition 2.2 ([9]). Let X be a non-empty set, a function

$$\omega : (0, \infty) \times X \times X \longrightarrow [0, \infty]$$

is said to be a metric modular on X if satisfying, for all $x, y, z \in X$ the following conditions hold:

(i) $\omega_\lambda(x, y) = 0$ for all $\lambda > 0 \Leftrightarrow x = y$;
(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$;
(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$.

The function $0 < \lambda \mapsto \omega_\lambda(x, y) \in [0, \infty)$ is [9] non-increasing on $(0, \infty)$. If $0 < \mu < \lambda$, then (i)-(iii) imply

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) \leq \omega_\mu(x, y).$$

Let's recall that definitions of two sets X_ω and X^*_ω [9]:

$$X_\omega \equiv X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \text{ as } \lambda \to \infty\}$$

and

$$X^*_\omega \equiv X^*_\omega(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x, x_0) < \infty\}.$$

Definition 2.3 ([21]). Let (X, ω) be a modular metric space.

- A sequence $(x_n)_{n\in\mathbb{N}}$ in X^*_ω is called ω-convergent to $x \in X^*_\omega$ if $\omega_\lambda(x_n, x) \to 0$, as $n \to \infty$ for all $\lambda > 0$.

- A sequence $(x_n)_{n\in\mathbb{N}} \subset X_\omega^*$ is said to be ω-Cauchy if and only if for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for each $n, m \geq n(\epsilon)$ and $\lambda > 0$ we have $\omega_\lambda(x_n, x_m) < \epsilon$.
- A subset C of X_ω^* is said to be ω-closed if the limit of ω-convergent sequence of C always belong to C.
- A subset C of X_ω^* is said to be ω-complete if any ω-Cauchy in C is ω-convergent sequence and its limit is in C.

Definition 2.4 ([11]). Let (X,ω) be a modular metric space and T be a self-mapping of X_ω^*. An orbit of T at the point $x \in X_\omega^*$ is the set

$$O(x,T) := \{x, Tx, \cdots, T^n x, \cdots\}.$$

Definition 2.5 ([1]). Let X_ω be a modular metric space. For $r > 0$ and $x \in X_\omega$, we define the open sphere $B_\omega(x,r)$ and the closed sphere $B_\omega[x,r]$ with center x and radius r as follows:

$$B_\omega(x,r) = \{y \in X_\omega : \omega_\lambda(x,y) < r\}$$
$$B_\omega[x,r] = \{y \in X_\omega : \omega_\lambda(x,y) \leq r\}.$$

3. Main Results

In this section, we first give some definitions about our study.

Definition 3.1. A subset U of X_ω^* is said to be ω-open if for each $x \in U$ there exists $r > 0$ such that $B_\omega(x,r) \subseteq U$.

Definition 3.2. Let (X,ω) be a modular metric space.

- (X,ω) is called T-orbitally w-complete if T is a self-mapping of X_ω^* and if any w-Cauchy subsequence $\{T^{n_i}x\}$ in orbit $O(x,T)$ for $x \in X_\omega^*$ converges in X_ω^*.
- An operator $T : X_\omega^* \to X_\omega^*$ on X_ω^* is called orbitally ω-continuous if

$$T^{n_i}x \to x_0 \Rightarrow T(T^{n_i}x) \to Tx_0 \text{ as } i \to \infty.$$

- A self-mapping T of a modular metric space X_ω^* is said to be a w-contraction type mapping if for every $x, y \in X_\omega^*$ there exist numbers $\alpha(x,y), 0 \leq \alpha(x,y) < 1$ and $\delta(x,y) > 0$ such that

$$\omega_\lambda(T^n x, T^n y) \leq [\alpha(x,y)]^n \delta(x,y);\ n = 1, 2, \cdots$$

Let's prove the first theorem.

