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Note on Aczél-type inequality and Bellman-type inequality

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Abstract

In this short note, by using the method of Vasić and Pečarić [P. M. Vasić, J. E. Pečarić, Mathematica Rev. D'Anal. Num. Th. L'Approx., **25** (1982), 95–103], we obtain some properties of Aczél-type inequality and Bellman-type inequality, and then we obtain some new refinements of Aczél-type inequality and Bellman-type inequality.

Keywords: Aczél-type inequality, Bellman-type inequality, refinement, property.
2010 MSC: 26D15, 26D10.

1. Introduction

The famous Aczél's inequality, which is of wide application in the theory of functional equations in non-Euclidean geometry, was given by Aczél [1] as follows.

Theorem 1.1. *Let n be a positive integer with $n \geq 2$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be real numbers such that $a_1^2 - \sum_{i=2}^n a_i^2 > 0$ and $b_1^2 - \sum_{i=2}^n b_i^2 > 0$. Then*

$$\left(a_1^2 - \sum_{i=2}^n a_i^2\right)\left(b_1^2 - \sum_{i=2}^n b_i^2\right) \leq \left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^2. \tag{1.1}$$

Later in 1959 Popoviciu [6] gave a generalization of the above inequality in the following theorem.

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Theorem 1.2. *Let n be a positive integer with $n \geq 2$, let $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be nonnegative real numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^q - \sum_{i=2}^n b_i^q > 0$. Then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \leq a_1 b_1 - \sum_{i=2}^n a_i b_i, \tag{1.2}$$

which is called as Aczél-Popoviciu inequality.

In 1979, Vasić and Pečarić [11] presented a further extension of inequality (1.1) as follows.

Theorem 1.3. *Let n, m be positive integers with $n \geq 2$, let $\lambda_j > 0$, $\sum_{j=1}^m \frac{1}{\lambda_j} \geq 1$, and let a_{rj} $(r = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$ be positive numbers such that $a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j} > 0$ for $j = 1, 2, \cdots, m$. Then*

$$\prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \leq \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}. \tag{1.3}$$

In 2012, Tian [7] presented the following reversed version of inequality (1.3) as follows.

Theorem 1.4. *Let n, m be positive integers with $n \geq 2$, let $\lambda_1 \neq 0$, $\lambda_j < 0$ $(j = 2, 3, \cdots, m)$, $\sum_{j=1}^m \frac{1}{\lambda_j} \leq 1$, and let a_{rj} $(r = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$ be positive numbers such that $a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j} > 0$ for $j = 1, 2, \cdots, m$. Then*

$$\prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} \geq \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}. \tag{1.4}$$

In 1990, Bjelica [3] obtained an interesting Aczél-type inequality as follows.

Theorem 1.5. *Let n be a positive integer with $n \geq 2$, let $0 < p \leq 2$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be nonnegative real numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^p - \sum_{i=2}^n b_i^p > 0$. Then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{\frac{1}{p}} \leq a_1 b_1 - \sum_{i=2}^n a_i b_i, \tag{1.5}$$

which is called as Aczél-Bjelica inequality.

If we set $m = 2$, $\lambda_1 = p = \lambda_2 = q < 0$, $a_{r1} = a_r$, $a_{r2} = b_r$ $(r = 1, 2, \cdots, n)$ in Theorem 1.4, then from Theorem 1.4 we obtain the following reversed version of Aczél-Bjelica inequality (1.5).

Theorem 1.6. *Let n be a positive integer with $n \geq 2$, let $p < 0$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be positive real numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^p - \sum_{i=2}^n b_i^p > 0$. Then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{\frac{1}{p}} \geq a_1 b_1 - \sum_{i=2}^n a_i b_i. \tag{1.6}$$

The well-known Bellman inequality is stated in the following theorem [2] (see also [5]).

