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Fixed point theorems in ordered cone metric spaces

Young-Oh Yang*, Hong Joon Choi

Department of Mathematics, Jeju National University, Jeju 690-756, Korea.

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Abstract

In this paper, we prove a new fixed point theorem of a nondecreasing and continuous mapping satisfying some type contractive condition in a partially ordered cone metric space by using $c-$distance. Also, we give a fixed point theorem without the assumption of continuity in a partially ordered cone metric space with normal cone.

Keywords: Cone metric space, normal cone, $c-$distance, common fixed point.
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1. Introduction

Since Huang and Zhang [5] introduced the cone metric space which is more general than the concept of a metric space, many fixed point theorems have been proved in normal or non-normal cone metric spaces by some authors [1, 4–6, 8, 10, 11]. Cho et al. [4] introduced the $c-$distance in a cone metric space which is a cone version of the $w-$distance of Kada et al. [7]. Recently the existence of fixed points for the given contractive mappings in partially ordered metric spaces was investigated by [2, 3].

In this paper, we prove a new fixed point theorem of a nondecreasing continuous mapping satisfying some type contractive condition in a partially ordered cone metric space by using $c-$distance.

Let E be a real Banach space and θ denote the zero element in E. A *cone* P is a subset of E such that

(i) P is closed, nonempty and $P \neq \{\theta\}$;

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, $x, y \in P \;\Rightarrow\; ax + by \in P$;

(iii) $P \cap (-P) = \{\theta\}$, i.e., $x \in P$ and $-x \in P$ imply $x = \theta$.

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*Corresponding author
Email addresses: yangyo@jejunu.ac.kr (Young-Oh Yang), next-xy@hanmail.net (Hong Joon Choi)

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For any cone $P \subseteq E$, the partial ordering $\preceq$ with respect to P is defined by $x \preceq y$ if and only if $y-x \in P$. The notation of $\prec$ stands for $x \preceq y$ but $x \neq y$. Also, we use $x \ll y$ to indicate that $y-x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. A cone P is called *normal* if there exists a number K such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \quad \text{implies} \quad \|x\| \leq K\|y\|. \tag{1.1}$$

Equivalently, the cone P is normal if

$$x_n \preceq y_n \preceq z_n \text{ and } \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x. \tag{1.2}$$

The least positive number K satisfying condition (1.1) is called the *normal constant* of P.

Definition 1.1. Let X be a nonempty set and let E be a real Banach space equipped with the partial ordering $\preceq$ with respect to the cone $P \subseteq E$. Suppose the mapping $d : X \times X \to E$ satisfies the following conditions:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$, and $d(x, y) = \theta$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, d is called a *cone metric* on X, and (X, d) is called a *cone metric space*.

Definition 1.2. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(1) If for every $c \in E$ with $\theta \ll c$, there exists a natural number N such that $d(x_n, x) \ll c$ for all $n > N$, then $\{x_n\}$ is said to be *convergent* and $\{x_n\}$ *converges to* x, and the point x is the *limit* of $\{x_n\}$. We denote this by
$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x \quad (n \to \infty).$$
(2) If for all $c \in E$ with $\theta \ll c$ there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $m, n > N$, then $\{x_n\}$ is called a *Cauchy sequence* in X.
(3) A cone metric space (X, d) is said to be *complete* if every Cauchy sequence in X is convergent.

Lemma 1.3 ([9])**.** *Let E be a real Banach space with a cone P. Then*

(1) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*
(2) *If $a \preceq b$ and $b \ll c$, then $a \ll c$.*

Lemma 1.4 ([9])**.** *Let E be a real Banach space with cone P. Then*

(1) *If $\theta \ll c$, then there exists $\delta > 0$ such that $\|b\| < \delta$ implies $b \ll c$.*
(2) *If $\{a_n\}, \{b_n\}$ are sequences in E such that $a_n \to a, b_n \to b$ and $a_n \preceq b_n$ for all $n \geq 1$, then $a \preceq b$.*

Lemma 1.5 ([5])**.** *Let (X, d) be a cone metric space, P a normal cone, $x \in X$, and $\{x_n\}$ a sequence in X. Then*

