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Fixed point results for generalized multi-valued contractions

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Abstract

Javahernia *et al.* [Fixed Point Theory and Applications 2014, 2014:195] introduced the concept of generalized Mizoguchi-Takahashi type contractions and established some common fixed point results for such contractions. In this paper, we define the notion of generalized α_*- Mizoguchi-Takahashi type contractions and obtain some new fixed point results which generalize various results existing in literature. An example is included to show that our results are genuine generalization of the corresponding known results.

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1. Introduction and Preliminaries

Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, we denote $d(x, A) = \inf\{d(x, y) : y \in A\}$. Let us denote by $N(X)$, the class of all nonempty subsets of X, $CL(X)$, the class of all nonempty closed subsets of X, $CB(X)$, the class of all nonempty closed and bounded subsets of X and $K(X)$, the class of all compact subsets of X. Let H be the Hausdorff-Pompeiu metric induced by metric d on X, that is,

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}$$

for every $A, B \in CB(X)$. Let $S : X \to CL(X)$ be a multivalued mapping. A point $q \in X$ is said to be a fixed point of S if $q \in Sq$. If, for $x_0 \in X$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $x_n \in Sx_{n-1}$, then

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the orbit of S is defined as $O(S, x_0) = \{x_0, x_1, x_2, ...\}$. A mapping $g : X \to \mathbb{R}$ is said to be S-orbitally lower semi-continuous if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $O(S, x_0)$ and $x_n \to v$ implies $g(v) \le \lim_n \inf g(x_n)$.

Nadler [16] extended the Banach contraction principle to multivalued mappings in the following way.

Theorem 1.1 ([16]). *Let (X, d) be a complete metric space and $S : X \to CB(X)$ be a multivalued mapping such that for all x , $y \in X$,*

$$H(S(x), S(y)) \le kd(x, y) \tag{1.1}$$

where $0 \le k < 1$. Then S has a fixed point.

Reich [18], established the following fixed point theorem for the case of multivalued mappings with compact range.

Theorem 1.2 ([18]). *Let (X, d) be a complete metric space and $\varphi : [0, \infty) \to [0, 1)$ be such that*

$$\lim_{v\to u^+} \sup \varphi(v) < 1$$

for each $u \in [0, \infty)$. If $S : X \to K(X)$ is a multivalued mapping satisfying

$$H(S(x), S(y)) \le \varphi(d(x, y))d(x, y) \tag{1.2}$$

for all $x, y \in X$, then S has a fixed point.

An open problem posed by Reich [18] asks wether the above theorem holds for mapping $S : X \to CB(X)$. Mizoguchi and Takahashi [15] proved the following famous result as a generalization of Nadler's fixed point theorem [16].

Theorem 1.3 ([15]). . *Let (X, d) be a complete metric space and $S : X \to CB(X)$ be a multivalued mapping. Assume that*

$$H(S(x), S(y)) \le \varphi(d(x, y))d(x, y) \tag{1.3}$$

for all $x, y \in X$, where $\varphi : [0, \infty) \to [0, 1)$ is such that

$$\lim_{v\to u^+} \sup \varphi(v) < 1$$

for each $u \in [0, \infty)$. Then S has a fixed point.

The above function $\varphi : [0, \infty) \to [0, 1)$ of Mizoguchi–Takahashi is named as MT-function. As in [15], we denote by Ω the set of all functions $\varphi : [0, \infty) \to [0, 1)$.

Kamran [12], generalized Mizoguchi and Takahashi's theorem in the following way.

Theorem 1.4 ([12]). *Let (X, d) be a complete metric space, $\varphi : [0, \infty) \to [0, 1)$ be an MT- function and $S : X \to CL(X)$ a multivalued mapping. Assume that*

$$d(y, S(y)) \le \varphi(d(x, y))d(x, y) \tag{1.4}$$

for each $x \in X$ and $y \in Sx$ where $\varphi \in \Omega$. Then

(i) *there exists an orbit $\{x_n\}$ of S and $w \in X$ such that $\lim_{n\to\infty} x_n = w$;*

(ii) *w is fixed point of S if and only if the function $g(x) = d(x, S(x))$ is S -orbitally lower semi-continuous at w.*

Asl et al. [2] defined the notion of α_*-admissible mappings as follows:

Definition 1.5. ([2]). Let (X,d) be a metric space, $\alpha : X \times X \to [0,+\infty)$ and $S : X \to CL(X)$ be given. We say that S is α_*-admissible if for all $x \in X$ and $y \in Sx$ with $\alpha(x,y) \geq 1$, we have that $\alpha_*(Sx,Sy) \geq 1$, where $\alpha_*(Sx,Sy) = \inf\{\alpha(a,b) : a \in Sx, b \in Sy\}$.