Theorem 1.7. *Let n be a positive integer with $n \geq 2$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be positive numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^p - \sum_{i=2}^n b_i^p > 0$. If $p \geq 1$ (or $p < 0$), then*

$$\left[\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} + \left(b_1^p - \sum_{i=2}^n b_i^p\right)^{\frac{1}{p}}\right]^p \leq (a_1 + b_1)^p - \sum_{i=2}^n (a_i + b_i)^p. \tag{1.7}$$

If $0 < p < 1$, then the reverse inequality in (1.7) holds.

Based on the mathematical induction, it is easy to see that the following generalized Bellman inequality is true.

Theorem 1.8. *Let n, m be positive integers with $n \geq 2$, let a_{rj} $(r = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$ be positive numbers such that $a_{1j}^p - \sum_{r=2}^{n} a_{rj}^p > 0$. If $p \geq 1$ (or $p < 0$), then*

$$\left[\sum_{j=1}^{m}\left(a_{1j}^p - \sum_{r=2}^{n} a_{rj}^p\right)^{\frac{1}{p}}\right]^p \geq \left(\sum_{j=1}^{m} a_{1j}\right)^p - \sum_{r=2}^{n}\left(\sum_{j=1}^{m} a_{rj}\right)^p. \tag{1.8}$$

If $0 < p < 1$, then the reverse inequality in (1.8) holds.

Remark 1.9. The case $p \geq 1$ of Theorem 1.8 was given by Yang [13].

As is well-known, an important research subject in analyzing inequality is to convert an univariate into the monotonicity of functions. For example, Hu in [4] solved the elaboration problems of the Opial-Hua inequality by using the monotonicity of Hu's inequality. Tian [9] solved the elaboration problems of the Opial-Beesack inequality and Singh's inequality by using a new monotonicity of generalized Hölder's inequality. Tian in [8] gave a new monotonicity property of reversed Hu's inequality, and then obtained some new refinements of Hölder's inequalities by using the property. Moreover, Tian in [10] obtained some new refinements of generalized Hölder's inequalities by using the monotonicity of generalized Hölder's inequalities.

In [12], Vasić and Pečarić gave the following monotonicity property of inequalities (1.2) and (1.7).

Theorem 1.10. *Let n be a positive integer with $n \geq 2$, let p, $q \neq 0$, $\frac{1}{p} + \frac{1}{q} = 1$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be nonnegative real numbers such that $a_1^p - \sum_{i=2}^{n} a_i^p > 0$ and $b_1^q - \sum_{i=2}^{n} b_i^q > 0$. Let one denote*

$$P(n) = \left(a_1^p - \sum_{k=2}^{n} a_k^p\right)^{\frac{1}{p}}\left(b_1^q - \sum_{k=2}^{n} b_k^q\right)^{\frac{1}{q}} - \left(a_1 b_1 - \sum_{k=2}^{n} a_k b_k\right).$$

If $p > 1$, then

$$P(n) \leq P(n-1),$$

and if $p < 1$ then the reverse inequality is valid.

Theorem 1.11. *Let n be a positive integer with $n \geq 2$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be positive numbers such that $a_1^p - \sum_{i=2}^{n} a_i^p > 0$ and $b_1^p - \sum_{i=2}^{n} b_i^p > 0$. Let one denote*

$$B(n) = \left(a_1^p - \sum_{k=2}^{n} a_k^p\right)^{\frac{1}{p}} + \left(b_1^q - \sum_{k=2}^{n} b_k^q\right)^{\frac{1}{q}} - \left((a_1 + b_1)^p - \sum_{k=2}^{n}(a_k + b_k)^p\right).$$

If $p > 1$ (or < 0) we have

$$B(n) \leq B(n-1),$$

and if $0 < p < 1$ then the reverse inequality is valid.

Stimulated by the works of Vasić and Pečarić [12], in this paper, using the method of Vasić and Pečarić [12], some similar properties of the above Aczél-type inequality and Bellman-type inequality are given, and then some new refinements of Aczél-type inequality and Bellman-type inequality are obtained.

