Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 895–905
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Banach fixed point theorem from the viewpoint of digital topology

Sang-Eon Han

Department of Mathematics Education, Institute of Pure and Applied Mathematics, Chonbuk National University, Jeonju-City Jeonbuk, 54896, Republic of Korea.

Communicated by Y. J. Cho

Abstract

The present paper studies the Banach contraction principle for digital metric spaces such as digital intervals, simple closed k-curves, simple closed 18-surfaces and so forth. Furthermore, we prove that a digital metric space is complete, which can strongly contribute to the study of Banach fixed point theorem for digital metric spaces. Although *Ege, et al.* [O. Ege, I. Karaca, J. Nonlinear Sci. Appl., **8** (2015), 237–245] studied "Banach fixed point theorem for digital images", the present paper makes many notions and assertions of the above mentioned paper refined and improved.

Keywords: Banach fixed point theorem, digital contraction map, Banach contraction principle, digital image, digital continuity, digital topology.
2010 MSC: 47H10, 54E35, 68U10.

1. Introduction

Fixed point theory plays an important role in many areas of mathematics such as mathematical analysis, general topology and functional analysis, which leads to lots of applications in mathematics and applied mathematics such as computer science, engineering, game theory, fuzzy theory, image processing and so forth [5, 14, 15]. In metric spaces, this theory begins with the Banach fixed-point theorem [1] (also known as the contraction mapping theorem or contraction mapping principle); it guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces [23, 24], and provides a constructive method of finding those fixed points. Hence the Banach fixed point theorem becomes an essential tool for solution of some problems in mathematics and engineering. Motivated by the Banach fixed point theorem [1], the other tools

Email address: `sehan@jbnu.ac.kr` (Sang-Eon Han)

Received 2015-07-16

[5] such as a generalized Banach contraction principle [19, 27] were established for studying metric spaces [18, 23, 24].
Digital topology has a focus on studying digital topological properties of nD digital images [25, 26], which have contributed to some areas of computer sciences such as image processing, computer graphics, mathematical morphology and so forth [2, 7, 20]. To be specific, the notion of digital continuity was developed by Rosenfeld [26] for studying 2D and 3D digital images. Then the concept was extended into the study of nD digital images, $n \in \mathbf{N}$ [7]. Up to now, several types of digital continuities have been developed [11] for studying digital images from the viewpoints of Khalimsky topology and Marcus Wyse topology and so forth [28, 29]. Based on this approach, the paper [26] firstly studied the almost fixed point property of digital images.
To establish digital versions of the Banach fixed point theorem [1] and a generalized Banach contraction principle [19, 27], the recent paper [6] studied "Banach fixed point theorem for digital images" and its applications. Thus some important results were obtained.

Before referring to the works, first of all, we need to recall the following: we say that a digital image (X, k) has the fixed point property [26] if every k-continuous map $f : (X, k) \to (X, k)$ has a fixed point $x \in X$, *i.e.* $f(x) = x$.
Based on this approach, the paper [6] mainly dealt with the following topics.

(1) Digital versions of both a classical Cauchy sequence and a limit point of a sequence in a metric space;
(2) a digital version of the Banach fixed point theorem (see Proposition 3.6 and Theorem 3.7 and Example 3.8 of the paper [6]);
(3) a digital version of a generalization of the Banach contraction principle (see Theorem 3.9 of the paper [6]);
(4) miscellaneous problems and applications related to the above issues.

To work these out in details, *Ege et al.* [6] used the notion of a digital metric space, a digital version of the Banach contraction principle and some properties from digital topology [2, 3, 8, 9, 10, 20, 21, 22, 26]. However, since the paper [6] has many things to be refined and corrected, the present paper makes these approaches from (1) to (4) above improved.

The rest of the paper proceeds as follows: Section 2 provides some basic notions from digital topology. Section 3 makes several notions proposed in the paper [6] refined and improved. More precisely, digital versions of both a Cauchy sequence and a limit of a sequence in a metric space are refined and improved. Besides we study the property of the completeness of digital metric spaces. Section 4 investigates some properties of the Banach contraction principle from the viewpoint of digital topology. To be specific, although a digital interval denoted by $[0, l]_{\mathbf{Z}}$ does not have the fixed point property [16, 26], we prove that any digital contraction map on $([0, l]_{\mathbf{Z}}, d, 2)$ is a constant map. Furthermore, we prove that any self digital contraction map on a simple closed k-curve with l elements in $\mathbf{Z}^n$, denoted by $SC_k^{n,l}$, has a fixed point on $SC_k^{n,l}$, it need not be a constant map. Besides, we prove that any digital contraction map on a simple closed 18-surface, denoted by MSS'_{18}, also has a fixed point on MSS'_{18}. Owing to these works, the present paper improves the paper [6]. Section 5 concludes the paper with a concluding remark.

2. Preliminaries

To study Banach fixed point theorem for digital images, we need to recall some basic notions from digital topology such as digital k-connectivity of n-dimensional integer grids, a digital k-neighborhood, digital continuity and so forth [7, 20, 25, 26].

Let $\mathbf{N}$ and $\mathbf{R}$ represent the sets of natural numbers and real numbers, respectively. Let $\mathbf{Z}^n, n \in \mathbf{N}$, be the set of points in the Euclidean nD space with integer coordinates.