Theorem 3.3. *Let X_ω^* be a T-orbitally w-complete and $T : X_\omega^* \to X_\omega^*$ be w-contraction type mapping and orbitally ω-continuous. Assume that there exists an element $x = x(\lambda) \in X_\omega^*$ such that $\omega_\lambda(x,Tx) < \infty$. Then we have the following statements:*

(i) *T has a unique fixed point $u \in X_\omega^*$,*
(ii) *$x_n = T^n x_0 \to u$ for every $x_0 \in X_\omega^*$,*
(iii) *There is an inequality*

$$\omega_\lambda(T^n x_0, u) \leq \frac{[\alpha(x_0,Tx_0)]^n}{1-\alpha(x_0,Tx_0)}\delta(x_0,Tx_0)$$

where $0 \leq \alpha(x_0,Tx_0) < 1$, $\delta(x_0,Tx_0) > 0$.

Proof. **(ii)** We establish a sequence $(x_n) \subset X_\omega^*$ such that $x_i = T^i x_0$ with $x_0 \in X_\omega^*$. Now, we must show that it is a ω-Cauchy.

$$\begin{aligned}\omega_\lambda(T^n x_0, T^{n+r}x_0) =& \omega_\lambda(T^n x_0, T^{n+r-1}Tx_0)\\ \leq& \omega_{\frac{\lambda}{r}}(T^n x_0, T^n Tx_0) + \omega_{\frac{\lambda}{r}}(T^{n+1}x_0, T^{n+1}Tx_0) + \cdots + \omega_{\frac{\lambda}{r}}(T^{n+r-1}x_0, T^{n+r-1}Tx_0).\end{aligned}$$

Since T is w-contraction type mapping, we obtain

$$\omega_\lambda(T^n x_0, T^{n+r}x_0) \le [\alpha(x_0,Tx_0)]^n\delta(x_0,Tx_0) + \cdots + [\alpha(x_0,Tx_0)]^{n+r-1}\delta(x_0,Tx_0)$$

and so

$$\omega_\lambda(T^n x_0, T^{n+r}x_0) \le \left(\sum_{k=n}^{n+r-1}[\alpha(x_0,Tx_0)]^k\right)\delta(x_0,Tx_0). \tag{3.1}$$

We have $\lim_{n\to\infty}\omega_\lambda(T^n x_0, T^{n+r}x_0) = 0$ because $\alpha(x_0,Tx_0) < 1$. Thus $(x_n) = (T^n x_0)$ is a ω-Cauchy and by the T-orbitally w-completeness of X^*_ω, there exists a point $u \in X^*_\omega$ such that

$$u = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x_0.$$

(i) Orbitally ω-continuity of T gives rise to

$$Tu = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = u.$$

As a result, u is a fixed point of T.

We indicate that u is the unique fixed point of T. Suppose that $u^{'}$ is another fixed point of T. Then

$$\omega_\lambda(u,u^{'}) = \omega_\lambda(T^n u, T^n u^{'}) \le [\alpha(u,u^{'})]^n\delta(u,u^{'}).$$

Since $\lim_{n\to\infty}[\alpha(u,u^{'})]^n = 0$, we conclude that $\omega_\lambda(u,u^{'}) = 0$. Therefore $u = u^{'}$.

(iii) Taking the limit as $r \to \infty$ in (3.1), we have the following:

$$\begin{aligned}\lim_{r\to\infty}\omega_\lambda(T^n x_0, T^{n+r}x_0) &\le \lim_{r\to\infty}\left(\sum_{k=n}^{n+r-1}[\alpha(x_0,Tx_0)]^k\right)\delta(x_0,Tx_0)\\ &= \lim_{r\to\infty}[\alpha(x_0,Tx_0)]^n\big(1+\alpha(x_0,Tx_0)+\cdots+[\alpha(x_0,Tx_0)]^{r-1}\big)\delta(x_0,Tx_0)\\ &= \lim_{r\to\infty}[\alpha(x_0,Tx_0)]^n\frac{1-[\alpha(x_0,Tx_0)]^r}{1-\alpha(x_0,Tx_0)}\delta(x_0,Tx_0)\\ &= \frac{[\alpha(x_0,Tx_0)]^n}{1-\alpha(x_0,Tx_0)}\delta(x_0,Tx_0).\end{aligned}$$