2. Main results

Theorem 2.1. *Let n, m be positive integers with $n \geq 3$, let $\lambda_1 \neq 0$, $\lambda_j < 0$ $(j = 2, 3, \cdots, m)$, $\sum_{j=1}^{m} \frac{1}{\lambda_j} \leq 1$, and let a_{rj} $(r = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$ be positive numbers such that $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$ for $j = 1, 2, \cdots, m$. Let one denote*

$$V_n = \prod_{j=1}^{m}\left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} - \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n}\prod_{j=1}^{m} a_{rj}\right).$$

Then

$$V_n \geq V_{n-1}. \tag{2.1}$$

Proof. Denoting

$$A_j = \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n-1} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}}, j = 1, 2, \cdots, m,$$

then

$$\prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} = \prod_{j=1}^{m} \left(A_j^{\lambda_j} - a_{nj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}}. \tag{2.2}$$

It is given that $A_j^{\lambda_j} - a_{nj}^{\lambda_j} > 0$, therefore from Theorem 1.4 for $n = 2$, on right-hand side of the above equation, we obtain

$$\prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} \geq \prod_{j=1}^{m} A_j - \prod_{j=1}^{m} a_{nj}, \tag{2.3}$$

that is

$$\prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} \geq \prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n-1} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} - \prod_{j=1}^{m} a_{nj}. \tag{2.4}$$

Thus, performing some simple computations immediately leads to the desired inequality. The proof of Theorem 2.1 is completed.

□

By the same method as in Theorem 2.1, but using Theorem 1.3 in place of Theorem 1.4, we can obtain the following Theorem.

Theorem 2.2. *Let n, m be positive integers with $n \geq 3$, let $\lambda_j > 0$, $j = 1, 2, \cdots, m$, $\sum_{j=1}^{m} \frac{1}{\lambda_j} \geq 1$, and let $a_{rj}(r = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$ be positive numbers such that $a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} > 0$ for $j = 1, 2, \cdots, m$. Let one denote*

$$\widetilde{V}_n = \prod_{j=1}^{m} \left(a_{1j}^{\lambda_j} - \sum_{r=2}^{n} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} - \left(\prod_{j=1}^{m} a_{1j} - \sum_{r=2}^{n} \prod_{j=1}^{m} a_{rj} \right).$$

Then

$$\widetilde{V}_n \leq \widetilde{V}_{n-1}. \tag{2.5}$$

Putting $m = 2$, $\lambda_1 = p \neq 0$, $\lambda_2 = q < 0$, $a_{r1} = a_r$, $a_{r2} = b_r$ $(r = 1, 2, \cdots, n)$ in Theorem 2.1, we obtain the following result.

Corollary 2.3. *Let n be a positive integer with $n \geq 3$, let $p \neq 0$, $q < 0$, $\frac{1}{p} + \frac{1}{q} \leq 1$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be positive numbers such that $a_1^p - \sum_{i=2}^{n} a_i^p > 0$ and $b_1^q - \sum_{i=2}^{n} b_i^q > 0$. Let one denote*

$$V_n^* = \left(a_1^p - \sum_{r=2}^{n} a_r^p \right)^{\frac{1}{p}} \left(b_1^q - \sum_{r=2}^{n} b_r^q \right)^{\frac{1}{q}} - \left(a_1 b_1 - \sum_{r=2}^{n} a_r b_r \right).$$

Then

$$V_n^* \geq V_{n-1}^*. \tag{2.6}$$

Similarly, putting $m = 2$, $\lambda_1 = p > 0$, $\lambda_2 = q > 0$, $a_{r1} = a_r$, $a_{r2} = b_r$ $(r = 1, 2, \cdots, n)$ in Theorem 2.2, we obtain the following Corollary.