To study nD digital images, for consistency with the nomenclature "4-adjacent" and "8-adjacent" (resp. "6-adjacent", "18-adjacent and "26-adjacent") well established in the context of 2-(resp. 3-)dimensional integer grids [20, 25], we will say that two distinct points $p, q \in \mathbf{Z}^n$ are k-(or $k(t, n)$-)adjacent if they satisfy the following property [7] (see also [10, 11]) as follows:
For a natural number t, $1 \le t \le n$, two distinct points

$$p = (p_1, p_2, \cdots, p_n) \text{ and } q = (q_1, q_2, \cdots, q_n) \in \mathbf{Z}^n \tag{2.1}$$

are $k(t, n)$-(k-, for short)adjacent if at most t of their coordinates differs by ± 1, and all others coincide. Concretely, these $k(t, n)$-adjacency relations of $\mathbf{Z}^n$ are determined according to the two numbers $t, n \in \mathbf{N}$ [7] (see also [13, 14]).

Using the operator of (2.1), we can obtain the k-adjacency relations of $\mathbf{Z}^n$ [7] (see also [13, 14]) as follows:

$$k := k(t, n) = \sum_{i=n-t}^{n-1} 2^{n-i} C_i^n, \tag{2.2}$$

where $C_i^n = \frac{n!}{(n-i)!\ i!}$, in the present paper we shall use the symbol " :=" in order to introduce new notions without mentioning the fact.

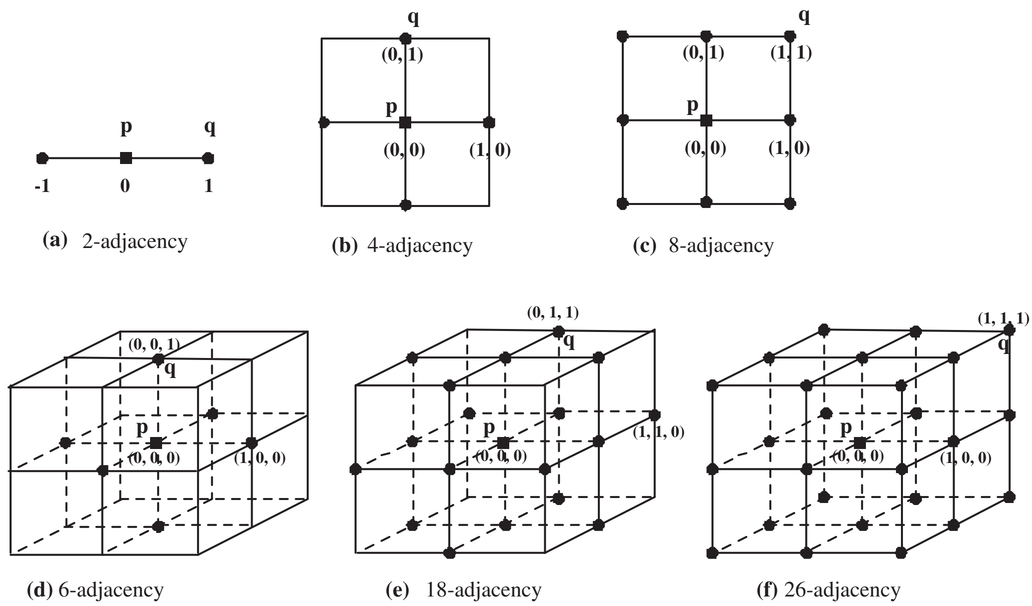

Figure 1: Configuration of the digital k-connectivity of $\mathbf{Z}^n, n \in \{1, 2, 3\}$ [25].

A. Rosenfeld [25] called a set $X \subset \mathbf{Z}^n$ with a k-adjacency a digital image denoted by (X, k). Indeed, to follow a graph theoretical approach of studying nD digital images [8, 17, 26], both the k-adjacency relations of $\mathbf{Z}^n$ (see the property (2.2)) and a digital k-neighborhood have been often used. More precisely, using the k-adjacency relations of $\mathbf{Z}^n$ of (2.2), we say that a digital k-neighborhood of p in $\mathbf{Z}^n$ is the set [25]

$$N_k(p) := \{q \,|\, p \text{ is } k\text{-adjacent to } q\}.$$

Furthermore, we often use the notation [20]

$$N_k^*(p) := N_k(p) \cup \{p\}.$$

For $a, b \in \mathbf{Z}$ with $a \lneq b$, the set $[a, b]_{\mathbf{Z}} = \{n \in \mathbf{Z} \,|\, a \le n \le b\}$ with 2-adjacency is called a digital interval [4, 20].

We say that two subsets (A,k) and (B,k) of (X,k) are k-adjacent to each other if $A \cap B = \emptyset$ and there are points $a \in A$ and $b \in B$ such that a and b are k-adjacent to each other [20]. We say that a set $X \subset \mathbf{Z}^n$ is k-connected if it is not a union of two disjoint non-empty sets that are not k-adjacent to each other [20]. For a digital image (X,k), we say the k-component of $x \in X$ is the largest k-connected subset of (X,k) containing the point x.

For a k-adjacency relation of $\mathbf{Z}^n$, a simple k-path with $l+1$ elements in $\mathbf{Z}^n$ is assumed to be an injective sequence $(x_i)_{i\in[0,l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if either $|\, i-j \,| = 1$ [20]. If $x_0 = x$ and $x_l = y$, then the length of the simple k-path, denoted by $l_k(x,y)$, is the number l. A simple closed k-curve with l elements in $\mathbf{Z}^n$, denoted by $SC_k^{n,l}$ [7, 20] (see Figure 2 (a),(b)), is the simple k-path $(x_i)_{i\in[0,l-1]_{\mathbf{Z}}}$, where x_i and x_j are k-adjacent if and only if $j = i+1 (mod\, l)$ or $i = j+1(mod\, l)$ [20].