As a consequence,

$$\omega_\lambda(T^n x_0, u) \le \frac{[\alpha(x_0,Tx_0)]^n}{1-\alpha(x_0,Tx_0)}\delta(x_0,Tx_0).$$

□

Definition 3.4. A mapping $T : X^*_\omega \to X^*_\omega$ is called almost weakly ω-contractive if for each $x, y \in X^*_\omega$ there exists a positive integer $m(x,y)$ such that for all $j, k \ge m(x,y)$

$$\omega_\lambda(T(T^j x), T(T^k y)) \le \alpha(\omega_\lambda(T^j x, T^k y))\omega_\lambda(T^j x, T^k y), \tag{3.2}$$

where $\alpha : (0,\infty) \to [0,1)$ is a real function satisfying $\sup\{\alpha(r) : p \le r \le q\} < 1$ for any $0 < p < q < +\infty$.

Theorem 3.5. *Let $T : X^*_\omega \to X^*_\omega$ be a self-mapping of a T-orbitally ω-complete X^*_ω. Suppose that there exists an element $x = x(\lambda) \in X^*_\omega$ such that $\omega_\lambda(x,Tx) < \infty$. If T is an almost weakly ω-contractive, then*

(1) *T has a unique fixed point $u \in X^*_\omega$,*

(2) *$\lim_{n\to\infty} T^n x = u$,*

(3) *$\omega_\lambda(T^n x, u) \le \varepsilon$ when $\omega_\lambda(T^{n-1}x, T^n x) \le [1-\alpha(\varepsilon)]\varepsilon$, $n \ge m(x,Tx)+1$ for every $x \in X^*_\omega$, where*

$$\alpha(\varepsilon) = \sup\{\alpha(r) : 0 < \varepsilon \le r \le 2\varepsilon\}.$$

Proof. **(2)** Let x and y be any two points in X_ω^*. For $n \geq m(x,y)$, we have

$$\omega_\lambda(T^{n+1}x, T^{n+1}y) \leq \alpha[\omega_\lambda(T^n x, T^n y)]\omega_\lambda(T^n x, T^n y) < \omega_\lambda(T^n x, T^n y)$$

and thus $(\omega_\lambda(T^n x, T^n y))$ is a non-increasing sequence. Assume that $\lim_{n\to\infty} \omega_\lambda(T^n x, T^n y) = t$ and $t > 0$. Define $\alpha(t) = \sup\{\alpha(r) : 0 < t \leq r \leq 2t\}$. Then there is an integer $s > m(x,y)$ such that

$$t \leq \omega_\lambda(T^s x, T^s y) < t + [1-\alpha(t)]t = [2-\alpha(t)]t.$$

Therefore

$$\begin{aligned}\omega_\lambda(T^{s+1}x, T^{s+1}y) \leq& \alpha[\omega_\lambda(T^s x, T^s y)]\omega_\lambda(T^s x, T^s y)\\ <& \alpha(t)[2-\alpha(t)]t\\ =& \big(1-[1-\alpha(t)]^2\big)t\\ <& t\end{aligned}$$

but it couldn't be true, as $\omega_\lambda(T^n x, T^n y) \geq t$ for each $n \geq m(x,y)$. As a result, we obtain

$$\lim_{n\to\infty} \omega_\lambda(T^n x, T^n y) = 0 \tag{3.3}$$

for each $x, y \in X_\omega^*$. Let x be a point of X_ω^*. Now we should show that the sequence $(T^n x)$ at x is a ω-Cauchy.