Corollary 2.4. *Let n be a positive integer with $n \geq 3$, let $p > 0$, $q > 0$, $\frac{1}{p} + \frac{1}{q} \geq 1$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be positive numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^q - \sum_{i=2}^n b_i^q > 0$. Let one denote*

$$\widetilde{V}_n^* = \left(a_1^p - \sum_{r=2}^n a_r^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{r=2}^n b_r^q\right)^{\frac{1}{q}} - \left(a_1 b_1 - \sum_{r=2}^n a_r b_r\right).$$

Then

$$\widetilde{V}_n^* \leq \widetilde{V}_{n-1}^*. \tag{2.7}$$

More particularly, if we set $p = q < 0$ in Corollary 2.3, then we have the following property of reversed Aczél-Bjelica inequality (1.6).

Corollary 2.5. *Under the assumptions of* Corollary 2.3, *and let $p = q < 0$, we have*

$$V_n^* \geq V_{n-1}^*. \tag{2.8}$$

Similarly, if we set $p = q > 0$ in Corollary 2.4, then we have the following property of Aczél-Bjelica inequality (1.6).

Corollary 2.6. *Under the assumptions of* Corollary 2.4, *and letting $p = q > 0$, we have*

$$\widetilde{V}_n^* \leq \widetilde{V}_{n-1}^*. \tag{2.9}$$

From Theorem 2.1, we obtain a new refinement of generalized Aczél inequality (1.4) as follows.

Corollary 2.7. *Under the assumptions of* Theorem 2.1, *we have*

$$\begin{aligned}\prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} &\geq \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj} + V_2 &(2.10)\\ &\geq \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}. &(2.11)\end{aligned}$$

Proof. From Theorem 2.1, we find

$$V_n \geq V_2 \geq 0. \tag{2.12}$$

Rearranging the terms of (2.12) immediately leads to the desired inequality. This completes the proof. □

Making similar technique as in the proof of Corollary 2.7, we get the following refinement of generalized Aczél inequality (1.3).

Corollary 2.8. *Under the assumptions of* Theorem 2.2, *we have*

$$\begin{aligned}\prod_{j=1}^m \left(a_{1j}^{\lambda_j} - \sum_{r=2}^n a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} &\leq \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj} + \widetilde{V}_2 \\ &\leq \prod_{j=1}^m a_{1j} - \sum_{r=2}^n \prod_{j=1}^m a_{rj}. &(2.13)\end{aligned}$$

Similarly, we have the following refinements of Aczél-type inequality.

Corollary 2.9. *Let n be a positive integer with $n \geq 3$, let $p \neq 0$, $q < 0$, $\frac{1}{p} + \frac{1}{q} \leq 1$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be positive numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^q - \sum_{i=2}^n b_i^q > 0$. Then*

$$\left(a_1^p - \sum_{r=2}^n a_r^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{r=2}^n b_r^q\right)^{\frac{1}{q}} \geq a_1 b_1 - \sum_{r=2}^n a_r b_r + V_2^* \geq a_1 b_1 - \sum_{r=2}^n a_r b_r, \tag{2.14}$$

where $V_2^* = \left(a_1^p - a_2^p\right)^{\frac{1}{p}} \left(b_1^q - b_2^q\right)^{\frac{1}{q}} - \left(a_1 b_1 - a_2 b_2\right) \geq 0$.

Corollary 2.10. *Let n be a positive integer with $n \geq 3$, let $p > 0$, $q > 0$, $\frac{1}{p} + \frac{1}{q} \geq 1$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be positive numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^q - \sum_{i=2}^n b_i^q > 0$. Then*

$$\left(a_1^p - \sum_{r=2}^n a_r^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{r=2}^n b_r^q\right)^{\frac{1}{q}} \leq a_1 b_1 - \sum_{r=2}^n a_r b_r + \widetilde{V}_2^* \leq a_1 b_1 - \sum_{r=2}^n a_r b_r, \tag{2.15}$$

where $\widetilde{V}_2^* = \left(a_1^p - a_2^p\right)^{\frac{1}{p}} \left(b_1^q - b_2^q\right)^{\frac{1}{q}} - \left(a_1 b_1 - a_2 b_2\right) \leq 0$.

