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Matrix Sturm-Liouville operators with boundary conditions dependent on the spectral parameter

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Abstract

Let L denote the operator generated in $L_2(\mathbb{R}_+, E)$ by the differential expression

$$l(y) = -y'' + Q(x)y, \quad x \in \mathbb{R}_+,$$

and the boundary condition $(A_0 + A_1\lambda)Y'(0,\lambda) - (B_0 + B_1\lambda)Y(0,\lambda) = 0$, where Q is a matrix-valued function and A_0, A_1, B_0, B_1 are non-singular matrices, with $A_0B_1 - A_1B_0 \neq 0$. In this paper, using the uniqueness theorems of analytic functions, we investigate the eigenvalues and the spectral singularities of L. In particular, we obtain the conditions on q under which the operator L has a finite number of the eigenvalues and the spectral singularities.

Keywords: Eigenvalues, spectral singularities, spectral analysis, Sturm-Liouville operator, non-selfadjoint matrix operator
2010 MSC: 34B24, 47A10, 34L40.

1. Introduction

Consider the boundary value problem (BVP)

$$-y'' + q(x)y = \lambda^2 y \,, \qquad 0 \leq x < \infty \tag{1.1}$$

$$y(0) = 0 \tag{1.2}$$

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in $L^2(\mathbb{R}_+)$, where q is a complex-valued function and $\lambda \in \mathbb{C}$ is a spectral parameter. The spectral theory of the above BVP was investigated by Naimark [22]. He showed the existence of the spectral singularities in the continuous spectrum of the (1.1)-(1.2). Also, the spectral singularities belong to the continuous spectrum and are the poles of the resolvent's kernel, but are not the eigenvalues of the BVP (1.1)-(1.2). Also he showed that if,

$$\int_0^\infty e^{\epsilon x} |q(x)| \, dx < \infty, \quad \epsilon > 0,$$

then the eigenvalues and spectral singularities are of a finite number and each of them is of a finite multiplicity. Pavlov [25] established the dependence of the structure of the spectral singularities of L_0 on the behavior of the potential function at infinity. He also proved that if

$$\sup_{x \in \mathbb{R}_+} \left[e^{\epsilon x^{1/2}} |q(x)| \right] < \infty, \ \epsilon > 0,$$

then the eigenvalues and spectral singularities are of a finite number and each of them is of a finite multiplicity.

In [19] the effect of the spectral singularities in the spectral expansion in terms of the principal vectors was considered. Some problems of spectral theory of differential and some other types of operators with spectral singularities were also studied in [1, 3, 4, 5, 6, 7, 16, 17]. The all above mentioned papers related with the differential and difference equations are of scalar coefficients. Spectral analysis of the selfadjoint differential and difference equations with matrix coefficients are studied in [10, 11, 14]. The spectral analysis of the non-selfadjoint operator, generated in $L^2(\mathbb{R}_+)$ by (1.1) and the boundary condition

$$\frac{y'(0)}{y(0)} = \frac{\beta_1 \lambda + \beta_0}{\alpha_1 \lambda + \alpha_0},$$

where $\alpha_i, \ \beta_i \in \mathbb{C}, \ i = 0, \ 1$ with $\alpha_0 \beta_1 - \alpha_1 \beta_0 \neq 0$ was investigated in detail by Bairamov *et al.* [8].

Let E be an n-dimensional ($n < \infty$) Euclidean space with the norm $\|.\|$ and let the Hilbert space of vector-valued functions with the values in E be denoted by $L^2(\mathbb{R}_+, E)$. In the $L^2(\mathbb{R}_+, E)$ space consider the BVP

$$-y'' + Q(x)\, y = \lambda^2 y \ , \qquad x \in \mathbb{R}_+ \tag{1.3}$$

$$y(0) = 0, \tag{1.4}$$

where Q is a non-selfadjoint matrix-valued function (i. e. $Q \neq Q^*$). It is clear that, the BVP (1.3), (1.4) is non-selfadjoint. In [24, 12] discrete spectrum of the non-selfadjoint matrix Sturm-Liouville operator was investigated.