Kiran et al. [13] generalized Theorem 1.4 of Kamran [12] for α_*-admissible mapping as follows.

Theorem 1.6 ([13]). *Let (X,d) be a complete metric space and $S : X \to CL(X)$ be α_*-admissible such that*

$$\alpha_*(S(x),S(y))d(y,S(y)) \leq \varphi(d(x,y))d(x,y) \tag{1.5}$$

$\forall x \in X$ with $y \in Sx$ and $\varphi \in \Omega$. Suppose that there exist $x_0 \in X$ and $x_1 \in Sx_0$ such that $\alpha(x_0,x_1) \geq 1$. Then

(i) *there exists an orbit $\{x_n\}$ of S and $w \in X$ such that $\lim_{n\to\infty} x_n = w$;*

(ii) *w is fixed point of S if and only if the function $g(x) = d(x,S(x))$ is S -orbitally lower semi-continuous at w.*

Very recently, Javahernia et al. [10] introduced the concept of generalized Mizoguchi–Takahashi function and proved some new common fixed point results.

Definition 1.7. ([10]). A function $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is said to be generalized Mizoguchi–Takahashi function if the following conditions hold:

(a_1) $0 < \beta(u,v) < 1$ for all $u, v > 0$;

(a_2) for any bounded sequence $\{u_n\} \subset (0,+\infty)$ and any non-increasing sequence $\{v_n\} \subset (0,+\infty)$, we have

$$\lim_{n\to\infty} \sup \beta(u_n, v_n) < 1.$$

Consistent with Javahernia et al. [10] , we denote by Λ the set of all functions $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying the conditions $(a_1) - (a_2)$.

They gave the following example of a generalized Mizoguchi–Takahashi function.

Example 1.8. Let $m(u) = \frac{\ln(u+10)}{u+9}$ for all $u > -9$. Define

$$\beta(t,s) = \begin{cases} \frac{t}{s^2+1}, & 1 < t < s, \\ m(s), & \text{otherwise.} \end{cases}$$

Then $\beta \in \Lambda$.

For more details, we refer the reader to [1, 3, 4, 5, 6, 7, 8, 9, 11, 14, 17, 19, 20].

In this paper, motivated by Javahernia et al. [10], we establish some new fixed point theorems. Our new results generalize and improve fixed point theorems due to Kiran-Ali-Kamran [13], Mizoguchi-Takahashi [15] and Nadler [16].

The following lemma is crucial for the proofs of our results.

Lemma 1.9 ([12]). *Let (X,d) be a metric space and B nonempty, closed subset of X and $q > 1$. Then, for each $x \in X$ with $d(x,B) > 0$ and $q > 1$, there exists $b \in B$ such that $d(x,b) < qd(x,B)$.*

Throughout this article, $\mathbb{N}, \mathbb{R}^+, \mathbb{R}$ stand for the set of: natural numbers, positive real numbers and real numbers, respectively.

2. Main Results

We start this section with the definition of generalized α_*-Mizoguchi-Takahashi type contraction.

Definition 2.1. Let (X, d) be a metric space. The mapping $S : X \to CL(X)$ is said to be generalized α_*-Mizoguchi-Takahashi type contraction if there exist functions $\alpha : X \times X \to [0, +\infty)$ and $\beta \in \Lambda$ such that

$$\alpha_*(S(x), S(y))d(y, S(y)) \leq \beta(d(y, S(y)), d(x, y))d(x, y) \tag{2.1}$$

for all $x \in X$, with $y \in Sx$.

Here is our main result.