(1) *$\{x_n\}$ converges to x if and only if $d(x_n, x) \to \theta$.*
(2) *The limit point of every sequence is unique.*
(3) *Every convergent sequence is a Cauchy sequence.*
(4) *$\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$ as $n, m \to \infty$.*
(5) *If $x_n \to x$ and $y_n \to y$, then, $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.*

Definition 1.6. Let (X, d) be a cone metric space. Then a mapping $q : X \times X \to E$ is called a *$c-$distance* on X if the followings are satisfied:

(q1) $\theta \preceq q(x, y)$ for all $x, y \in X$;
(q2) $q(x, z) \preceq q(x, y) + q(y, z)$ for all $x, y, z \in X$;

(q3) for all $x \in X$ and all $n \geq 1$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;

(q4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that

$$q(z, x) \ll e \text{ and } q(z, y) \ll e \text{ imply } d(x, y) \ll c.$$

Example 1.7 ([4])**.** Let (X, d) be a cone metric space and let P be a normal cone. Put $q(x, y) = d(x, y)$ for all $x, y \in X$. Then, q is a $c-$distance.

Example 1.8 ([4])**.** Let (X, d) be a cone metric space and let P be a normal cone. Put $q(x, y) = d(u, y)$ for all $x, y \in X$, where $u \in X$ is constant. Then, q is a $c-$distance.

Example 1.9 ([4])**.** Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then, q is a $c-$distance.

Remark 1.10.

(1) $q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$.

(2) $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Lemma 1.11 ([4])**.** *Let (X, d) be a cone metric space and let q be a c-distance on X. Let $\{x_n\}$ and $\{y_n\}$be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ is a sequence in P converging to θ. Then the following facts hold:*

(1) *If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $y = z$.*

(2) *If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq u_n$, then $\{y_n\}$ converges to z.*

(3) *If $q(x_n, x_m) \preceq u_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*

(4) *If $q(y, x_n) \preceq u_n$, then $\{x_n\}$is a Cauchy sequence in X.*

Definition 1.12. The mapping $T : X \to X$ is *continuous* if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} Tx_n = Tx$.

2. Main results

In this section, we prove a new fixed point theorem by using $c-$distance in partially ordered cone metric spaces.

Theorem 2.1 ([3])**.** *Let $(X,\sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a $c-$distance on X and $f : X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$ (without the assumption of continuity of f). Suppose that the following three assertions hold:*

(i) *there exist nonnegative numbers $a_i, i = 1, 2$ with $a_1 + a_2 < 1$ such that*

$$q(fx, fy) \preceq a_1 q(x, y) + a_2 q(x, fx)$$

for all $x, y \in X$ with $x \sqsubseteq y$;

(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$;*

(iii) *if $\{x_n\}$ is nondecreasing mapping with respect to $\sqsubseteq$ and converges to x then $x_n \sqsubseteq x$ as $n \to \infty$.*

Then, f has a fixed point $x \in X$. If $v = fv$ then $q(v, v) = \theta$.

Theorem 2.2 ([4]). *Let $(X,\sqsubseteq)$ be a partially ordered set and suppose that (X,d) is a complete cone metric space. Let q be a $c-$distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) *there exist $a_i \geq 0, i = 1,2,3$ with $a_1 + a_2 + a_3 < 1$ such that*

$$q(fx, fy) \preceq a_1 q(x,y) + a_2 q(x, fx) + a_3 q(y, fy)$$

for all $x, y \in X$ with $x \sqsubseteq y$;

(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.*

Then, f has a fixed point $x \in X$. If $v = fv$, then $q(v,v) = \theta$.

Theorem 2.3 ([3]). *Let $(X,\sqsubseteq)$ be a partially ordered set and suppose that (X,d) is a complete cone metric space. Let q be a $c-$distance on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) *there exist $a_i \geq 0, i = 1,2,3,4$ with $a_1 + a_2 + a_3 + 2a_4 < 1$ such that*

$$q(fx, fy) \preceq a_1 q(x,y) + a_2 q(x, fx) + a_3 q(y, fy) + a_4 q(x, fy)$$

for all $x, y \in X$ with $x \sqsubseteq y$;

(ii) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.*

Then, f has a fixed point $x \in X$. If $v = fv$, then, $q(v,v) = \theta$.