Let us consider the BVP

$$-\, y'' + Q(x) y = \lambda^2 y, \ x \in \mathbb{R}_+, \tag{1.5}$$

$$(A_0 + A_1 \lambda) Y'(0, \lambda) - (B_0 + B_1 \lambda) Y(0, \lambda) = 0, \tag{1.6}$$

where Q is a non-singular matrix-valued function and A_0, A_1, B_0, B_1 are non-singular matrices such $A_0 B_1 - A_1 B_0 \neq 0$ in $L_2(\mathbb{R}_+, E)$. We will denote the operator generated in $L_2(\mathbb{R}_+)$ by (1.5)-(1.6). In this paper we discuss the discrete spectrum of L and prove that the operator L has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity if

$$Q \in AC(\mathbb{R}_+) \ , \ \lim_{x \to \infty} Q(x) = 0 \ , \ \int_0^\infty e^{\epsilon x^\delta} \left\| Q'(x) \right\| dx < \infty \tag{1.7}$$

for some $\epsilon > 0$ and $1/2 \leq \delta < 1$ holds. In particular, we show that the analogue of the Naimark condition for L is in the form

$$Q \in AC(\mathbb{R}_+)\ ,\ \lim_{x\to\infty} Q(x) = 0\ ,\ \int_0^\infty e^{\epsilon x} \left\|Q'(x)\right\| dx < \infty, \quad \epsilon > 0. \tag{1.8}$$

2. Jost Solution of (1.5)

We will denote the solution of (1.5) satisfying the condition

$$\lim_{x\to\infty} Y(x,\lambda)e^{-i\lambda x} = I, \qquad \lambda \in \overline{\mathbb{C}}_+ := \{\lambda : \lambda \in \mathbb{C}, funcIm\lambda \geq 0\} \tag{2.1}$$

by $E(x,\lambda)$. The solution $E(x,\lambda)$ is called the Jost solution of (1.5).

Under the condition

$$\int_0^\infty x \left\|Q(x)\right\| dx < \infty \tag{2.2}$$

the Jost solution has a representation

$$E(x,\lambda) = e^{i\lambda x} I + \int_x^\infty K(x,t)e^{i\lambda t} dt \tag{2.3}$$

for $\lambda \in \bar{\mathbb{C}}_+$, where the kernel matrix function $K(x,t)$ satisfies

$$K(x,t) = \frac{1}{2}\int_{\frac{x+t}{2}}^\infty Q(s)ds + \frac{1}{2}\int_x^{\frac{x+t}{2}} \int_{t+x-s}^{t+s-x} Q(s)K(s,v)dvds + \frac{1}{2}\int_{\frac{x+t}{2}}^\infty \int_s^{t+s-x} Q(s)K(s,v)dvds. \tag{2.4}$$

Moreover, $K(x,t)$ is continuously differentiable with respect to its arguments and

$$\left\|K(x,t)\right\| \leq c\sigma\left(\frac{x+t}{2}\right) \tag{2.5}$$

$$\left\|K_x(x,t)\right\| \leq \frac{1}{4}\left\|Q\left(\frac{x+t}{2}\right)\right\| + c\sigma\left(\frac{x+t}{2}\right) \tag{2.6}$$

$$\left\|K_t(x,t)\right\| \leq \frac{1}{4}\left\|Q\left(\frac{x+t}{2}\right)\right\| + c\sigma\left(\frac{x+t}{2}\right), \tag{2.7}$$

where $\sigma(x) = \int_x^\infty \left\|Q(s)\right\| ds$ and $c > 0$ is a constant.

Therefore, $E(x,\lambda)$ is analytic with respect to λ in $\mathbb{C}_+ := \{\lambda : \lambda \in \mathbb{C}_+,\ Im\lambda > 0\}$ and continuous on the real axis([2]. Chap.1; see also [18]. Chap.4; [20]. Chap.3).