Let $\alpha(\varepsilon) = \sup\{\alpha(r) : \varepsilon \leq r \leq 2\varepsilon\}$ be defined for arbitrary $\varepsilon > 0$. By (3.3), there exists an integer $n_0 \geq m(x, Tx) + 1$ such that for every $k \geq n_0$,

$$\omega_\lambda(T^{k-1}x, T^k x) < [1-\alpha(\varepsilon)]\varepsilon. \tag{3.4}$$

From induction on r, we must show that

$$\omega_\lambda(T^k x, T^{k+r}x) < \varepsilon. \tag{3.5}$$

For the case $r = 1$, it is clear that (3.5) holds by (3.4). Assume that (3.5) holds for some $r \geq 1$. (3.4) and the induction hypothesis give the following:

$$\omega_\lambda(T^{k-1}x, T^{k+r}x) \leq \omega_{\frac{\lambda}{2}}(T^{k-1}x, T^k x) + \omega_{\frac{\lambda}{2}}(T^k x, T^{k+r}x) < [1-\alpha(\varepsilon)]\varepsilon + \varepsilon = [2-\alpha(\varepsilon)]\varepsilon. \tag{3.6}$$

If $\omega_\lambda(T^{k-1}x, T^{k+r}x) < \varepsilon$ and $k-1 \geq m(x, Tx)$, then we have

$$\omega_\lambda(T^k x, T^{k+r+1}x) < \varepsilon$$

from the hypothesis for T. Assuming $\varepsilon \leq \omega_\lambda(T^{k-1}x, T^{k+r}x)$, we find

$$\begin{aligned}\omega_\lambda(T^k x, T^{k+r+1}x) \leq& \alpha[\omega_\lambda(T^{k-1}x, T^{k+r}x)]\omega_\lambda(T^{k-1}x, T^{k+r}x)\\ \leq& \alpha(\varepsilon)\omega_\lambda(T^{k-1}x, T^{k+r}x)\\ <& \alpha(\varepsilon)[2-\alpha(\varepsilon)]\varepsilon\\ =& (1-(1-\alpha(\varepsilon))^2)\varepsilon\\ <& \varepsilon\end{aligned}$$

by (3.2) and (3.6). As a result, we conclude that (3.5) holds for each $r \in \mathbb{N}$ by induction. Thus, $(T^n x)$ is a ω-Cauchy and by T-orbitally ω-completeness of X_ω^*, there is an element $u \in X_\omega^*$ such that $\lim_{n\to\infty} T^n x = u$.

(1) By orbitally ω-continuity of T, we get $Tu = u$. We shall show that u is the unique fixed point of T. Suppose that u' is another fixed point such that $Tu' = u'$. Then

$$\begin{aligned}\omega_\lambda(u, u') =& \omega_\lambda(T^{n+1}u, T^{n+1}u')\\ \leq& \alpha[\omega_\lambda(T^n u, T^n u')]\omega_\lambda(T^n u, T^n u')\\ =& \alpha[\omega_\lambda(T^n u, T^n u')]\omega_\lambda(u, u')\end{aligned}$$

$$\Rightarrow \big(1-\alpha[\omega_\lambda(T^n u, T^n u^{'})]\big)\omega_\lambda(u,u^{'}) \leq 0.$$

Since $\alpha[\omega_\lambda(T^n u, T^n u^{'})] < 1$, we have $\omega_\lambda(u,u^{'}) = 0$. Therefore $u = u^{'}$.

Taking the limit as r approaches infinity in (3.5), we obtain (3). □

Theorem 3.6. *Let X^*_ω be ω-complete metric space and $T : X^*_\omega \to X^*_\omega$ be a map such that*

$$\omega_\lambda(Tx, Ty) \leq \phi[\omega_{3\lambda}(x,y)] \tag{3.7}$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a real function, upper semicontinuous from the right and satisfying

$$\phi(t) < t \qquad for \ \ t > 0. \tag{3.8}$$

Suppose that there exists an element $x = x(\lambda) \in X^*_\omega$ such that $\omega_\lambda(x, Tx) < \infty$. Then T has a unique fixed point $y \in X^*_\omega$ and $T^n x \to y$ as $n \to \infty$ for each $x \in X^*_\omega$.

