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Stability of derivations in fuzzy normed algebras

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Abstract

In this paper, we find a fuzzy approximation of derivation for an m-variable additive functional equation. In fact, using the fixed point method, we prove the Hyers-Ulam stability of derivations on fuzzy Lie C^*-algebras for the the following additive functional equation

$$\sum_{i=1}^{m} f\Big(mx_i + \sum_{j=1,\, j\neq i}^{m} x_j\Big) + f\Big(\sum_{i=1}^{m} x_i\Big) = 2f\Big(\sum_{i=1}^{m} mx_i\Big)$$

for a given integer m with $m \geq 2$. ©2015 All rights reserved.

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1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [9] concerning the stability of group homomorphisms:

Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x * y), h(x) \diamond h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

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If the answer is affirmative, we would say that the equation of homomorphism $H(x * y) = H(x) \diamond H(y)$ is *stable*.

We recall a fundamental result in fixed point theory.

Let Ω be a set. A function $d : \Omega \times \Omega \to [0, \infty]$ is called a *generalized metric* on Ω if d satisfies the following:

(1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in \Omega$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in \Omega$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in \Omega$.

Theorem 1.1. [3] *Let (Ω, d) be a complete generalized metric space and let $J : \Omega \to \Omega$ be a contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in \Omega$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n \ge n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $\Gamma = \{y \in \Omega : d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in \Gamma$.*

In this paper, using the fixed point method, we prove the Hyers-Ulam stability of homomorphisms and derivations in fuzzy Lie C^*-algebras for the the following additive functional equation (see [10])

$$\sum_{i=1}^{m} f\Big(mx_i + \sum_{j=1,\, j\neq i}^{m} x_j\Big) + f\Big(\sum_{i=1}^{m} x_i\Big) = 2f\Big(\sum_{i=1}^{m} mx_i\Big) \tag{1.1}$$

for all $m \in \mathbb{N}$ with $m \ge 2$.

We use the definition of fuzzy normed spaces given in [1, 4, 6, 7, 8] to investigate a fuzzy version of the Hyers-Ulam stability for the Cauchy-Jensen functional equation in the fuzzy normed algebra setting.

Definition 1.2. [6] Let X be a vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on X if

(N1) $N(x, t) = 0$ for all $x \in X$ and $t \in \mathbb{R}$ with $t \le 0$;
(N2) $x = 0$ if and only if $N(x, t) = 1$ for all $x \in X$ and $t > 0$;
(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ for all $x \in X$ and $c \neq 0$;
(N4) $N(x + y, s + t) \ge \min\{N(x, s), N(y, t)\}$ for all $x, y \in X$ and $s, t \in \mathbb{R}$;
(N5) $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$ for all $x \in X$ $t \in \mathbb{R}$;
(N6) for all $x \in X$ with $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a *fuzzy normed vector space*.

Definition 1.3. [6] (1) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to *be convergent* to a point $x \in X$ or *converges* if there exists $x \in X$ such that

$$\lim_{n\to\infty} N(x_n - x, t) = 1$$

for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

(2) Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if, for each $\varepsilon > 0$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy normed vector space is said to be *complete* and the complete fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is *continuous* at a point $x_0 \in X$ if, for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $\{f(x_n)\}$ converges to $f(x_0)$.

If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [6]).

Definition 1.4. [5] A *fuzzy normed algebra* (X, N) is a fuzzy normed space (X, N) with the algebraic structure such that

(N7) $N(xy, ts) \geq N(x, t)N(y, s)$ for all $x, y \in X$ and $t, s > 0$.

Every normed algebra $(X, \|\cdot\|)$ defines a fuzzy normed algebra (X, N), where N is defined by

$$N(x,t) = \frac{t}{t + \|x\|}$$

for all $t > 0$. This space is called the *induced fuzzy normed algebra.*

Definition 1.5. Let (X, N) and (Y, N) be fuzzy normed algebras. (1) An $\mathbb{C}$-linear mapping $f : X \to Y$ is called a *homomorphism* if

$$f(xy) = f(x)f(y)$$

for all $x, y \in X$.

(2) An $\mathbb{C}$-linear mapping $f : X \to X$ is called a *derivation* if

$$f(xy) = f(x)y + xf(y)$$

for all $x, y \in X$.