Theorem 2.4. *Let $(X,\sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a $c-$distance on X. Let $f : X \to X$ be a continuous and nondecreasing with respect to $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) *there exist nonnegative constants $a_i \in [0,1)$ $i = 1,2,3,4,5$ with $a_1 + 2a_2 + 2a_3 + 3a_4 + a_5 < 1$ such that*

$$q(fx, fy) \preceq a_1 q(x,y) + a_2 q(x, fx) + a_3 q(y, fy) + a_4 q(x, fy) + a_5 q(y, fx)$$

for all $x, y \in X$ with $x \sqsubseteq y$;

(ii) *there exist $x_0, x_1 \in X$ such that $x_0 \sqsubseteq x_1 \sqsubseteq fx_0$.*

Then, f has a fixed point in X. If $v = fv$, then, $q(v,v) = \theta$.

Proof. Since f is nondecreasing with respect to $\sqsubseteq$, we have

$$x_0 \sqsubseteq x_1 \sqsubseteq fx_0 = x_2 \sqsubseteq fx_1 = x_3 \sqsubseteq \cdots .$$

Then, we have

$$\begin{aligned}
&q(x_{2n}, x_{2n+1})\\
&\quad = q(fx_{2n-2}, fx_{2n-1})\\
&\quad \preceq a_1 q(x_{2n-2}, x_{2n-1}) + a_2 q(x_{2n-2}, fx_{2n-2}) + a_3 q(x_{2n-1}, fx_{2n-1})\\
&\qquad + a_4 q(x_{2n-2}, fx_{2n-1}) + a_5 q(x_{2n-1}, fx_{2n-2})\\
&\quad = a_1 q(x_{2n-2}, x_{2n-1}) + a_2 q(x_{2n-2}, x_{2n}) + a_3 q(x_{2n-1}, x_{2n+1}) + a_4 q(x_{2n-2}, x_{2n+1}) + a_5 q(x_{2n-1}, x_{2n})\\
&\quad \preceq a_1 q(x_{2n-2}, x_{2n-1}) + a_2 \{q(x_{2n-2}, x_{2n-1}) + q(x_{2n-1}, x_{2n})\} + a_3 \{q(x_{2n-1}, x_{2n}) + q(x_{2n}, x_{2n+1})\}\\
&\qquad + a_4 \{q(x_{2n-2}, x_{2n-1}) + q(x_{2n-1}, x_{2n}) + q(x_{2n}, x_{2n+1})\} + a_5 q(x_{2n-1}, x_{2n}).
\end{aligned}$$

Hence,

$$q(x_{2n}, x_{2n+1}) \preceq \alpha q(x_{2n-1}, x_{2n}) + \beta q(x_{2n-2}, x_{2n-1}),$$

where, $\alpha = \frac{a_2+a_3+a_4+a_5}{1-a_3-a_4}$ and $\beta = \frac{a_1+a_2+a_4}{1-a_3-a_4}$.

Similarly,

$$q(x_{2n-1}, x_{2n}) \preceq \alpha q(x_{2n-2}, x_{2n-1}) + \beta q(x_{2n-3}, x_{2n-2}).$$

Clearly $0 \le \alpha, \beta < 1$. Set $b_1 = \alpha$ and $c_1 = \beta$. By applying the above inequalities and putting $b_2 = c_1 + \alpha b_1 = \beta + \alpha b_1,\ c_2 = \beta b_1$, we obtain

$$\begin{aligned} q(x_{2n}, x_{2n+1}) &\preceq b_1 q(x_{2n-1}, x_{2n}) + c_1 q(x_{2n-2}, x_{2n-1}) \\ &\preceq b_2 q(x_{2n-2}, x_{2n-1}) + c_2 q(x_{2n-3}, x_{2n-2}) \\ &\ \ \vdots \\ &\preceq b_{2n-1} q(x_1, x_2) + c_{2n-1} q(x_0, x_1), \end{aligned} \tag{2.1}$$

where, $b_{2n-1} = \beta b_{2n-3} + \alpha b_{2n-2}$ and $c_{2n-1} = \beta b_{2n-2}$.