We will denote the class of complex valued absolutely continuous functions in $\mathbb{R}_+$ by $AC(\mathbb{R}_+)$.

Lemma 2.1. *If*

$$Q \in AC(\mathbb{R}_+)\ ,\ \lim_{x\to\infty} Q(x) = 0\ \ ,\ \int_0^\infty x^2 \left\|Q'(x)\right\| dx < \infty \tag{2.8}$$

then $K_{xt}(x,t)$ exists.

$$K_{xt}(x,t) = -\frac{1}{8}Q'\left(\frac{x+t}{2}\right) - \frac{1}{2}\int\limits_{\frac{x+t}{2}}^{\infty} Q(s)K_t(s,t+s-x)ds - \frac{1}{4}Q\left(\frac{x+t}{2}\right)K\left(\frac{x+t}{2},\frac{x+t}{2}\right)$$
$$-\frac{1}{2}\int\limits_{x}^{\frac{x+t}{2}} Q(s)\left[K_t(s,x+t-s)+K_t\left(t-x+s\right)\right]ds. \tag{2.9}$$

The proof of the lemma is the direct consequence of (2.4). From (2.5)-(2.7) and (2.9) we obtain that

$$\|K_{xt}(0,t)\| \le c\left\{\left\|Q'\left(\frac{t}{2}\right)\right\| + \left\|Q\left(\frac{t}{2}\right)\right\| + \sigma\left(\frac{t}{2}\right)\right\}, \tag{2.10}$$

where $c > 0$ is a constant.

3. The Green function and the continuous spectrum

Let $\varphi(x,\lambda)$ denote the solution of (1.5) subject to the initial conditions $\varphi(0,\lambda) = A_0 + A_1\lambda$, $\varphi'(0,\lambda) = B_0 + B_1\lambda$. Therefore $\varphi(x,\lambda)$ is entire function of λ.

Let us define the following functions:

$$D_\pm(\lambda) = \varphi(0,\lambda)E_x\left(0,\pm\lambda\right) - \varphi'(0,\lambda)E(0,\pm\lambda) \qquad \lambda \in \bar{\mathbb{C}}_\pm, \tag{3.1}$$

where $\bar{\mathbb{C}}_\pm = \{\lambda : \lambda \in \mathbb{C},\ \pm\ Im\lambda \ge 0\}$. It is obvious that the functions $D_+(\lambda)$ and $D_-(\lambda)$ are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively and continuous on the real axis.

The resolvent of L defined by the following

$$R_\lambda(L)f = \int\limits_0^\infty G(x,t;\lambda)g(t)dt, \qquad g \in L_2(\mathbb{R}_+, E), \tag{3.2}$$

where

$$G(x,t;\lambda) = \begin{cases} G_+(x,t;\lambda), & \lambda \in \mathbb{C}_+ \\ G_-(x,t;\lambda), & \lambda \in \mathbb{C}_- \end{cases} \tag{3.3}$$

and

$$G_\pm(x,t;\lambda) = \begin{cases} -E(x,\pm\lambda)D_\pm^{-1}(\lambda)\varphi^T(t,\lambda), & 0 \le t \le x \\ -\varphi(x,\lambda)\left[D_\mp^T(\pm\lambda)\right]^{-1}E^T(t,\pm\lambda), & x \le t < \infty. \end{cases} \tag{3.4}$$

$D_+(\lambda)$ has the form of

$$D_+(\lambda) = iA_1\lambda^2 + A\lambda + B + \int\limits_0^\infty F(t)e^{i\lambda t}dt, \tag{3.5}$$

where

$$\begin{aligned} A &= iA_0 - A_1K(0,0) - B_1, \\ B &= -\left(A_0 + iB_1\right)K(0,0) - B_0 + iA_1K_x(0,0), \\ F(t) &= -B_0K(0,t) - iB_1K_t(0,t) + A_0K_x(0,t) + iA_1K_{xt}(0,t). \end{aligned} \tag{3.6}$$

Also using (2.5)-(2.7) and (2.10) we obtain that $F \in L_1(\mathbb{R}_+)$.