Definition 1.6. Let $(\mathcal{U}, N)$ be a fuzzy Banach algebra. Then an *involution* on $\mathcal{U}$ is a mapping $u \to u^*$ from $\mathcal{U}$ into $\mathcal{U}$ which satisfies the following:

(a) $u^{**} = u$ for any $u \in \mathcal{U}$;
(b) $(\alpha u + \beta v)^* = \overline{\alpha} u^* + \overline{\beta} v^*$;
(c) $(uv)^* = v^* u^*$ for any $u, v \in \mathcal{U}$.

If, in addition, $N(u^* u, ts) = N(u, t)N(u, s)$ and $N(u^*, t) = N(u, t)$ for all $u \in \mathcal{U}$ and $t, s > 0$, then $\mathcal{U}$ is a *fuzzy C^*–algebra.*

2. Stability of derivations on fuzzy C^*-algebras

Throughout this section, assume that A is a fuzzy C^*-algebra with the norm N_A.

For any mapping $f : A \to A$, we define

$$D_\mu f(x_1, \cdots, x_m) := \sum_{i=1}^{m} \mu f\Big(m x_i + \sum_{j=1, j\neq i}^{m} x_j\Big) + f\Big(\mu \sum_{i=1}^{m} x_i\Big) - 2f\Big(\mu \sum_{i=1}^{m} m x_i\Big)$$

for all $\mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and $x_1, \cdots, x_m \in A$.

Note that a $\mathbb{C}$-linear mapping

$$\delta : A \to A$$

is called a *fuzzy C^*-algebra derivation* on fuzzy C^*-algebras if δ satisfies the following:

$$\delta(xy) = y\delta(x) + x\delta(y)$$

and

$$\delta(x^*) = \delta(x)^*$$

for all $x, y \in A$.

Now, we prove the Hyers-Ulam stability of fuzzy C^*-algebra derivations on fuzzy C^*-algebras for the functional equation

$$D_\mu f(x_1, \cdots, x_m) = 0.$$

Theorem 2.1. *Let $f : A \to A$ be a mapping for which there are functions $\varphi : A^m \times (0,\infty) \to [0,1]$, $\psi : A^2 \times (0,\infty) \to [0,1]$ and $\eta : A \times (0,\infty) \to [0,1]$ such that*

$$N_A(D_\mu f(x_1,\cdots,x_m),t) \geq \varphi(x_1,\cdots,x_m,t), \tag{2.1}$$

$$\lim_{j\to\infty} \varphi(m^j x_1,\cdots,m^j x_m, m^j t) = 1, \tag{2.2}$$

$$N_A(f(xy) - xf(y) - xf(y),t) \geq \psi(x,y,t), \tag{2.3}$$

$$\lim_{j\to\infty} \psi(m^j x, m^j y, m^{2j} t) = 1, \tag{2.4}$$

$$N_A(f(x^*) - f(x)^*, t) \geq \eta(x,t), \tag{2.5}$$

$$\lim_{j\to\infty} \eta(m^j x, m^j t) = 1 \tag{2.6}$$

for all $\mu \in \mathbb{T}^1$, $x_1,\cdots,x_m,x,y \in A$ and $t>0$. If there exists an $L<1$ such that

$$\varphi(mx,0,\cdots,0,mLt) \geq \varphi(x,0,\cdots,0,t) \tag{2.7}$$

for all $x \in A$ and $t>0$, then there exists a unique fuzzy C^*-algebra derivation $\delta : A \to A$ such that

$$N_A(f(x) - \delta(x),t) \geq \varphi(x,0,\cdots,0,(m-mL)t) \tag{2.8}$$

for all $x \in A$ and $t>0$.