Proof. Set $\alpha_n = \omega_\lambda(T^{n-1}x, T^n x)$ for arbitrary $x \in X^*_\omega$. Then we have

$$\begin{aligned}\alpha_{n+1} =& \omega_\lambda(T^n x, T^{n+1}x)\\ =&\omega_\lambda(TT^{n-1}x, TT^n x)\\ \leq&\phi[\omega_{3\lambda}(T^{n-1}x, T^n x)]\\ <&\omega_{3\lambda}(T^{n-1}x, T^n x)\\ <&\omega_\lambda(T^{n-1}x, T^n x)\\ =&\alpha_n.\end{aligned}$$

Therefore we conclude that $\{\alpha_n\}$ is a decreasing sequence and so it has a limit a. Assume that $a > 0$. From $\alpha_{n+1} \leq \phi(\alpha_n)$ and upper semicontinuity from the right of ϕ, we obtain

$$a \leq \lim_{\alpha_n \to a+} \sup \phi(\alpha_n) \leq \phi(a).$$

But the last statement is in contradiction in (3.8). Thus, we get

$$\lim_{n\to\infty} \omega_\lambda(T^{n-1}x, T^n x) = 0.$$

We now show that $\{T^n x\}$ is ω-Cauchy. If we suppose that $\{T^n x\}$ is not ω-Cauchy, then there exists an $\epsilon > 0$ such that for every $n \in \mathbb{N}$ there is $m = m(n) > n$ such that

$$\omega_\lambda(T^n x, T^m x) \geq \epsilon. \tag{3.9}$$

We can assume that $m(n)$ is the smallest integer for which (3.9) holds. It means

$$\omega_\lambda(T^n x, T^{m-1}x) < \epsilon.$$

Using the triangle inequality, we have

$$\begin{aligned}\epsilon \leq \omega_\lambda(T^n x, T^m x) \leq& \omega_{\frac{\lambda}{2}}(T^n x, T^{m-1}x) + \omega_{\frac{\lambda}{2}}(T^{m-1}x, T^m x)\\ \leq& \frac{\epsilon}{2} + \omega_{\frac{\lambda}{2}}(T^{m-1}x, T^m x)\\ <&\epsilon + \omega_{\frac{\lambda}{2}}(T^{m-1}x, T^m x).\end{aligned}$$

As $\lim_{m\to\infty} \omega_{\frac{\lambda}{2}}(T^{m-1}x, T^m x) = 0$, we obtain

$$\gamma_n = \omega_\lambda(T^n, T^m x) \to \epsilon + \quad \text{as} \quad m \to \infty.$$

From the fact that $m > n$ implies $\omega_\lambda(T^m x, T^{m+1}x) \leq \omega_\lambda(T^n x, T^{n+1}x)$, we have

$$\begin{aligned}\epsilon \leq \gamma_n = \omega_\lambda(T^n x, T^m x) &\leq \omega_{\frac{\lambda}{3}}(T^n x, T^{n+1}x) + \omega_{\frac{\lambda}{3}}(TT^n x, TT^m x) + \omega_{\frac{\lambda}{3}}(T^m x, T^{m+1}x)\\ &\leq 2\omega_{\frac{\lambda}{3}}(T^n x, T^{n+1}x) + \phi[\omega_\lambda(T^n x, T^m x)]\\ &= 2\omega_{\frac{\lambda}{3}}(T^n x, T^{n+1}x) + \phi(\gamma_n).\end{aligned}$$

By the continuity of ϕ, we conclude

$$\epsilon \leq \lim_{n\to\infty} 2\omega_{\frac{\lambda}{3}}(T^n x, T^{n+1}x) + \lim_{n\to\infty} \sup \phi(\gamma_n) < \phi(\epsilon)$$

which contradicts with (3.8). As a result, $\{T^n x\}$ is ω-Cauchy and as X^*_ω is ω-complete, $\{T^n x\}$ ω-converges to x_0 in X^*_ω. From (3.7) and (3.8), as T is ω-continuous, we get

$$Tx_0 = T(\lim_{n\to\infty} T^n x) = \lim_{n\to\infty} T(T^n x) = \lim_{n\to\infty} T^{n+1}x = x_0.$$

Thus, the limit point x_0 of $\{T^n x\}$ is a fixed point of T.