For a digital image (X,k), as a generalization of $N_k^*(p)$ the digital k-neighborhood of $x_0 \in X$ with radius ε is defined in X to be the following subset [7] of X.

$$N_k(x_0,\varepsilon) := \{x \in X \,|\, l_k(x_0,x) \leq \varepsilon\} \cup \{x_0\}, \tag{2.3}$$

where $l_k(x_0,x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbf{N}$. Concretely, for $X \subset \mathbf{Z}^n$ we obtain [12]

$$N_k(x,1) = N_k^*(x) \cap X. \tag{2.4}$$

Since the notion of digital continuity is an essential notion in digital topology, the paper [25] established the notion of digital continuity of a map from $f : (X,k_0) \to (Y,k_1)$ by saying that f maps every k_0-connected subset of (X,k_0) into a k_1-connected subset of (Y,k_1). Motivated by this approach, since the notion of a digital k-neighborhood of a point with radius 1 (see the property (2.4)) is very useful in digital topology, the digital continuity of maps between digital images was represented with the following version, which can be substantially used for studying digital spaces X in $\mathbf{Z}^n$.

Proposition 2.1 ([7, 9])**.** *Let (X,k_0) and (Y,k_1) be digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : (X,k_0) \to (Y,k_1)$ is digitally (k_0,k_1)-continuous if and only if for every $x \in X$ $f(N_{k_0}(x,1)) \subset N_{k_1}(f(x),1)$.*

According to Proposition 2.1, we see that the point $y \in N_{k_0}(x,1)$ is mapped into the point $f(y) \in N_{k_1}(f(x),1)$, which implies that for the points x,y which are k_0-adjacent a (k_0,k_1)-continuous map f has the property

$$f(x) = f(y) \text{ or } f(y) \in N_{k_1}(f(x)) \cap Y. \tag{2.5}$$

Hereafter, we will use the term "(k_0,k_1)-continuous" for short instead of "digitally (k_0,k_1)-continuous". In Proposition 2.1 in case $n_0 = n_1$ and $k_0 = k_1 := k$, the map f is called a "k-continuous" map instead of a "(k,k)-continuous" map. Besides, we need to point out that the digital continuity is different from the continuity for studying metric spaces because the digital neighborhood $N_k(x,1)$ of (2.4) is not a smallest open set containing the point x (see Remark 3.2).

Since an nD digital image (X,k) is considered to be a set $X \subset \mathbf{Z}^n$ with one of the k-adjacency relations of (2.2) (or a digital k-graph [8, 17]), in relation to the classification of nD digital images, we use the term a (k_0,k_1)-isomorphism as in [8] (see also [17]) rather than a (k_0,k_1)-homeomorphism as in [2].

Definition 2.2 ([2, 17, 8])**.** Consider two digital images (X,k_0) and (Y,k_1) in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Then a map $h : X \to Y$ is called a (k_0,k_1)-isomorphism if h is a (k_0,k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1,k_0)-continuous.

In Definition 2.2, in case $n_0 = n_1$ and $k_0 = k_1 := k$, we call it a *k-isomorphism* [8, 9]. In addition, we denote by $X \approx_k Y$ a k-isomorphism from X to Y.

In relation to the study of a Banach contraction principle of a closed 18-surface [6], we need to recall basic notions from digital k-surface theory [2, 3, 9, 10, 21, 22]. In a digital image (X,k), a point $x \in X$ is called a *k-corner* if x is k-adjacent to two and only two points y, $z \in X$ such that y and z are k-adjacent to each other [3]. The k-corner x is called *simple* if y, z are not k-corners and if x is the only point k-adjacent to both y, z. X is called a *generalized simple closed k-curve* if what is obtained by removing all simple k-corners of X is a simple closed k-curve [3]. For a k-connected digital image (X, k) in $\mathbf{Z}^3$, we recall $|X|^x = N^*_{26}(x) \cap X$, $N^*_{26}(x) = \{x' \,|\, x \text{ and } x' \text{ are 26-adjacent}\}$ [2, 3]. Thus we restate $|X|^x = N_{26}(x, 1) - \{x\}$ in $\mathbf{Z}^3$. More generally, for a k-connected digital image (X, k) in $\mathbf{Z}^n, n \geq 3$, we can state $|X|^x = N^*_{3^n-1}(x) \cap X$, where $N^*_{3^n-1}(x) = \{x' \,|\, x \text{ and } x' \text{ are } (3^n-1)\text{-adjacent}\}$. In other words, $|X|^x = N_{3^n-1}(x, 1) - \{x\}$ in $\mathbf{Z}^n$.

In the paper [10], a (simple) closed k-surface in $\mathbf{Z}^n$ was studied, where $(k, \bar{k}) = (3^n - 2^n - 1, 2n)$ such as $(18, 6)$ in $\mathbf{Z}^3$, which is the generalization of *Malgouyres' simple closed* 18*-surface* [3, 22]. Also, we recall that for a digital image (X, k), a *simple k-point* [2] is one whose removal does not change the digital topological property of (X, k) [2].

Definition 2.3 ([9])**.** Let (X, k) and $(\bar{X} := \mathbf{Z}^n \setminus X, \bar{k})$ be digital images in $\mathbf{Z}^n, n \geq 3$ and $k \neq \bar{k}$. Then, X is called a *closed k-surface* if it satisfies the following.

(1) In case $(k, \bar{k}) \in \{(k, 2n), (2n, 3^n-1)\}$, where the k-adjacency is taken from (2.2) with $k \notin \{3^n-2^n-1\}$, then

(a) for each point $x \in X$, $|X|^x$ has exactly one k-component k-adjacent to x;
(b) $|\bar{X}|^x$ has exactly two $\bar{k}$-components $\bar{k}$-adjacent to x; we denote by C^{xx} and D^{xx} these two components;
(c) for any point $y \in N_k(x) \cap X$, $N_{\bar{k}}(y) \cap C^{xx} \neq \phi$ and $N_{\bar{k}}(y) \cap D^{xx} \neq \phi$.