Theorem 2.2. *Let (X, d) be a complete metric space and $S : X \to CL(X)$ be generalized α_*-Mizoguchi-Takahashi type contraction and α_*-admissible. Suppose that there exist $x_0 \in X$ and $x_1 \in Sx_0$ such that $\alpha(x_0, x_1) \geq 1$. Then*

(i) *there exists an orbit $\{x_n\}$ of S and $w \in X$ such that $\lim_{n\to\infty} x_n = w$;*

(ii) *w is fixed point of S if and only if the function $g(x) = d(x, S(x))$ is S -orbitally lower semi-continuous at w.*

Proof. By hypothesis, we have $x_0 \in X$ and $x_1 \in Sx_0$ with $\alpha(x_0, x_1) \geq 1$. As S is α_*-admissible, so we have $\alpha_*(Sx_0, Sx_1) \geq 1$. If $x_0 = x_1$, then x_0 is fixed point of S. Let $x_0 \neq x_1$ i.e $d(x_0, x_1) > 0$. If $x_1 \in Sx_1$, then x_1 is fixed point of S. Assume that $x_1 \notin Sx_1$, that is, $d(x_1, S(x_1)) > 0$. Since $d(x_0, x_1) > 0$ and $d(x_1, S(x_1)) > 0$ so by taking $h = \frac{1}{\sqrt{\beta(d(x_1,S(x_1)),d(x_0,x_1))}} > 1$, it follows by Lemma 1.9, that there exists $x_2 \in Sx_1$ such that

$$\begin{aligned} d(x_1, x_2) &\leq \frac{d(x_1, S(x_1))}{\sqrt{\beta(d(x_1, S(x_1)), d(x_0, x_1))}} \\ &\leq \frac{\alpha_*(S(x_0), S(x_1))d(x_1, S(x_1))}{\sqrt{\beta(d(x_1, S(x_1)), d(x_0, x_1))}}. \end{aligned} \tag{2.2}$$

From (2.1), we have

$$\alpha_*(S(x_0), S(x_1))d(x_1, S(x_1)) \leq \sqrt{\beta(d(x_1, S(x_1)), d(x_0, x_1))}d(x_0, x_1)). \tag{2.3}$$

As S is α_*-admissible, so $\alpha(x_1, x_2) \geq \alpha_*(S(x_0), S(x_1)) \geq 1$ implies $\alpha_*(S(x_1), S(x_2)) \geq 1$. If $x_1 = x_2$, then x_1 is fixed point of S. Let $x_1 \neq x_2$ i.e $d(x_1, x_2) > 0$. If $x_2 \in S(x_2)$, then x_2 is fixed point of S. Assume that $x_2 \notin Sx_2$, that is, $d(x_2, S(x_2)) > 0$. Since $d(x_1, x_2) > 0$ and $d(x_2, S(x_2)) > 0$ so by taking $h = \frac{1}{\sqrt{\beta(d(x_2,S(x_2)),d(x_1,x_2))}} > 1$, it follows by Lemma 1.9, that there exists $x_3 \in Sx_2$ such that

$$\begin{aligned} d(x_2, x_3) &\leq \frac{d(x_2, S(x_2))}{\sqrt{\beta(d(x_2, S(x_2)), d(x_1, x_2))}} \\ &\leq \frac{\alpha_*(S(x_1), S(x_2))d(x_2, S(x_2))}{\sqrt{\beta(d(x_2, S(x_2)), d(x_1, x_2))}}. \end{aligned} \tag{2.4}$$

From (2.1), we have

$$\alpha_*(S(x_1), S(x_2))d(x_2, S(x_2)) \leq \sqrt{\beta(d(x_2, S(x_2)), d(x_1, x_2))}d(x_1, x_2)). \tag{2.5}$$

Repeating the above procedure, we obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $x_n \in Sx_{n-1}$, $\alpha_*(Sx_{n-1}, Sx_n) \geq 1$ for each $n \in \mathbb{N}$ and