Now we prove the uniqueness. For this purpose, let u be another fixed point of T. Then

$$\begin{aligned}\omega_\lambda(u - x_0) &= \omega_\lambda(Tu, Tx_0)\\ &\leq \phi[\omega_{3\lambda}(u, x_0)]\\ &< \omega_{3\lambda}(u, x_0)\\ &< \omega_\lambda(u, x_0).\end{aligned}$$

Since this is contradiction, $u = x_0$. Thus T has a unique fixed point. □

4. An Application to Homotopy

Theorem 4.1. *Let X^*_ω be ω-complete metric space, U, V be an open and a closed subsets of X^*_ω with $U \subset V$, respectively. Let the operator $H : V \times [0, 1] \to X^*_\omega$ be satisfied the following conditions:*

(a) $x \neq H(x, t)$ for every $x \in V \setminus U$ and every $t \in [0, 1]$,

(b) there exists $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous non-decreasing function satisfying $\phi(t) < t$ such that for each $t \in [0, 1]$ and each $x, y \in V$ we have

$$\omega_\lambda(H(x, t), H(y, t)) \leq \phi[\omega_{3\lambda}(x, y)],$$

(c) there is a continuous function $\alpha : [0, 1] \to \mathbb{R}$ such that

$$\omega_\lambda(H(x, t), H(x, s)) \leq |\alpha(t) - \alpha(s)|$$

for all $t, s \in [0, 1]$ and every $x \in V$,

(d) $\psi : [0, +\infty) \to [0, +\infty)$ is strictly non-decreasing mapping where $\psi(x) = x - \phi(x)$.

Then $H(., 0)$ has a fixed point if and only if $H(., 1)$ has a fixed point.

Proof. Define the following set:

$$G := \{t \in [0, 1] \mid x = H(x, t) \text{ for some } x \in U\}.$$

($\Rightarrow$) Assume that $H(., 0)$ has a fixed point. Since (a) holds, we have $0 \in G$ and hence G is a non-empty set. We would like to show that G is both closed and open in $[0, 1]$. From the connectedness of $[0, 1]$, we have the required result because $G = [0, 1]$.

We begin with proving that G is open in $[0,1]$. Let $t_0 \in G$ and $x_0 \in U$ with $x_0 = H(x_0,t_0)$. There exists $r>0$ such that $B_\omega(x_0,r) \subseteq U$ as U is open in X^*_ω. Considering $\varepsilon = \psi(r+\omega_\lambda(x,x_0))>0$, then there exists $\beta(\varepsilon)>0$ such that $|\alpha(t)-\alpha(t_0)|<\varepsilon$ for all $t\in(t_0-\beta(\varepsilon),t_0+\beta(\varepsilon))$ because α is continuous on t_0.

Let $t\in(t_0-\beta(\varepsilon),t_0+\beta(\varepsilon))$, for $x\in\overline{B_\omega(x_0,r)} = \{x\in X^*_\omega | \omega_\lambda(x,x_0)\le r\}$, we obtain

$$\begin{aligned}\omega_\lambda(H(x,t),x_0) =&\omega_\lambda(H(x,t),H(x_0,t_0))\\ \le&\omega_{\frac{\lambda}{2}}(H(x,t),H(x,t_0))+\omega_{\frac{\lambda}{2}}(H(x,t_0),H(x_0,t_0))\\ \le&|\alpha(t)-\alpha(t_0)|+\phi[\omega_{\frac{3\lambda}{2}}(x,x_0)]\\ \le&\varepsilon+\omega_{\frac{3\lambda}{2}}(x,x_0)\\ \le&\varepsilon+\omega_\lambda(x,x_0)\\ =&\psi(r+\omega_\lambda(x,x_0))+\omega_\lambda(x,x_0)\\ =&r+\omega_\lambda(x,x_0)-\phi(r+\omega_\lambda(x,x_0))+\omega_\lambda(x,x_0)\\ \le&r+2\omega_\lambda(x,x_0)-r-\omega_\lambda(x,x_0)\\ =&\omega_\lambda(x,x_0)\\ \le&r\end{aligned}$$