Furthermore, if a closed k-surface X does not have a simple k-point, then X is called simple.

(2) In case $(k, \bar{k}) = (3^n - 2^n - 1, 2n)$, then

(a) X is k-connected;
(b) for each point $x \in X$, $|X|^x$ is a generalized simple closed k-curve.

Furthermore, if the image $|X|^x$ is a simple closed k-curve, then the closed k-surface X is called simple.

Example 2.4 ([7, 9, 10])**.** In Figure 2 (c), we see a simple closed 18-surface in $\mathbf{Z}^3$.

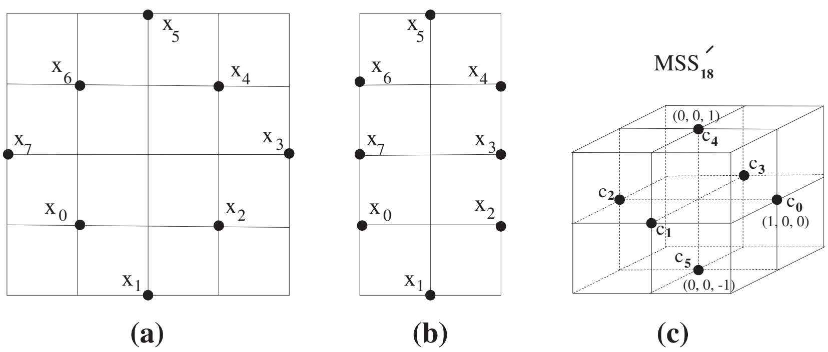

Figure 2: (a), (b): Two simple closed 8-curves $SC_8^{2,8}$[7, 11] (c): Simple closed 18-surface, MSS'_{18} [9, 10].

3. Remarks on the completeness of digital metric spaces

Before studying the Banach fixed point theorem from the viewpoint of digital topology, first of all, we say again that a digital image (X, k) has the fixed point property [26] if every k-continuous map $f : (X, k) \to (X, k)$ has a fixed point $x \in X$, *i.e.* $f(x) = x$. Although the recent paper [6] studied a digital version of the Banach fixed point theorem, we need to make many notions and assertions more simplified and improved.

While the paper [6] studied "Banach fixed point theorem for digital images" on digital metric spaces, there is no precise definition of a digital metric space in the paper. Hence, to delete the ambiguity, we need to define the notion as follows:

Definition 3.1 ([6])**.** We say that (X, d, k) is a digital metric space if $X \subset \mathbf{Z}^n$, (X, d) is a metric space inherited from the metric space $(\mathbf{R}^n, d)$ with the standard Euclidean metric function d on $\mathbf{R}^n$ and (X, k) is a digital image, $k := k(t, n)$ is the k-adjacency of (2.2).

In view of Definition 3.1, we need to recall the following:

Remark 3.2. For a digital metric space (X, d, k), the set $X \subset \mathbf{Z}^n$ has the discrete topological structure induced by the metric topological space (X, T_d) generated by the metric space (or a metric function) $(\mathbf{R}^n, d)$. Hence we need to consider X with one of the k-adjacency relations of (2.2) so that we have a relation among points in $X \subset \mathbf{Z}^n$ in terms of the digital k-connectivity of (2.2). Thus we conclude that a digital image (X, k) can be considered to be a digital graph with k-adjacency (or a digital k-graph) in $\mathbf{Z}^n$ (see [8, 9]).

In relation to the study of a digital version of the Banach fixed point theorem, first of all we need to recall the following:

Even an integer interval $([0, l]_{\mathbf{Z}}, 2)$ does not have the fixed point property [15, 26]. For instance, consider the map $f : [0, 5]_{\mathbf{Z}} \to [0, 5]_{\mathbf{Z}}$ given by $f(x) = 5 - x$, which is a self 2-continuous map on $[0, 5]_{\mathbf{Z}}$. Then it cannot have any fixed point in $[0, 5]_{\mathbf{Z}} := \{0, 1, 2, 3, 4, 5\}$. Indeed, this approach can be extended to some other digital metric spaces. Motivated by the formal Banach fixed point theorem [1], the paper [6] tried to propose its digital version as well as a digital version of the Cauchy sequence, as follows:

Definition 3.3 ([6])**.** We say that a sequence $\{x_n\}$ of points of a digital metric space $(X, d, k := k(t, n))$ is a Cauchy sequence if for all $\varepsilon \gneq 0$, there is $\alpha \in \mathbf{N}$ such that for all $n, m \gneq \alpha$, then $d(x_n, x_m) \lneq \varepsilon$.

This seems to be motivated by the original version of the Cauchy sequence in a metric space. However, a moment's reflection urges that we need to make it more clear or refined from the viewpoint of digital topology. Since the sequence $\{x_n\}$ is considered in the digital metric space $(X, d, k := k(t, n))$, in view of the property (2.1), Definition 2.2 and Remark 3.2, we observe that the Euclidean distance between any two distinct points $p, q \in X$ is greater than or equal to 1 as follows:

Proposition 3.4. *In a digital metric space $(X, d, k := k(t, n))$, consider two points x_i, x_j in a sequence $\{x_n\} \subset X$ such that they are k-adjacent, i.e. $x_i \in N_k(x_j, 1)$ or $x_j \in N_k(x_i, 1)$, and $x_i \neq x_j$. Then they have the Euclidean distance $d(x_i, x_j)$ which is greater than or equal to 1 and at most $\sqrt{t}$ depending on the position of the two points, i.e. $d(x_i, x_j) \in \{\sqrt{l} \,|\, l \in [1, t]_{\mathbf{Z}}\}$.*