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{d(x_n, S(x_n))}{\sqrt{\beta(d(x_n, S(x_n)), d(x_{n-1}, x_n))}} \\ &\leq \frac{\alpha_*(S(x_{n-1}), S(x_n))d(x_n, S(x_n))}{\sqrt{\beta(d(x_n, S(x_n)), d(x_{n-1}, x_n))}} \end{aligned} \tag{2.6}$$

for all $n = 1, 2, ...$ We have assumed that $x_{n-1} \neq x_n$, otherwise x_{n-1} is a fixed point of S. Also $x_n \notin Sx_n$ for all $n = 1, 2, ...$ From (2.1), we have

$$\alpha_*(S(x_{n-1}), S(x_n))d(x_n, S(x_n)) \leq \sqrt{\beta(d(x_n, S(x_n)), d(x_{n-1}, x_n))}d(x_{n-1}, x_n)) \tag{2.7}$$

for all $n = 1, 2, ...$, which implies that $\{d(x_n, Sx_n)\}_{n\in\mathbb{N}}$ is a bounded sequence. Combining (2.6) and (2.7), we have

$$d(x_n, x_{n+1}) \leq \sqrt{\beta(d(x_n, S(x_n)), d(x_{n-1}, x_n))}d(x_{n-1}, x_n) < d(x_{n-1}, x_n) \tag{2.8}$$

for $n = 1, 2, ...$ It means that $\{d(x_{n-1}, x_n)\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence of positive real numbers. So

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = l. \tag{2.9}$$

By $(a_{2)}$, we have

$$\lim_{n\to\infty} \sup \beta(d(x_n, Sx_n), d(x_{n-1}, x_n)) < 1. \tag{2.10}$$

Now we claim that $l = 0$. Otherwise, by taking limit in (2.8), we get

$$l \leq \sqrt{\lim_{n\to\infty} \sup \beta(d(x_n, Sx_n), d(x_{n-1}, x_n))}l < l$$

which is a contradiction. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0. \tag{2.11}$$

Now we prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
For each $n \in \mathbb{N}$, let $\lambda_n := \sqrt{\beta(d(x_n, Sx_n), d(x_{n-1}, x_n))}$. Then $\lambda_n \in (0, 1)$ for all $n \in \mathbb{N}$. By (2.8), we obtain

$$d(x_n, Sx_n) \leq \alpha_*(S(x_{n-1}), S(x_n))d(x_n, S(x_n)) \leq \beta_n d(x_{n-1}, x_n) \tag{2.12}$$

for all $n \in \mathbb{N}$. By (a_2), we have $\lim_{n\to\infty} \sup \lambda_n < 1$, so there exist $c \in [0, 1)$ and $n_0 \in \mathbb{N}$, such that

$$\lambda_n \leq c, \text{ for all } n \in \mathbb{N} \text{ with } n \geq n_{0.} \tag{2.13}$$

Thus for any $n \geq n_0$, from (2.12) and (2.13), we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \lambda_n d(x_{n-1}, x_n) \\ &\leq \lambda_n \lambda_{n-1} d(x_{n-2}, x_{n-1}) \\ &\;\;\vdots \\ &\leq \lambda_n \lambda_{n-1} \lambda_{n-2} ... \lambda_{n_0} d(x_0, x_1) \\ &\leq c^{n-n_0+1} d(x_0, x_1). \end{aligned}$$

Put $\delta_n = \frac{c^{n-n_0+1}}{1-c} d(x_0, x_1)$ for $n \in \mathbb{N}$ and $n \geq n_0$. For $m, n \in \mathbb{N}$ with $m > n \geq n_0$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\leq c^{n-n_0+1} d(x_0, x_1) + c^{n-n_0+2} d(x_0, x_1) + ... + c^{n-n_0+m} d(x_0, x_1) \\ &\leq c^{n-n_0+1}(1 + c + c^2 + ... + c^{m-1}) d(x_0, x_1) \\ &\leq \delta_n. \end{aligned} \tag{2.14}$$

Since $c \in [0,1)$, $\lim_{n\to\infty} \delta_n = 0$ and hence $\lim_{n\to\infty} \sup\{d(x_n, x_m) : m > n\} = 0$. Thus $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete so there exists $w \in X$ such that $x_n \to w$. Since $x_n \in Sx_{n-1}$, it follows from (2.12) that

$$d(x_n, Sx_n) \le \alpha_*(S(x_{n-1}), S(x_n))d(x_n, Sx_n) \le \beta(d(x_n, Sx_n), d(x_{n-1}, x_n))d(x_{n-1}, x_n). \tag{2.15}$$