Similarly,

$$q(x_{2n-1}, x_{2n}) \preceq b_{2n-2} q(x_1, x_2) + c_{2n-2} q(x_0, x_1), \tag{2.2}$$

where $b_{2n-2} = \beta b_{2n-4} + \alpha b_{2n-3}$ and $c_{2n-2} = \beta b_{2n-3}$. From (2.1) and (2.2),

$$q(x_{n+1}, x_{n+2}) \preceq b_n q(x_1, x_2) + c_n q(x_0, x_1),$$

where, $b_n = \beta b_{n-2} + \alpha b_{n-1}$ and $c_n = \beta b_{n-1}$. Thus

$$b_{n+2} = \alpha b_{n+1} + \beta b_n \quad (0 \le \alpha, \beta \le 1, b_1, b_2 \ge 0)$$

and $b_n \ge 0$ for all $n \in \mathbb{N}$. Its characteristic equation is $t^2 - \alpha t - \beta = 0$. If $1 - \alpha - \beta > 0$ and $1 + \alpha - \beta > 0$, then it has two roots t_1, t_2 such that $-1 < t_1 \le 0 \le t_2 < 1$. Also the hypothesis $a_1 + 2a_2 + 2a_3 + 3a_4 + a_5 < 1$ implies $1 - \alpha - \beta > 0$ and $1 + \alpha - \beta > 0$. For such t_1 and t_2, we obtain $b_n = k_1(t_1)^n + k_2(t_2)^n$ for some $k_1, k_2 \in \mathbb{R}$.

Let $m > n \ge 1$. It follows that

$$\begin{aligned} q(x_n, x_m) \preceq& q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ \preceq& (b_{n-1} + b_n + \cdots + b_{m-2}) q(x_1, x_2) + (c_{n-1} + c_n + \cdots + c_{m-2}) q(x_0, x_1) \\ \preceq& \{k_1(t_1^{n-1} + t_1^n + \cdots + t_1^{m-2}) + k_2(t_2^{n-1} + \cdots + t_2^{m-2})\} q(x_1, x_2) \\ &+ \beta\{k_1(t_1^{n-2} + \cdots + t_1^{m-3}) + k_2(t_2^{n-2} + \cdots + t_2^{m-3})\} q(x_0, x_1) \\ \preceq& \left(\frac{k_1 t_1^{n-1}}{1 - t_1} + \frac{k_2 t_2^{n-1}}{1 - t_2}\right) q(x_1, x_2) + \beta\left(\frac{k_1 t_1^{n-2}}{1 - t_1} + \frac{k_2 t_2^{n-2}}{1 - t_2}\right) q(x_0, x_1) \\ \to& \theta \end{aligned}$$

as $n \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence in X by Lemma 1.11 (3). Since X is complete, there exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. Using the continuity of f,

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f x_{n-2} = f x.$$

Therefore, x is a fixed point of f. Moreover, suppose that $v = fv$. Then we have

$$\begin{aligned} q(v, v) =& q(fv, fv) \\ \preceq& a_1 q(v, v) + a_2 q(v, fv) + a_3 q(v, fv) + a_4 q(v, fv) + a_5 q(v, fv) \\ =& (a_1 + a_2 + a_3 + a_4 + a_5) q(v, v). \end{aligned}$$

Since $0 \le a_1 + a_2 + a_3 + a_4 + a_5 < 1$, we have $q(v, v) = \theta$. □

The following corollaries can be obtained as consequences of Theorem 2.4.

Corollary 2.5. *Let $(X,\sqsubseteq)$ be a partially ordered set and suppose that $(X,\ d)$ is a complete cone metric space. Let q be a $c-$distance on X. Let $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) *there exist nonnegative constants $a \in [0, 1/4)$ such that*

$$q(fx, fy) \preceq aq(x, fx) + aq(y, fy)$$

for all $x, y \in X$ with $x \sqsubseteq y$;

(ii) *there exist $x_0, x_1 \in X$ such that $x_0 \sqsubseteq x_1 \sqsubseteq fx_0$.*

Then, f has a fixed point in X. If $v = fv$, then, $q(v, v) = \theta$.

Corollary 2.6. *Let $(X,\sqsubseteq)$ be a partially ordered set and suppose that $(X,\ d)$ is a complete cone metric space. Let q be a $c-$distance on X. Let $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) *there exist nonnegative constants $a_i \in [0, 1)$ $i = 1, 2$ with $a_1 + a_2 < 1$ such that*

$$q(fx, fy) \preceq a_1q(x, y) + a_2q(y, fx)$$

for all $x, y \in X$ with $x \sqsubseteq y$;

(ii) *there exist $x_0, x_1 \in X$ such that $x_0 \sqsubseteq x_1 \sqsubseteq fx_0$.*

Then, f has a fixed point in X. If $v = fv$, then, $q(v, v) = \theta$.