Proof. According to the property (2.1), it is clear that any two distinct points $x_i, x_j \in \{x_n\}$ in the digital metric space $(X, d, k := k(t, n))$ has at most t of their coordinates which differs by ± 1 and all others coincide. Thus the standard Euclidean distance $d(x_i, x_j)$ inherited by embedding these points x_i and x_j in $(\mathbf{R}^n, d)$ places between 1 and $\sqrt{t}$ depending on the position of the two points. More precisely, $d(x_i, x_j) \in \{1, \sqrt{2}, \cdots, \sqrt{t}\}$. For instance, in Figure 1 (a) the Euclidean distance of any two points 2-adjacent in $\mathbf{Z}$ is 1. In Figure 1 (b) the Euclidean distance of any two points $4 := k(1, 2)$-adjacent in $\mathbf{Z}^2$ is 1. In Figure 1 (c) the Euclidean distance of any two points $8 := k(2, 2)$-adjacent in $\mathbf{Z}^2$ is either 1 or $\sqrt{2}$ depending on the position of the given two points. More precisely, we can consider the following cases: the points $p := (0, 0)$ and $p_1 := (0, 1)$ and $p_2 := (1, 1)$. It is clear that these points are 8-adjacent to each other. The Euclidean distances of these two points are $d(p, p_1) = 1$ and $d(p, p_2) = \sqrt{2}$. Similarly, in three dimensional case, depending on the situation of the given points, their Euclidean distances are taken among $\{\sqrt{l} \,|\, l \in [1, 3]_{\mathbf{Z}}\}$. For instance, in Figure 1 (f) consider the points $p := (0, 0, 0)$, $q_1 := (0, 0, 1)$, $q_2 := (0, 1, 1)$ and $q_3 := (1, 1, 1)$. It is clear that p is 26- (or $k(3, 3)$-) adjacent to each of the points $q_i, i \in [1, 3]_{\mathbf{Z}}$. Then we obtain the Euclidean distances between two points in such a way: $d(p, q_1) = 1$, $d(p, q_2) = \sqrt{2}$ and $d(p, q_3) = \sqrt{3}$. □

In relation to the notion of Cauchy sequence proposed in Definition 3.3, it requires the following condition "if for all $\varepsilon \gneq 0$, there is $\alpha \in \mathbf{N}$ such that for all $n, m \gneq \alpha$, then $d(x_n, x_m) \lneq \varepsilon$". In view of Proposition 3.4,we may take $\varepsilon = 1$ because two distinct points p, q in a digital metric space is greater than or equal to 1.

Thus we obtain the following:

Proposition 3.5. *We say that a sequence $\{x_n\}$ of points of a digital metric space $(X, d, k := k(t, n))$ is a Cauchy sequence if and only if there is $\alpha \in \mathbf{N}$ such that for all $n, m \gneqq \alpha$, then $d(x_n, x_m) \lneqq 1$.*

In terms of Proposition 3.4, we obtain the following from the digital topological point of view.

Theorem 3.6. *For a digital metric space $(X, d, k := k(t, n))$, if a sequence $\{x_n\} \subset X \subset \mathbf{Z}^n$ is a Cauchy sequence then there is $\alpha \in \mathbf{N}$ such that for all $n, m \gneqq \alpha$, we have $x_n = x_m$.*

Proof. According to Definition 3.1 and Proposition 3.5, if a sequence $\{x_n\}$ of points of a digital metric space (X, d, k) is a Cauchy sequence, then there is $\alpha \in \mathbf{N}$ such that for all $n, m \gneqq \alpha$, then $d(x_n, x_m) \lneqq 1$. Thus, by Proposition 3.4, the elements x_n and x_m should be equal to each other. □

Definition 3.7 ([6])**.** A sequence $\{x_n\}$ of points of a digital metric space $(X, d, k := k(t, n))$ converges to a limit $L \in X$ if for all $\varepsilon \gneqq 0$, there is $\alpha \in \mathbf{N}$ such that for all $n \gneqq \alpha$, then $d(x_n, L) \lneqq \varepsilon$.

By Proposition 3.4, Definition 3.7 can be represented in such a way.

Proposition 3.8. *A sequence $\{x_n\}$ of points of a digital metric space $(X, d, k := k(t, n))$ converges to a limit $L \in X$ if there is $\alpha \in \mathbf{N}$ such that for all $n \gneqq \alpha$, then $d(x_n, L) \lneqq 1$.*

Since the digital metric space (X, d, k) is a kind of digital image with both a k-adjacency and the metric d, by Proposition 3.4, the limit of the sequence $\{x_n\}$ in Definition 3.7 has the following property.

Proposition 3.9. *A sequence $\{x_n\}$ of points of a digital metric space $(X, d, k := k(t, n))$ converges to a limit $L \in X$ if there is $\alpha \in \mathbf{N}$ such that for all $n \gneqq \alpha$, then $x_n = L$, i.e. $x_n = x_{n+1} = x_{n+2} = \cdots = L$.*

Motivated by the notion of completeness of a sequence in a metric space, its digital version was established in [6], as follows:

Definition 3.10 ([6])**.** A digital metric space $(X, d, k := k(t, n))$ is complete if any Cauchy sequence $\{x_n\}$ converges to a point L of (X, d, k).