Letting $n \to +\infty$ in (2.15), we have

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0. \tag{2.16}$$

Suppose $g(x) = d(x_n, Sx_n)$ is S-orbitally lower continuous at u. Then

$$d(w, Sw) = g(w) \le \lim_{n\to\infty} \inf g(x_n) = \lim_{n\to\infty} \inf d(x_n, Sx_n) = 0.$$

Since Sw is closed, so $w \in Sw$. Conversely, if w is fixed point of S, then

$$g(w) = 0 \le \lim_{n\to\infty} \inf g(x_n).$$

□

The proofs of the following theorems are similar to the proof of Theorem 2.2 and so are omitted.

Theorem 2.3. *Let (X, d) be a complete metric space and $S : X \to CL(X)$ be α_*-admissible such that*

$$\alpha_*(y, S(y))d(y, S(y)) \le \beta(d(y, S(y)), d(x, y))d(x, y) \tag{2.17}$$

for all $x \in X$, with $y \in Sx$ and $\beta \in \Lambda$. Suppose that there exist $x_0 \in X$ and $x_1 \in Sx_0$ such that $\alpha(x_0, x_1) \ge 1$. Then

(i) *there exists an orbit $\{x_n\}$ of S and $w \in X$ such that $\lim_{n\to\infty} x_n = w$;*

(ii) *w is fixed point of S if and only if the function $g(x) = d(x, S(x))$ is S -orbitally lower semi-continuous at w.*

Theorem 2.4. *Let (X, d) be a complete metric space and $S : X \to CL(X)$ be α_*-admissible such that*

$$\alpha(x, y)d(y, S(y)) \le \beta(d(y, S(y)), d(x, y))d(x, y) \tag{2.18}$$

for all $x \in X$, with $y \in Sx$ and $\beta \in \Lambda$. Suppose that there exist $x_0 \in X$ and $x_1 \in Sx_0$ such that $\alpha(x_0, x_1) \ge 1$. Then

(i) there exists an orbit $\{x_n\}$ of S and $w \in X$ such that $\lim_{n\to\infty} x_n = w$;

(ii) w is fixed point of S if and only if the function $g(x) = d(x, S(x))$ is S -orbitally lower semi-continuous at w.

3. Example

In this section, we construct an example which shows that Theorem 2.2 is a proper generalization of Theorem 1.6.

Example 3.1. Let $X = [0, 1]$ and $d : X \times X \to \mathbb{R}$ be the usual metric. Define $S : X \to CL(X)$ by

$$S(x) = \begin{cases} \frac{1}{3}x^2, & \text{for } x \in [0, \frac{6}{11}) \cup (\frac{6}{11}, 1] \\ \{\frac{27}{110}\} & \text{for } x = \frac{6}{11} \end{cases}$$

and $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, \frac{6}{11}) \cup (\frac{6}{11}, 1] \\ 0 & \text{otherwise.} \end{cases}$$

Define $\varphi : [0, \infty) \to [0, 1)$ by

$$\varphi(t) = \begin{cases} \frac{4}{9}x, & \text{for } x \in [0, \frac{3}{10}) \cup (\frac{3}{10}, \frac{1}{3}) \\ \frac{9}{11} & \text{for } x = \frac{3}{10} \\ \frac{1}{3} & \text{for } x \in [\frac{1}{3}, \infty) \end{cases}$$

and $\beta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\beta(u, v) = 1 - \frac{\varphi(v)}{v}$ for all $u, v > 0$. For any bounded sequence $\{u_n\} \subset (0, +\infty)$ and any non-increasing sequence $\{v_n\} \subset (0, +\infty)$, we have

$$\lim_{n\to\infty} \sup \beta(u_n, v_n) = \lim_{n\to\infty} \sup(1 - \frac{\varphi(v_n)}{v_n}) < 1.$$

We show that S satisfies all the hypotheses of our Theorem 2.2 . It is easy to see that the function $g(x) = d(x, S(x))$ is lower semi-continuous. Moreover, for each $x \in [0, \frac{6}{11}) \cup (\frac{6}{11}, 1]$, we have $S(x) = \{\frac{1}{3}x^2\}$ and therefore $y = \frac{1}{3}x^2$.