Corollary 2.7. *Let $(X,\sqsubseteq)$ be a partially ordered set and suppose that $(X,\ d)$ is a complete cone metric space. Let q be a $c-$distance on X. Let $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assertions hold:*

(i) *there exist nonnegative constants $a \in [0, 1)$ such that*

$$q(fx, fy) \preceq aq(x, y)$$

for all $x, y \in X$ with $x \sqsubseteq y$;

(ii) *there exist $x_0, x_1 \in X$ such that $x_0 \sqsubseteq x_1 \sqsubseteq fx_0$.*

Then, f has a fixed point in X. If $v = fv$, then, $q(v, v) = \theta$.

We obtain the following fixed point theorem without the assumption of continuity in a partially ordered cone metric space with normal cone.

Theorem 2.8. *Let $(X,\sqsubseteq)$ be a partially ordered set. Suppose that $(X,\ d)$ is a complete cone metric space and P is a normal cone with normal constant K. Let q be a $c-$distance on X. Let $f : X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following three assertions hold:*

(i) *there exist nonnegative constants $a_i \in [0, 1)$ $i = 1, 2, 3, 4, 5$ with $a_1 + 2a_2 + 2a_3 + 3a_4 + a_5 < 1$ such that*

$$q(fx, fy) \preceq a_1q(x, y) + a_2q(x, fx) + a_3q(y, fy) + a_4q(x, fy) + a_5q(y, fx)$$

for all $x, y \in X$ with $x \sqsubseteq y$;

(ii) *there exist $x_0, x_1 \in X$ such that $x_0 \sqsubseteq x_1 \sqsubseteq fx_0$;*

(iii) *for all $y \in X$ with $fy \neq y$,*

$$\inf\{\|q(x, y)\| + \|q(x, fx)\| + \|q(fx, y)\| : x \in X\} > 0.$$

Then, f has a fixed point in X. If $v = fv$, then, $q(v, v) = \theta$.

Proof. Since f is nondecreasing with respect to $\sqsubseteq$, we have

$$x_0 \sqsubseteq x_1 \sqsubseteq fx_0 = x_2 \sqsubseteq fx_1 = x_3 \sqsubseteq \cdots .$$

If $m > n \geq 1$, then by the proof of Theorem 2.4,

$$\begin{aligned} q(x_n, x_m) \preceq & q(x_n, x_{n+1}) + q(x_{n+1}, x_{n+2}) + \cdots + q(x_{m-1}, x_m) \\ \preceq & (\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2}) q(x_1, x_2) + \beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2}) q(x_0, x_1) \\ \to & \theta \end{aligned}$$

as $n \to \infty$. Therefore, $\{x_n\}$ is a Cauchy sequence in X by Lemma 1.11 (3). Since X is complete, there exists $x' \in X$ such that $x_n \to x'$ as $n \to \infty$. By (q3),

$$q(x_n, x') \preceq (\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2}) q(x_1, x_2) + \beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2}) q(x_0, x_1).$$

Since P is a normal cone with normal constant K, we have

$$\begin{aligned} \|q(x_n, x_m)\| \leq & K\|(\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2}) q(x_1, x_2) + \beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2}) q(x_0, x_1)\| \\ \leq & K(\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2}) \|q(x_1, x_2)\| + K\beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2}) \|q(x_0, x_1)\| \\ \to & 0 \end{aligned}$$

as $n \to \infty$. Also

$$\begin{aligned} \|q(x_n, x')\| \leq & K\|(\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2}) q(x_1, x_2) + \beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2}) q(x_0, x_1)\| \\ \leq & K(\frac{k_1 t_1^{n-1}}{1-t_1} + \frac{k_2 t_2^{n-1}}{1-t_2}) \|q(x_1, x_2)\| + K\beta(\frac{k_1 t_1^{n-2}}{1-t_1} + \frac{k_2 t_2^{n-2}}{1-t_2}) \|q(x_0, x_1)\| \\ \to & 0 \end{aligned}$$

as $n \to \infty$.