However, this digital version has its own feature. A sequence in a digital metric space is quite different from that in a metric space. More precisely, according to Definitions 3.1 and 3.7, Theorem 3.6, Propositions 3.8 and 3.9, we obtain the following:

Theorem 3.11. *A digital metric space (X, d, k) is complete, where $k := k(t, n)$.*

Proof. Let us consider any Cauchy sequence $\{x_n\}$ of points of a digital metric space (X, d, k). Then, by Theorem 3.6 and Proposition 3.9, we obtain $\alpha \in \mathbf{N}$ such that for all $n, m \gneqq \alpha$ we have $x_n = x_{n+1} = \cdots = x_m$ in X. □

4. A digital contraction map and the Banach contraction principle from the viewpoint of digital topology

At this moment we need to recall the original version of a contraction map [1] as follows: Let (X, d) be a metric space and f a continuous self-map on (X, d). If there exists $\gamma \in [0, 1)$ such that for all $x, y \in X$ $d(f(x), f(y)) \leq \gamma d(x, y)$, then f is called a contraction map. Unlike the ordinary contraction map, its digital version has its own features. As proposed in Theorem 3.11, although every digital metric space is complete, the digital version of the Banach contraction principle is different from the classical one. Thus, to study a digital version of Banach contraction principle [1], the paper [6] introduced the following:

Definition 4.1 ([6])**.** Let (X, d, k) be a digital metric space and $f : (X, d, k) \to (X, d, k)$ be a k-continuous self-map, $k := k(t, n)$. If there exists $\gamma \in (0, 1)$ such that for all $x, y \in X$ $d(f(x), f(y)) \leq \gamma d(x, y)$, then f is called a digital contraction map.

Indeed, the notion of a digital contraction map is certainly inherited from both the digital continuity in $\mathbf{Z}^n$ and a classical contraction map on the metric space $(\mathbf{Z}^n, d)$. Comparing the original version of a contraction map with its digital version of Definition 4.1, we observe a little difference between $\gamma \in [0, 1)$ and $\gamma \in (0, 1)$. Indeed, in view of Proposition 2.1 and the property (2.5), in Definition 4.1 we may take $\gamma \in [0, 1)$ instead of $\gamma \in (0, 1)$.

Based on Definition 4.1, the authors of [6] obtained the following:

Proposition 4.2 (Proposition 3.6 of [6])**.** *Every digital contraction map is digitally continuous.*

Namely, we can restate Proposition 4.2 more precisely, as follows: Every digital contraction map $f : (X, d, k) \to (X, d, k)$ is k-continuous.

Theorem 4.3 (Banach contraction principle: Theorem 3.7 of [6])**.** *Let (X, d, k) be a complete digital metric space which has a usual Euclidean metric in $\mathbf{Z}^n$. Let $f : (X, d, k) \to (X, d, k)$ be a digital contraction map, where $k := k(t, n)$. Then f has a unique fixed point.*

The assertions of Proposition 4.2 and Theorem 4.3 are valid. However, by Theorem 3.11 since every digital metric space is proved complete, the hypothesis of "complete" in Theorem 4.3 is redundant.
In addition, to guarantee Theorem 4.3, the authors of [6] used the following example. However, this approach is incorrect.

Example 4.4 (Example 3.8 of [6])**.** Let $X := [0, 2]_{\mathbf{Z}}$ be a digital interval with 2-adjacency. Consider the map $f : X \to X$ defined by $f(x) = \frac{x}{2}$. It is clear that f has a fixed point.

A moment's reflection shows that it is easy to see that Example 4.4 is incorrect, as follows:

Remark 4.5. In Example 4.4, we see that the given map $f(x) = \frac{x}{2}$ cannot be a self digital contraction map on $[0, 2]_{\mathbf{Z}}$. More precisely, we see that the given map $f : X := \{0, 1, 2\} \to X := \{0, 1, 2\}$ defined by $f(x) = \frac{x}{2}$ cannot be a map because $f(X)$ is the set $\{f(0) = 0, f(1) = \frac{1}{2}, f(2) = 1\}$ which is not even a subset of X.

For a digital contraction map f on a digital metric space (X, d, k), while $Im(f)$ need not be a singleton (see Figure 3), a digital interval has the following property.

Corollary 4.6. (Banach contraction principle for a digital interval $([0, l]_{\mathbf{Z}}, 2)$): *Any digital contraction map on $([0, l]_{\mathbf{Z}}, 2)$) is a constant map, which implies that f has a fixed point.*

Proof. Let $f : ([0, l]_{\mathbf{Z}}, 2) \to ([0, l]_{\mathbf{Z}}, 2)$ be a digital contraction map. Take any point $x_0 \in [0, l]_{\mathbf{Z}}$. Assume $f(x_0) := t_0 \in [0, l]_{\mathbf{Z}}$. Owing to the hypothesis of the digital contraction of f, for the points $x \in N_2(x_0, 1)$, $f(x)$ should be mapped into the point t_0 and further, another points $x' \in N_2(x, 1)$ have the image $f(x') = t_0$. Consecutively, by induction, we see that the image $f([0, l]_{\mathbf{Z}})$ should be the singleton $\{t_0\}$, which completes the proof. □

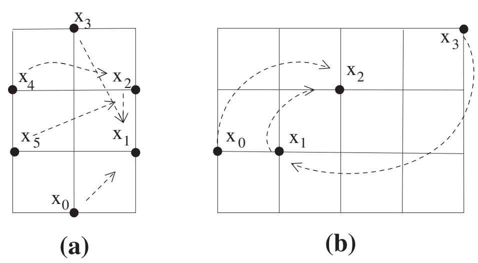

Figure 3: Configuration of a fixed point of a digital contraction map on $SC_8^{2,6}$ or the digital metric space $(X, d, 8)$ which is not 8-connected.