Proof. Consider the set $X := \{g : A \to A\}$ and introduce the *generalized metric* on X:

$$d(g,h) = \inf\{C \in \mathbb{R}_+ : N_A(g(x)-h(x),Ct) \geq \varphi(x,0,\cdots,0,t),\ \forall x \in A,\ t>0\}.$$

It is easy to show that (X,d) is complete. Now, we consider the linear mapping $J : X \to X$ such that $Jg(x) := \frac{1}{m} g(mx)$ for all $x \in A$. By [2, Theorem 3.1], we have

$$d(Jg,Jh) \leq Ld(g,h)$$

for all $g,h \in X$. Letting $\mu = 1$, $x = x_1$ and $x_2 = \cdots = x_m = 0$ in (2.1), we get

$$N_A(f(mx) - mf(x),t) \geq \varphi(x,0,\cdots,0,t) \tag{2.9}$$

for all $x \in A$ and $t>0$. So

$$N_A(f(x) - \frac{1}{m} f(mx),t) \geq \varphi(x,0,\cdots,0,mt)$$

for all $x \in A$ and $t>0$. Hence $d(f,Jf) \leq \frac{1}{m}$. By Theorem 1.1, there exists a mapping $\delta : A \to A$ such that

(1) δ is a fixed point of J, i.e.,

$$\delta(mx) = m\delta(x) \tag{2.10}$$

for all $x \in A$. The mapping δ is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \infty\}.$$

This implies that δ is a unique mapping satisfying (2.10) such that there exists $C \in (0,\infty)$ satisfying

$$N_A(\delta(x) - f(x),Ct) \geq \varphi(x,0,\cdots,0,t)$$

for all $x \in A$ and $t>0$.

(2) $d(J^n f, \delta) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(m^n x)}{m^n} = \delta(x) \tag{2.11}$$

for all $x \in A$.

(3) $d(f,\delta) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality $d(f,\delta) \le \frac{1}{m-mL}$. This implies that the inequality (2.8) holds.

It follows from (2.1), (2.2) and (2.11) that

$$\begin{aligned}
& N_A\Big(\sum_{i=1}^m \delta\Big(mx_i + \sum_{j=1,\, j\neq i}^m x_j\Big) + \delta\Big(\sum_{i=1}^m x_i\Big) - 2\delta\Big(\sum_{i=1}^m mx_i\Big), t\Big) \\
= & \lim_{n\to\infty} N_A\Big(\sum_{i=1}^m f\Big(m^{n+1}x_i + \sum_{j=1,\, j\neq i}^m m^n x_j\Big) + f\Big(\sum_{i=1}^m m^n x_i\Big) - 2f\Big(\sum_{i=1}^m m^{n+1} x_i\Big), m^n t\Big) \\
\le & \lim_{n\to\infty} \varphi(m^n x_1, \cdots, m^n x_m, m^n t) \\
= & 1
\end{aligned}$$

for all $x_1, \cdots, x_m \in A$ and $t > 0$ and so

$$\sum_{i=1}^m \delta\Big(mx_i + \sum_{j=1,\, j\neq i}^m x_j\Big) + \delta\Big(\sum_{i=1}^m x_i\Big) = 2\delta\Big(\sum_{i=1}^m mx_i\Big)$$

for all $x_1, \cdots, x_m \in A$.

By the similar method to above, we get $\mu\delta(mx) = \delta(m\mu x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Thus one can show that the mapping $\delta : A \to A$ is $\mathbb{C}$-linear.

It follows from (2.3), (2.4) and (2.11) that

$$\begin{aligned}
& N_A(\delta(xy) - y\delta(x) - x\delta(y), t) \\
= & \lim_{n\to\infty} N_A(f(m^n xy) - m^n y f(m^n x) - m^n x f(m^n y), m^n t) \\
\le & \lim_{n\to\infty} \psi(m^n x, m^n y, m^{2n} t) \\
= & 1
\end{aligned}$$

for all $x, y \in A$. So $\delta(xy) = y\delta(x) + x\delta(y)$ for all $x, y \in A$. Thus $\delta : A \to A$ is a derivation satisfying (2.7).

Also, by (2.5), (2.6), (2.11) and a similar method, we have $\delta(x^*) = \delta(x)^*$. □

3. Stability of derivations on fuzzy Lie C^*-algebras

A fuzzy C^*-algebra $\mathcal{C}$, endowed with the Lie product

$$[x, y] := \frac{xy - yx}{2}$$

on $\mathcal{C}$, is called a *fuzzy Lie C^*-algebra*.

Definition 3.1. Let A be a fuzzy Lie C^*-algebra. A $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *fuzzy Lie C^*-algebra derivation* if

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$$

for all $x, y \in \mathcal{A}$.