Suppose that x' is not a fixed point of f. Then by assumption,

$$\begin{aligned} 0 < & \inf\{\|q(x, x')\| + \|q(x, fx)\| + \|q(fx, x')\| : x \in X\} \\ \leq & \inf\{\|q(x_n, x')\| + \|q(x_n, fx_n)\| + \|q(fx_n, x')\| : n \in \mathbb{N}\} \\ = & \inf\{\|q(x_n, x')\| + \|q(x_n, x_{n+2})\| + \|q(x_{n+2}, x')\| : x \in \mathbb{N}\} \\ = & 0, \end{aligned}$$

which is a contradiction. Therefore, x' is a fixed point of f.

Moreover, suppose that $v = fv$. Then we have

$$\begin{aligned} q(v, v) = q(fv, fv) \preceq & a_1 q(v, v) + a_2 q(v, fv) + a_3 q(v, fv) + a_4 q(v, fv) + a_5 q(v, fv) \\ = & (a_1 + a_2 + a_3 + a_4 + a_5) q(v, v). \end{aligned}$$

Since $0 \leq a_1 + a_2 + a_3 + a_4 + a_5 < 1$, we have $q(v, v) = \theta$. □

We give an example which can not be applied to Theorem 2.3 and Theorem 2.2, but can be applied to Theorem 2.4.

Example 2.9. Let $X = \{0, 1, 2, 3\}$, $E = \mathbb{R}$, and $P = \{x \in \mathbb{R} : x \geq 0\}$. Define $d : X \times X \to E$ by $d(x, y) = |x - y|$ and define $\sqsubseteq$ by

$$x \sqsubseteq y \quad \Leftrightarrow \quad x \geq y.$$

Then, (X, d) is a complete cone metric space and X is a partially ordered set. Define $q : X \times X \to E$ by the following :

$$\begin{aligned}
q(0,0) &= 0, & q(0,1) &= 1, & q(0,2) &= 1.1, & q(0,3) &= 0.5,\\
q(1,0) &= 1, & q(1,1) &= 0, & q(1,2) &= 0.1, & q(1,3) &= 0.5,\\
q(2,0) &= 1, & q(2,1) &= 1, & q(2,2) &= 0, & q(2,3) &= 0.5,\\
q(3,0) &= 1, & q(3,1) &= 0.5, & q(3,2) &= 0.6, & q(3,3) &= 0.
\end{aligned}$$

Then, it is easy to show that q is a $c-$distance.

Define $f : X \to X$ by $f0 = 1, f1 = 2, f2 = 2, f3 = 2$. Then, f is nondecreasing. If we take $x = 2, y = 0$, then, $q(f2, f0) = q(2, 1) = 1$ and

$$\begin{aligned}
a_1 q(2,0) + a_2 q(2, f2) + a_3 q(0, f0) + a_4 q(2, f0) &= a_1 q(2,0) + a_2 q(2,2) + a_3 q(0,1) + a_4 q(2,1)\\
&= a_1 + a_3 + a_4 \leq a_1 + a_3 + 2a_4 < 1
\end{aligned}$$

for any nonnegative real numbers $a_i\,(i = 1, 2, 3, 4)$ with $a_1 + a_2 + a_3 + 2a_4 < 1$. Hence, the contractive conditions of Theorem 2.3 and Theorem 2.2 are not satisfied and so Theorem 2.3 and Theorem 2.2 can not be applied to this example.

But Theorem 2.4 can be applied to this example. In fact we take $a_1 = 0.14, a_2 = a_3 = a_4 = 0$ and $a_5 = 0.85$. Then,

$$\begin{aligned}
1 &= q(f1, f0) < a_1 q(1,0) + a_5 q(0, f1) = 1.075,\\
1 &= q(f2, f0) < a_1 q(2,0) + a_5 q(0, f2) = 1.075,\\
1 &= q(f3, f0) < a_1 q(3,0) + a_5 q(0, f3) = 1.075.
\end{aligned}$$

If we take $x_0 = 3$ and $x_1 = 2$, then, $x_0 \sqsubseteq x_1 \sqsubseteq fx_0$. Clearly f is continuous. Hence, the hypotheses are satisfied and so by Theorem 2.4 f has a fixed point 2.

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