Rosenfeld [26] was first come up with the fixed point theorem for digital images (X,k) in a graph theoretical approach. Finally, he proved that (see Theorems 3.3 and 4.1 of [26], for more details, see [14, 15]) a digital image (X,k) with $|\,X\,| \geq 2$ does not have the fixed point property. This means that only a singleton has the fixed point property in digital topology in a graph theoretical approach. In particular, he intensively studied this property with $k = 2n$. Thus we obtain the following (see Theorem 4.7 (1) below).

Theorem 4.7. *Let us consider a digital metric space (X,d,k) in $\mathbf{Z}^n$ and (X,k) be k-connected, $k := k(t,n)$.*
(1) If $k = 2n$, then a digital contraction map $f : (X,d,2n) \to (X,d,2n)$ is a constant map.
(2) If $k := k(t,n) \neq 2n$, then a digital contraction map $f : (X,d,k) \to (X,d,k)$ need not be a constant map, where $|\,X\,| \geq 3$.

Proof. (1) Let $f : (X,d,2n) \to (X,d,2n)$ be a digital contraction map. Take any point $x_0 \in X$ and put $f(x_0) = x'_0$. First, consider a point $x_1 \in N_{2n}(x_0,1)$ such that $x_1 \neq x_0$, then owing to the digital contraction of f, for $\lambda \in (0,1)$ we obtain

$$d(f(x_1), f(x_0)) \leq \lambda d(x_1, x_0) = \lambda,$$

because $d(x_1,x_0) = 1$ (see Proposition 3.4). By Proposition 3.4, we have $d(f((x_1), f(x_0)) = 0$, which implies $f(x_1) = f(x_0)$.
Second, consider a point $x_2 \in N_{2n}(x_1,1)$ such that $x_2 \notin \{x_0,x_1\}$, then owing to the digital contraction of f, for $\lambda \in (0,1)$ we obtain

$$d(f((x_2), f(x_1)) \leq \lambda d(x_2, x_1) = \lambda,$$

because $d(x_2,x_1) = 1$. By Proposition 3.4, we obtain $d(f(x_2), f(x_1)) = 0$, which implies $f(x_2) = f(x_1)$. Consecutively, by induction, we see that the image $f(X)$ should be the singleton $\{t_0\}$, which completes the proof.

(2) To guarantee the assertion, we may pose two examples:
(Case 1: k-connected case) Let us consider the digital contraction map f on $SC_8^{2,6} := (x_i)_{i\in[0,5]_{\mathbf{Z}}}$ in Figure 3 (a) *e.g.* $f : SC_8^{2,6} \to SC_8^{2,6}$ given by $f(\{x_0,x_1,x_2,x_3\}) = \{x_1\}, f(\{x_4,x_5\}) = \{x_2\}$. Then it is clear that the digital contraction map f has the fixed point x_1 and $|\,Im(f)\,| = 2$.

(Case 2: Non-k-connected case) Let us consider the digital contraction map g on $(X := \{x_i\,|\,i \in [0,3]_{\mathbf{Z}}\}, d, 8)$ in Figure 3 (b), *e.g.* $f : (X,d,8) \to (X,d,8)$ given by $g(\{x_0,x_1\}) = \{x_2\}, f(\{x_3\}) = \{x_1\}$. Then it is clear that the contraction map g has the fixed point x_2 and $|\,Im(f)\,| = 2$. □

Let us apply these works to digital geometry, as follows:

Corollary 4.8. *Any digital contraction map $f : SC_{2n}^{n,l} \to SC_{2n}^{n,l}$ is a constant map, which implies that f has a fixed point.*

By the method similar to the proof of Theorem 4.7 (1), we obtain the following property of the minimal simple closed 18-surface (see Figure 2 (c)):

Corollary 4.9. *Any digital contraction map $f : MSS'_{18} \to MSS'_{18}$ is a constant map, which implies that f has a fixed point.*

Proof. Let us consider a digital contraction map f on $MSS'_{18} := \{c_i\,|\,i \in [0,5]_{\mathbf{Z}}\}$ (see Figure 2 (c)). Take any point in MSS'_{18}, for convenience $c_0 \in MSS'_{18}$ and put $f(c_0) := c' \in MSS'_{18}$. First, consider the point $c_i \in N_{18}(c_0,1), i \in \{1,3,4,5\}$, then owing to the digital contraction of f, for $\lambda \in (0,1)$ we obtain

$$d(f(c_i), f(c_0)) \leq \lambda d(c_i, c_0) = \lambda\sqrt{2},$$

because $d(c_i,c_0) = \sqrt{2}$ (see Proposition 3.4). Hence $d(f(c_i), f(c_0)) \lneq \sqrt{2}$. Owing to the property of MSS'_{18}, we have $d(f(c_i), f(c_0)) = 0$, which implies $f(c_i) = f(c_0) = c'$.
Consecutively, by induction, we see that the image $f(MSS'_{18})$ should be the singleton $\{c'\}$, which completes the proof. □

5. Further remarks

We have studied a digital version of the Banach contraction principle. Making many notions in the paper [6] refined, we obtained digital versions of both a Cauchy sequence and a limit of a sequence in a digital metric space. In particular, proving that every digital metric space is complete, we can make many assertions in [6] refined and improved, as follows:

(1) According to Theorem 3.11 of the present paper, since every digital metric space is proved to be complete, the term "complete digital metric space" in Theorem 3.7 and 3.9 of [6] should be replaced into "digital metric space". Hence the proofs of the theorems can be extremely shortened.
(2) In Definitions 3.1 and 3.2 of [6], the number ε can be replaced by 1.
(3) In Definition 3.4 of [6], the authors defined the notion of "right-continuous" in the digital image (X, k) as follows: Let (X, k) be any digital image. A function $f : (X, k) \to (X, k)$ is called right-continuous if $f(a) = lim_{x \to a^+} f(x)$, where $a \in X$ (see Definition 3.4 of the paper [6]). But this approach can be reconsidered because in the set $X \subset \mathbf{Z}^n$ such a kind of approach using "$lim_{x \to a^+}$" cannot be hold.
(4) In Definition 3.5 of [6], we may take $\gamma \in [0, 1)$ instead of $\gamma \in (0, 1)$.