Now $d(x, y) = d(x, S(x)) = x - \frac{1}{3}x^2$. Further,

$$\begin{aligned} \alpha_*(S(x), S(y))d(y, S(y)) &= d(\frac{1}{3}x^2, \frac{1}{27}x^4) = \frac{1}{3}(x^2 - (\frac{1}{3}x^2)^2) = \frac{1}{3}(x + \frac{1}{3}x^2)(x - \frac{1}{3}x^2) \\ &\le \frac{5}{9}d(x, y) \\ &= \beta(d(y, S(y)), d(x, y))d(x, y). \end{aligned}$$

Take $x = \frac{6}{11}$. Then we have have $S(x) = \{\frac{27}{110}\}$ and $d(x, y) = d(x, S(x)) = \frac{3}{10}$. The contractive condition (2.1) is satisfied trivially. Now we show that a given map S does not satisfy hypotheses of Theorem 1.6 of Kiran et al. [13].

For $x = 1$, we have we have $S(x) = \{\frac{1}{3}\}$, $y = \frac{1}{3}$ and $S(y) = \{\frac{1}{27}\}$. Then $d(x, y) = \frac{2}{3}$ and $d(y, S(y)) = \frac{8}{27}$. Consequently,

$$\alpha_*(S(x), S(y))d(y, S(y)) = \frac{4}{9}.\frac{2}{3} > \frac{1}{3}.\frac{2}{3} = \varphi(d(x, y))d(x, y).$$

Therefore, for $x = 1$, the inequality (1.5) in Theorem 1.6 is not satisfied.

4. Consequences

Remark 4.1. Theorem 2.2 improves Theorem 1.6, since S may take values in $CL(X)$ and $d(y, S(y)) \le H(S(x), S(y))$ for $y \in S(x)$.

Corollary 4.2. *Let (X, d) be a complete metric space and $S : X \to CL(X)$ be α_*-admissible such that*

$$\alpha_*(S(x), S(y))H(S(x), S(y)) \le \beta(d(y, S(y)), d(x, y))d(x, y)$$

for each $x \in X$ and $y \in Sx$ where $\beta \in \Lambda$. Suppose that there exist $x_0 \in X$ and $x_1 \in Sx_0$ such that $\alpha(x_0, x_1) \ge 1$. Then

(i) *there exists an orbit $\{x_n\}$ of S and $w \in X$ such that $\lim_{n\to\infty} x_n = w$;*

(ii) *w is fixed point of S if and only if the function $g(x) = d(x, S(x))$ is S -orbitally lower semi-continuous at w.*

Corollary 4.3. *Theorem 1.6 follows from Theorem 2.2 by putting $\beta(u, v) = \varphi(v)$.*

Remark 4.4. Taking $\beta(u, v) = \varphi(v)$ in Theorem 2.3 and Theorem 2.4, we obtain Theorem 2.6 and Theorem 2.7 in [13], respectively.

Corollary 4.5. *Let (X,d) be a complete metric space and $S : X \to CL(X)$ satisfies*

$$d(y,S(y)) \leq \beta(d(y,S(y)), d(x,y))d(x,y)$$

for each $x \in X$ and $y \in Sx$ where $\beta \in \Lambda$. Then

(i) *there exists an orbit $\{x_n\}$ of S and $w \in X$ such that $\lim_{n\to\infty} x_n = w$;*

(ii) *w is fixed point of S if and only if the function $g(x) = d(x,S(x))$ is S -orbitally lower semi-continuous at w.*

Proof. Define $\alpha : X \times X \to [0,+\infty)$ by $\alpha(x,y) = 1$ for each $x, y \in X$. Then the proof follows from Theorem 2.2 . □

Remark 4.6. Taking $\beta(u,v) = \varphi(v)$ in Corollary 4.5, we can get Theorem 1.6 which is Theorem 2.1 in [12].