Theorem 3.1. *$D_+(\lambda)$ has the asymptotic behavior:*

$$D_+(\lambda) = iA_1\lambda^2 + A\lambda + B + o(1) \qquad , \ |\lambda| \to \infty \tag{3.7}$$

for $\lambda \in \bar{\mathbb{C}}_+$.

Proof. From K, K_x, K_t, $K_{xt} \in L_1(\mathbb{R}_+)$ and Riemann-Lebesque lemma we obtain (3.7). □

We will denote the continuous spectrum of L by σ_c. From [23, Theorem 2] we have

$$\sigma_c = \mathbb{R}. \tag{3.8}$$

4. The discrete spectrum of the operator L

Assume that the eigenvalues and the spectral singularities of the operator L by σ_d and σ_{ss} respectively. Let us suppose that

$$H_{\pm}(\lambda) = \det D_{\pm}(\lambda). \tag{4.1}$$

From (2.3) and (3.1)-(3.8)

$$\begin{aligned} \sigma_d &= \{\lambda : \lambda \in \mathbb{C}_+,\ H_+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbb{C}_-,\ H_-(\lambda) = 0\} \\ \sigma_{ss} &= \{\lambda : \lambda \in \mathbb{R}^*,\ H_+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbb{R}^*,\ H_-(\lambda) = 0\}, \end{aligned} \tag{4.2}$$

where $\mathbb{R}^* = \mathbb{R}\backslash \{0\}$.

Definition 4.1. The multiplicity of a zero of D_+ (or D_-) in $\bar{\mathbb{C}_+}$ (or $\bar{\mathbb{C}_-}$) is defined as the multiplicity of the corresponding eigenvalue and spectral singularity of L.

In order to investigate the quantitative properties of the eigenvalues and the spectral singularities of L, we observe the quantitative properties of the zeros of D_+ and D_- in $\bar{\mathbb{C}}_+$ and $\bar{\mathbb{C}}_-$, respectively. We will consider only the zeros of D_+ in $\bar{\mathbb{C}}_+$. A similar procedure may also be employed for zeros of D_- in $\bar{\mathbb{C}}_-$.

Let us define

$$M_1^{\pm} = \{\lambda : \lambda \in \mathbb{C}_{\pm},\ H_{\pm}(\lambda) = 0\}, M_2^{\pm} = \{\lambda : \lambda \in \mathbb{R},\ H_{\pm}(\lambda) = 0\}. \tag{4.3}$$

So from (4.2) we get

$$\sigma_d = M_1^+ \cup M_1^-, \quad \sigma_{ss} = M_2^+ \cup M_2^- - \{0\}. \tag{4.4}$$

Theorem 4.2. *Under the conditions in* (2.8)

(i) The discrete spectrum σ_d is a bounded, at most countable set and its limit points lie on the bounded subinterval of the real axis;

(ii) The set σ_{ss} is a bounded and its linear Lebesque measure is zero.

Proof. From Theorem 3.1 and uniqueness theorem of analytic functions [13] we have (i) and (ii). □

Theorem 4.3. *If*

$$Q \in AC(\mathbb{R}_+) \quad , \quad \lim_{x\to\infty} Q(x) = 0 \quad , \quad \int_0^{\infty} x^3 \left\| Q'(x) \right\| dx < \infty, \tag{4.5}$$

then

$$\sum_v |l_v|\, ln\, |l_v| < \infty, \tag{4.6}$$

where $|l_v|$ is the lengths of the boundary complementary intervals of σ_{ss}.