Corollary 2.11. *Let n be a positive integer with $n \geq 3$, let $p < 0$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be positive numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^p - \sum_{i=2}^n b_i^p > 0$. Then*

$$\left(a_1^p - \sum_{r=2}^n a_r^p\right)^{\frac{1}{p}} \left(b_1^p - \sum_{r=2}^n b_r^p\right)^{\frac{1}{p}} \geq a_1 b_1 - \sum_{r=2}^n a_r b_r + V_2^* \geq a_1 b_1 - \sum_{r=2}^n a_r b_r, \tag{2.16}$$

where $V_2^* = \left(a_1^p - a_2^p\right)^{\frac{1}{p}} \left(b_1^p - b_2^p\right)^{\frac{1}{p}} - \left(a_1 b_1 - a_2 b_2\right) \geq 0$.

Corollary 2.12. *Let n be a positive integer with $n \geq 3$, let $0 < p \leq 2$, and let a_i, b_i $(i = 1, 2, \cdots, n)$ be positive numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^p - \sum_{i=2}^n b_i^p > 0$. Then*

$$\left(a_1^p - \sum_{r=2}^n a_r^p\right)^{\frac{1}{p}} \left(b_1^p - \sum_{r=2}^n b_r^p\right)^{\frac{1}{p}} \leq a_1 b_1 - \sum_{r=2}^n a_r b_r + \widetilde{V}_2^* \leq a_1 b_1 - \sum_{r=2}^n a_r b_r, \tag{2.17}$$

where $\widetilde{V}_2^* = \left(a_1^p - a_2^p\right)^{\frac{1}{p}} \left(b_1^p - b_2^p\right)^{\frac{1}{p}} - \left(a_1 b_1 - a_2 b_2\right) \leq 0$.

Next, we give a new property of generalized Bellman inequality (1.8) as follows.

Theorem 2.13. *Let m and n be positive integers with $n \geq 3$, let a_{rj} $(r = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$ be positive numbers such that $a_{1j}^p - \sum_{r=2}^n a_{rj}^p > 0$ for $j = 1, 2, \cdots, m$. Let one denote*

$$B_n = \left[\sum_{j=1}^m \left(a_{1j}^p - \sum_{r=2}^n a_{rj}^p\right)^{\frac{1}{p}}\right]^p - \left[\left(\sum_{j=1}^m a_{1j}\right)^p - \sum_{r=2}^n \left(\sum_{j=1}^m a_{rj}\right)^p\right].$$

If $p > 1$ (or $p < 0$) we have

$$B_n \leq B_{n-1}, \tag{2.18}$$

and if $0 < p < 1$ then the reverse inequality is valid.

Proof. Similar to the proof of Theorem 2.1 but using Theorem 1.8 in place of Theorem 1.4, we immediately obtain the desired results. □

Finally, we give the following refinement of generalized Bellman inequality (1.8).

Corollary 2.14. *Under the assumptions of* Theorem 2.13. *If $p > 1$ (or $p < 0$) we have*

$$\left[\prod_{j=1}^{m}\left(a_{1j}^{p}-\sum_{r=2}^{n}a_{rj}^{p}\right)^{\frac{1}{p}}\right]^{p} \le \left(\sum_{j=1}^{m}a_{1j}\right)^{p}-\sum_{r=2}^{n}\left(\sum_{j=1}^{m}a_{rj}\right)^{p}+B_2 \le \left(\sum_{j=1}^{m}a_{1j}\right)^{p}-\sum_{r=2}^{n}\left(\sum_{j=1}^{m}a_{rj}\right)^{p} \tag{2.19}$$

and if $0 < p < 1$ then the reverse inequality is valid, where $B_2 \le 0$ for $p > 1$ (or $p < 0$), $B_2 \ge 0$ for $0 < p < 1$.

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