Corollary 4.7. *Let (X,d) be a complete metric space and $S : X \to CL(X)$ satisfies*

$$d(y,S(y)) \leq \mu d(x,y)$$

for all $x \in X$, with $y \in Sx$ where $\mu \in [0,1)$. Then

(i) *there exists an orbit $\{x_n\}$ of S and $w \in X$ such that $\lim_{n\to\infty} x_n = w$;*

(ii) *w is fixed point of S if and only if the function $g(x) = d(x,S(x))$ is S -orbitally lower semi-continuous at w.*

Proof. Take $\beta(u,v) = \mu$ and apply Corollary 4.5. □

Corollary 4.8. *Let (X,d) be a complete metric space and $S : X \to CL(X)$ satisfies*

$$d(y,S(y)) \leq \varphi(d(x,y))$$

for each $x \in X$ and $y \in Sx$ where $\varphi : [0,\infty) \to [0,1)$ is a function such that $\varphi(v) < v$ and $\lim_{v\to u^+} \sup \frac{\varphi(v)}{v} < 1$. Then

(i) *there exists an orbit $\{x_n\}$ of S and $w \in X$ such that $\lim_{n\to\infty} x_n = w$;*

(ii) *w is fixed point of S if and only if the function $g(x) = d(x,S(x))$ is S -orbitally lower semi-continuous at w.*

Proof. Take $\beta(u,v) = \frac{\varphi(v)}{v}$ and apply Corollary 4.5. □

Javahernia et al. [10] also introduced the concept of weak l.s.c. in the following way.

Definition 4.9. A function $\phi : [0,\infty) \to [0,\infty)$ is said to be weak l.s.c. function if for each bounded sequence $\{u_n\} \subset (0,+\infty)$, we have

$$\lim_{n\to\infty} \inf \phi(u_n) > 0.$$

Consistent with Javahernia et al. [10] , we denote by $\digamma$, the set of all functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the above condition.

Theorem 4.10. *Let (X, d) be a complete metric space and $S : X \to CL(X)$ be α_*-admissible such that*

$$\alpha_*(S(x), S(y))d(y, S(y)) \leq d(x, y) - \phi(d(x, y))$$

for each $x \in X$ and $y \in Sx$ where $\phi : [0, \infty) \to [0, \infty)$ is such that $\phi(0) = 0$, $\phi(v) < v$ and $\phi \in F$. Suppose that there exist $x_0 \in X$ and $x_1 \in Sx_0$ such that $\alpha(x_0, x_1) \geq 1$. Then

(i) *there exists an orbit $\{x_n\}$ of S and $w \in X$ such that $\lim_{n\to\infty} x_n = w$;*

(ii) *w is fixed point of S if and only if the function $g(x) = d(x, S(x))$ is S -orbitally lower semi-continuous at w.*

Proof. Define $\beta(u, v) = 1 - \frac{\phi(u)}{u}$ for all $u, v > 0$. Since for each bounded sequence $\{u_n\} \subset (0, +\infty)$, we have $\lim_{n\to\infty} \inf \phi(u_n) > 0$. So $\lim_{n\to\infty} \inf \frac{\phi(u_n)}{u_n} > 0$. Thus

$$\lim_{n\to\infty} \sup(1 - \frac{\phi(u_n)}{u_n}) = 1 - \lim_{n\to\infty} \inf \frac{\phi(u_n)}{u_n} < 0.$$

This shows that $\beta \in \Lambda$. Also

$$\alpha_*(S(x), S(y))d(y, S(y)) \leq \beta(d(y, S(y)), d(x, y))d(x, y).$$

Thus by Theorem 2.2 , w is fixed point of S. □

Corollary 4.11. *Let (X, d) be a complete metric space and $S : X \to CL(X)$ be such that*

$$d(y, S(y)) \leq d(x, y) - \phi(d(x, y))$$

for each $x \in X$ and $y \in Sx$, where $\phi : [0, \infty) \to [0, \infty)$ is such that $\phi(0) = 0$, $\phi(v) < v$ and $\phi \in F$. Then

(i) *there exists an orbit $\{x_n\}$ of S and $w \in X$ such that $\lim_{n\to\infty} x_n = w$;*

(ii) *w is fixed point of S if and only if the function $g(x) = d(x, S(x))$ is S -orbitally lower semi-continuous at w.*

Proof. Define $\alpha : X \times X \to [0, +\infty)$ by $\alpha(x, y) = 1$ for each $x, y \in X$. Then the proof follows from Theorem 4.10. □

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