Proof. Let

$$r_{\pm}(\lambda) = (\lambda + i)^{-1} H_{\pm}(\lambda) \tag{4.7}$$

where $H_{\pm}(\lambda) = \det D_{\pm}(\lambda)$. $r_{\pm}$ has the same properties, since the function $H_{\pm}$ is analytic on $\mathbb{C}_{\pm}$ and continuous on $\bar{\mathbb{C}}_{\pm}$. From (3.7) we find that

$$\begin{aligned} \left| \frac{d}{d\lambda} r_{\pm}(\lambda) \right| &= \left| \frac{-1}{(\lambda+i)^2} H_{\pm}(\lambda) + \frac{1}{(\lambda+i)} \frac{d}{d\lambda} H_{\pm}(\lambda) \right| \le \frac{1}{|\lambda+i|^2} |H_{\pm}(\lambda)| + \frac{1}{|\lambda+i|} \left| \frac{d}{d\lambda} H_{\pm}(\lambda) \right| \\ &\le \frac{1}{|\lambda+i|^2} M |\lambda|^2 + \frac{1}{|\lambda+i|} S |\lambda| \le M + S, \end{aligned} \tag{4.8}$$

where $M, S > 0$ are constants. So $r_{\pm}$ satisfies Lipschitz condition and is not identically equal to zero, by Beurling's theorem we obtain (4.6) [9]. □

Theorem 4.4. *If*

$$Q \in AC(\mathbb{R}_+) \quad , \quad \lim_{x\to\infty} Q(x) = 0 \quad , \quad \int_0^\infty e^{\epsilon x} \left\| Q'(x) \right\| dx < \infty, \quad \epsilon > 0, \tag{4.9}$$

the operator L has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.

Proof. (2.5), (2.8), (2.10), (3.5) and (4.9) imply that the function D_+ has analytic continuation to the half-plane $\operatorname{Im}\lambda > -\epsilon/2$. Therefore, H_+ is analytic for $\operatorname{Im}\lambda > -\epsilon/2$ The limit points of its zeros on $\bar{\mathbb{C}}_+$ cannot lie in $\mathbb{R}$. So using Theorem 4.2, we have the finiteness of zeros of H_+ in $\bar{\mathbb{C}}_+$. A similar consequence holds for the function H_- in $\bar{\mathbb{C}}_-$. Then the proof of the theorem is the direct consequence of (4.4).

It is seen that the condition (4.9) guarantees the analytic continuation of H_+ and H_- from the real axis to the lower and the upper half-planes respectively. So the finiteness of the eigenvalues and the spectral singularities of L are obtained as a result of these analytic continuations. Consequently eigenvalues and spectral singularities have a finite number of elements with a finite multiplicity. □

Let us denote the sets of limit points of M_1^+ and M_2^+ by M_3^+ and M_4^+ respectively and the set of all zeros of D_+ with infinite multiplicity in $\bar{\mathbb{C}}_+$ by M_5^+. Analogously define the sets M_3^-, M_4^- and M_5^-.

It is explicit from the boundary uniqueness theorem of analytic functions that [13]

$$\begin{gathered} M_1^\pm \cap M_5^\pm = \varnothing, \quad M_3^\pm \subset M_2^\pm, \quad M_4^\pm \subset M_2^\pm, \\ M_5^\pm \subset M_2^\pm, \quad M_3^\pm \subset M_5^\pm, \quad M_4^\pm \subset M_5^\pm \end{gathered} \tag{4.10}$$

and $\mu(M_3^\pm) = \mu(M_4^\pm) = \mu(M_5^\pm) = 0$, where μ denote the Lebesgue measure on the real axis.

Theorem 4.5. *If*

$$Q \in AC(\mathbb{R}_+) \ , \ \lim_{x\to\infty} Q(x) = 0 \ , \ \int_0^\infty e^{\epsilon x^\delta} \left\| Q'(x) \right\| dx < \infty \tag{4.11}$$

for some $\epsilon > 0$ and $1/2 \le \delta < 1$ holds, then $M_5^+ = M_5^- = \varnothing$.