Throughout this section, assume that A is a fuzzy Lie C^*-algebra with norm N_A. We prove the Hyers-Ulam stability of fuzzy Lie C^*-algebra derivations on fuzzy Lie C^*-algebras for the functional equation

$$D_\mu f(x_1, \cdots, x_m) = 0.$$

Theorem 3.2. *Let $f : A \to A$ be a mapping for which there are two functions $\varphi : A^m \times (0, \infty) \to [0, 1]$ and $\psi : A^2 \times (0, \infty) \to [0, 1]$ such that*

$$\lim_{j\to\infty} \varphi(m^j x_1, \cdots, m^j x_m, m^j t) \quad = \quad 1, \tag{3.1}$$

$$N_A(D_\mu f(x_1, \cdots, x_m), t) \quad \geq \quad \varphi(x_1, \cdots, x_m, t), \tag{3.2}$$

$$N_A(f([x, y]) - [f(x), y] - [x, f(y)], t) \quad \geq \quad \psi(x, y, t), \tag{3.3}$$

$$\lim_{j\to\infty} \psi(m^j x, m^j y, m^{2j} t) \quad = \quad 1 \tag{3.4}$$

for all $\mu \in \mathbb{T}^1$, $x_1, \cdots, x_m$, $x, y \in A$ and $t > 0$. If there exists an $L < 1$ such that

$$\varphi(mx, 0, \cdots, 0, mx) \geq \varphi(x, 0, \cdots, 0, t)$$

for all $x \in A$ and $t > 0$, then there exists a unique fuzzy Lie C^*-algebra derivation $\delta : A \to A$ such that

$$N_A(f(x) - \delta(x), t) \geq \varphi(x, 0, \cdots, 0, (m - mL)t) \tag{3.5}$$

for all $x \in A$ and $t > 0$.

Proof. By the same reasoning as in the proof of Theorem 2.1, we can find the mapping $\delta : A \to A$ given by

$$\delta(x) = \lim_{n\to\infty} \frac{f(m^n x)}{m^n}$$

for all $x \in A$. It follows from (3.3) that

$$\begin{aligned} & N_A(\delta([x, y]) - [\delta(x), y] - [x, \delta(y)], t) \\ = \; & \lim_{n\to\infty} N_A(f(m^{2n}[x, y]) - [f(m^n x), \cdot m^n y] - [m^n x, f(m^n y)], m^{2n} t) \\ \geq \; & \lim_{n\to\infty} \psi(m^n x, m^n y, m^{2n} t) = 1 \end{aligned}$$

for all $x, y \in A$ and $t > 0$. So

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$$

for all $x, y \in A$. Thus $\delta : A \to B$ is a fuzzy Lie C^*-algebra derivation satisfying (3.5). This completes the proof. □

Corollary 3.3. *Let A be a normed fuzzy Lie C^*-algebra with norm $\|\cdot\|$. Let $0 < r < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping such that*

$$N_A(D_\mu f(x_1, \cdots, x_m), t) \geq \frac{t}{t + \theta(\|x_1\|_A^r + \|x_2\|_A^r + \cdots + \|x_m\|_A^r)}$$

and

$$N_A(f([x, y]) - [f(x), y] - [x, f(y)], t) \geq \frac{t}{t + \theta \cdot \|x\|_A^r \cdot \|y\|_A^r}$$

for all $\mu \in \mathbb{T}^1$, all $x_1, \cdots, x_m, x, y \in A$ and $t > 0$. Then there exists a unique fuzzy Lie C^*-algebra derivation $\delta : A \to A$ such that

$$N_A(f(x) - \delta(x), t) \leq \frac{t}{t + \frac{\theta}{m - m^r}\|x\|_A^r}$$

for all $x \in A$ and $t > 0$.

Proof. The proof follows from Theorem 3.2 by taking

$$\varphi(x_1, \cdots, x_m, t) = \frac{t}{t + \theta(\|x_1\|_A^r + \|x_2\|_A^r + \cdots + \|x_m\|_A^r)}$$

and

$$\psi(x, y, t) := \frac{t}{t + \theta \cdot \|x\|_A^r \cdot \|y\|_A^r}$$

for all $x_1, \cdots, x_m, x, y \in A$ and $t > 0$. Putting $L = m^{r-1}$, we get the desired result. □

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