and $H(x,t)\in\overline{B_\omega(x_0,r)}$. Therefore,

$$H(.,t):\overline{B_\omega(x_0,r)}\to\overline{B_\omega(x_0,r)}$$

for every fixed $t\in(t_0-\beta(\varepsilon),t_0+\beta(\varepsilon))$. Since all hypotheses of Theorem 3.6 hold, $H(.,t)$ has a fixed point in V, but it must be in U as (a) holds. So $(t_0-\beta(\varepsilon),t_0+\beta(\varepsilon))\subseteq G$ and thus we conclude that G is open in $[0,1]$.

We now show that G is closed in $[0,1]$. Let $\{t_n\}_{n\in\mathbb{N}^*}$ be a sequence in G where $t_n\to t^*\in[0,1]$ as $n\to+\infty$. Our aim is to show that $t^*\in G$. From the definition of G, there exists $x_n\in U$ with $x_n=H(x_n,t_n)$ for all $n\in\mathbb{N}^*$. Moreover we have

$$\begin{aligned}\omega_\lambda(x_n,x_m) =&\omega_\lambda(H(x_n,t_n),H(x_m,t_m))\\ \le&\omega_{\frac{\lambda}{2}}(H(x_n,t_n,H(x_n,t_m))+\omega_{\frac{\lambda}{2}}(H(x_n,t_m),H(x_m,t_m))\\ \le&|\alpha(t_n)-\alpha(t_m)|+\phi[\omega_{\frac{3\lambda}{2}}(x_n,x_m)]\\ \le&|\alpha(t_n)-\alpha(t_m)|+\phi[\omega_\lambda(x_n,x_m)]\end{aligned}$$

for $m,n\in\mathbb{N}^*$. Last statement gives rise to

$$\psi(\omega_\lambda(x_n,x_m))\le|\alpha(t_n)-\alpha(t_m)|,$$

and so we obtain

$$\omega_\lambda(x_n,x_m)\le\psi^{-1}(|\alpha(t_n)-\alpha(t_m)|)$$

by (d). If we use continuity of ψ^{-1} and α, convergence of $\{t_n\}_{n\in\mathbb{N}^*}$ with $n,m\to+\infty$ in the last inequality, we obtain $\lim\limits_{n,m\to+\infty}\omega_\lambda(x_n,x_m)=0$. It means that $\{x_n\}_{n\in\mathbb{N}^*}$ is ω-Cauchy sequence in X^*_ω. As X^*_ω is ω-complete, there exists $x^*\in V$ such that

$$\lim_{n\to+\infty}\omega_\lambda(x^*,x_n)=0.$$

Letting $n\to+\infty$ in the following inequality,

$$\begin{aligned}\omega_\lambda(x_n,H(x^*,t^*)) =&\omega_\lambda(H(x_n,t_n),H(x^*,t^*))\\ \le&\omega_{\frac{\lambda}{2}}(H(x_n,t_n,H(x_n,t^*))+\omega_{\frac{\lambda}{2}}(H(x_n,t^*),H(x^*,t^*))\\ \le&|\alpha(t_n)-\alpha(t^*)|+\phi[\omega_{\frac{3\lambda}{2}}(x_n,x^*)],\end{aligned}$$

we find $\lim_{n\to+\infty} \omega_\lambda(x_n, H(x^*, t^*)) = 0$ and hence

$$\omega_\lambda(x^*, H(x^*, t^*)) = \lim_{n\to+\infty} \omega_\lambda(x_n, H(x^*, t^*)) = 0.$$

It implies that $x^* = H(x^*, t^*)$. Since (a) holds, we have $x^* \in U$. Thus $t^* \in G$ and G is closed in $[0, 1]$.
($\Leftarrow$) It can be shown similarly same argument in above. □

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