Proof. We will prove that $M_5^+ = \varnothing$. The case $M_5^- = \varnothing$ is similar. For sufficiently large $N > 0$ such that

$$\left| \int_{-\infty}^{-N} \frac{In\left|H_+(\lambda)\right|}{1+\lambda^2} d\lambda \right| < \infty, \quad \left| \int_N^\infty \frac{In\left|H_+(\lambda)\right|}{1+\lambda^2} d\lambda \right| < \infty. \tag{4.12}$$

H_+ is analytic in $\mathbb{C}_+$ and all of its derivatives are continuous on the real axis and

$$\left| \frac{d^n}{d\lambda^n} H_+(\lambda) \right| \le R_n, \quad n = 0, 1, 2, \cdots, \lambda \in \bar{\mathbb{C}}_+, |\lambda| < 2N, \tag{4.13}$$

where

$$R_0 = 4\left\|A_1\right\| N^2 + 2\left\|A\right\| N + \left\|B\right\| + \int_0^\infty \left\|F(t)\right\| dt,$$

$$R_1 = 4\left\|A_1\right\| N + \left\|A\right\| + \int_0^\infty t\left\|F(t)\right\| dt$$

$$R_2 = 2\|A_1\| + \int_0^\infty t^2 \|F(t)\|\, dt, \tag{4.14}$$

$$R_n = \int_0^\infty t^n \|F(t)\|\, dt, \quad n \geq 3.$$

Using (4.12), (4.13) and Pavlov's theorem [26], we get that M_5^+ satisfies

$$\int_0^h InT(s) d\mu(M_{5,s}^+) > -\infty \tag{4.15}$$

where $h > 0$, $T(s) = \inf_n \dfrac{R_n s^n}{n!}$, $\mu(M_{5,s}^+)$ is the linear Lebesque measure of $s-$neighborhood of M_5^+. We obtain that

$$R_n \leq R d^n n! n^{n(1/\delta - 1)}, \tag{4.16}$$

where R and d are constants depending on ϵ and δ. Substituting (4.16) in the definition of $T(s)$, we arrive at

$$T(s) = \inf_n \frac{R_n s^n}{n!} \leq R \exp\left(-\left(\frac{1}{\delta} - 1\right) e^{-1} d^{-\delta/(1-\delta)} s^{-\delta/(1-\delta)}\right). \tag{4.17}$$

Now by (4.15) and (4.17), we get

$$\int_0^h s^{-\delta/(1-\delta)} d\mu(M_{5,s}^+) < \infty. \tag{4.18}$$

Since $\delta/(1-\delta) \geq 1$, consequently (4.18) holds for arbitrary s if and only if $\mu(M_{5,s}^+) = 0$ or $M_5^+ = \varnothing$. □

Theorem 4.6. *Under the condition* (4.11) *the operator L has a finite number of the eigenvalues and the spectral singularities and each of them is of a finite multiplicity.*

Proof. To be able to prove the theorem we have to show that the functions D_+ and D_- have finite number of zeros with finite multiplicities in $\bar{\mathbb{C}}_+$ and $\bar{\mathbb{C}}_-$, respectively. We will prove it only for D_+. The case of D_- is similar.

It follows from (4.10) that $M_3^+ = M_4^+ = \varnothing$. So the bounded sets M_1^+ and M_2^+ have no limit points, that is, the D_+ has only a finite number of zeros in $\bar{\mathbb{C}}_+$. Since $M_5^+ = \varnothing$ these zeros are of a finite multiplicity. □

Theorem 4.7. *If the condition* (2.8) *is satisfied then the set σ_{ss} is of the first category.*

Proof. From the continuity of H_+ it is clear that the set M_2^+ is closed and is a set of Lebesgue measure zero which is of type $F\sigma$. According to Martin's theorem [21] there is a measurable set whose metric density exists and is different from 0 and 1 at every point M_2^+.

So, M_2^+ is of the category from the theorem due to Goffman [15]. We also have obviously same things for M_2^-. Consequently σ_{ss} *is of the first category.* □

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