Using these new digital versions of several notions relevant to a Cauchy sequence and the notion of "completeness" and a limit of a sequence in a digital metric space, we can further study digital metric spaces from the viewpoints of digital k-curve and digital k-surface theory.

Acknowledgements:

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2013R1A1A4A01007577). This paper was supported by research funds of Chonbuk National University in 2015.

References

[1] S. Banach, *Sur les operations dans les ensembles abstraits et leurs applications aux equations integrales*, Fund. Math., **3** (1922), 133–181. 1, 3, 4
[2] G. Bertrand, *Simple points, topological numbers and geodesic neighborhoods in cubic grids*, Pattern Recognition Letters, **15** (1994), 1003–1011. 1, 2, 2.2, 2
[3] G. Bertrand, M. Malgouyres, *Some topological properties of discrete surfaces in Z3*, J. Math. Imaging. Vision., **11** (1999), 207–221.1, 2
[4] L. Boxer, *Digitally continuous functions*, Pattern Recognition Letters, **15** (1994), 833–839. 2
[5] R. Brown, *The Lefschetz Fixed Point Theorem* Scott, Foresman and Co., (1971). 1
[6] O. Ege, I. Karaca, *Banach fixed point theorem for digital images*, J. Nonlinear. Sci. Appl., **8** (2015), 237–245.1, 2, 3, 3.1, 3, 3.3, 3.7, 3, 3.10, 4, 4.1, 4, 4.2, 4.3, 4, 4.4, 5
[7] S. E. Han, Non-product property of the digital fundamental group, Inform. Sci., **171** (2005), 73–91. 1, 2, 2, 2, 2.1, 2.4, 2
[8] S. E. Han, *On the simplicial complex stemmed from a digital graph*, Honam Math. J., **27** (2005), 115–129. 1, 2, 2, 2.2, 2, 3.2
[9] S. E. Han, *Connected sum of digital closed surfaces*, Inform. Sci., **176** (2006), 332–348. 1, 2.1, 2, 2.3, 2.4, 2, 3.2
[10] S. E. Han, *Minimal simple closed* 18*-surfaces and a topological preservation of* 3*D surfaces*, Inform. Sci., **176** (2006), 120–134. 1, 2, 2, 2.4, 2
[11] S. E. Han, *Continuities and homeomorphisms in computer topology and their applications*, J. Korean Math. Soc., **45** (2008), 923–952. 1, 2, 2
[12] S. E. Han, *Equivalent* (k_0, k_1)*-covering and generalized digital lifting*, Inform. Sci., **178** (2008), 550–561. 2
[13] S. E. Han, *The k-homotopic thinning and a torus-like digital image in* $\mathbf{Z}^n$, J. Math. Imaging. Vision., **31** (2008), 1–16. 2
[14] S. E. Han, *KD-*(k_0, k_1)*-homotopy equivalence and its applications*, J. Korean Math. Soc., **47** (2010), 1031–1054. 1, 2, 4
[15] S. E. Han, *Digital version of the fixed point theory*, Proceedings of 11th ICFPTA (Abstracts) (2015). 1, 3, 4
[16] S. E. Han, *Remarks on the Lefschetz fixed point theorem for digital images*, (submitted). 1

[17] S. E. Han, B. G. Park, *Digital graph* (k_0, k_1)*-isomorphism and its applications*, atlas-conference, (2003). 2, 2, 2.2

[18] S. Jain, S. Jain, L. B. Jain, *On Banach contraction principle in a cone metric space*, J. Nonlinear Sci. Appl., **5** (2012), 252–258. 1

[19] M. Jleli, B. Samet, *A new generalization of the Banach contraction principle*, J. Inequal. Appl., **2014** (2014), 8 pages. 1

[20] T. Y. Kong, A. Rosenfeld, *Topological Algorithms for the Digital Image Processing*, Elsevier Sci., Amsterdam, (1996). 1, 2, 2

[21] R. Malgouyres, *Computing the Fundamental Group in Digital Spaces*, Int. J. Patt. Recogn. Artif. Intell., **15** (2001), 1075–1088. 1, 2

[22] R. Malgouyres, G. Bertrand, *A new local property of strong n-surfaces*, Pattern Recognition Letters, **20** (1999), 417–428. 1, 2

[23] J. Merryfield, *A generalization of the Banach contraction principle*, J. Math. Anal. Appl., **273** (2002), 112–120. 1

[24] R. S. Palais, *A simple proof of the Banach contraction principle*, J. Fixed point theory appl., **2** (2007), 221–223. 1

[25] A. Rosenfeld, *Digital topology*, Amer. Math. Monthly, **86** (1979), 621-630. 1, 2, 1, 2, 2

[26] A. Rosenfeld, *Continuous functions on digital pictures*, Pattern Recognition Letters, **4** (1986), 177–184. 1, 2, 2, 3, 3, 4

[27] W. Shatanawi, H. K. Nashine, *A generalization of Banach's contraction principle for nonlinear contraction in a partial metric space*, J. Nonlinear Sci. Appl., **5** (2012), 37–43. 1

[28] J. Šlapal, *Topological structuring of the digital plane*, Discrete Math. Theor. Comput. Sci., **15** (2013), 165–176. 1

[29] F. Wyse, D. Marcus , *Solution to problem 5712*, Am. Math. Monthly, **77** (1970), 1119. 1