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Functional inequalities in generalized quasi-Banach spaces

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Abstract

In this paper, we investigate the Hyers-Ulam stability of the following function inequalities

$$\|af(x) + bg(y) + ch(z)\| \leq \left\|Kp\left(\frac{ax + by + cz}{K}\right)\right\|,$$
$$\|af(x) + bg(y) + Kh(z)\| \leq \left\|Kp\left(\frac{ax + by}{K} + z\right)\right\|,$$

in generalized quasi-Banach spaces, where a, b, c, K are nonzero real numbers.

Keywords: Hyers-Ulam stability, additive functional inequality, generalized quasi-Banach space, additive mapping.
2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [14] in 1940, concerning the stability of group homomorphisms. Let $(G_1, .)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(., .)$. Given $\epsilon > 0$, does there exists a $\delta > 0$, such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x.y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with

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$d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the other words, Under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [5] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear. In 1978, Th. M. Rassias [9] proved the following theorem.

Theorem 1.1. *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for all $x, y \neq 0$, and (1.2) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear.

In 1991, Gajda [4] answered the question for the case $p > 1$, which was raised by Th. M. Rassias. On the other hand, J. M. Rassias [11] generalized the Hyers-Ulam stability result by presenting a weaker condition controlled by a product of different powers of norms.

Theorem 1.2 ([10, 12])**.** *If it is assumed that there exists constants $\Theta \geq 0$ and $p_1, p_2 \in \mathbb{R}$ such that $p = p_1 + p_2 \neq 1$, and $f : E \to E'$ is a mapping from a norm space E into a Banach space E' such that the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \Theta\|x\|^{p_1}\|y\|^{p_2}$$

for all $x, y \in E$, then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \frac{\Theta}{2-2^p}\|x\|^p,$$

for all $x \in E$. If, in addition, $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear.

In [8], Park et al. investigated the following inequalities

$$\begin{aligned}
\|f(x) + f(y) + f(z)\| &\leq \left\|2f\left(\frac{x+y+z}{2}\right)\right\|, \\
\|f(x) + f(y) + f(z)\| &\leq \|f(x+y+z)\|, \\
\|f(x) + f(y) + 2f(z)\| &\leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\|
\end{aligned}$$

in Banach spaces. Recently, Cho et al. [3] investigated the following functional inequality

$$\|f(x) + f(y) + f(z) \leq \left\|Kf\left(\frac{x+y+z}{K}\right)\right\| \qquad (0 < |K| < |3|)$$

in non-Archimedean Banach spaces. Lu and Park [6] investigated the following functional inequality

$$\left\|\sum_{i=1}^{N} f(x_i)\right\| \le \left\|Kf\left(\frac{\sum_{i=1}^{N}(x_i)}{K}\right)\right\| \qquad (0 < |K| \le N)$$

in Fréchet spaces.

In [7], we investigated the following functional inequalities

$$\|f(x)+f(y)+f(z)\| \le \left\|Kf\left(\frac{x+y+z}{K}\right)\right\| \qquad (0<|K|<3), \tag{1.3}$$

$$\|f(x)+f(y)+Kf(z)\| \le \left\|Kf\left(\frac{x+y}{K}+z\right)\right\| \qquad (0<K\ne 2) \tag{1.4}$$

and proved the Hyers-Ulam stability of the functional inequalities (1.3) and (1.4) in Banach spaces.

We consider the following functional inequalities

$$\|af(x)+bg(y)+ch(z)\| \le \left\|Kp\left(\frac{ax+by+cz}{K}\right)\right\|, \tag{1.5}$$

$$\|af(x)+bg(y)+Kh(z)\| \le \left\|Kp\left(\frac{ax+by}{K}+z\right)\right\|, \tag{1.6}$$

where a, b, c, K are nonzero scalars.

Now, we recall some basic facts concerning quasi-Banach spaces and some preliminary results.

Definition 1.3 ([2, 13])**.** Let X be a linear space. A quasi-norm is a real-valued function on X satisfying the following:

(1) $\|x\| \ge 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $\beta \ge 1$ such that $\|x+y\| \le \beta(\|x\|+\|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X.

A *quasi-Banach space* is a complete quasi-normed space.

Baak [1] generalized the concept of quasi-normed spaces.

Definition 1.4 ([1])**.** Let X be a linear space. A **generalized quasi-norm** is a real-valued function on X satisfying the following:

(1) $\|x\| \ge 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.

(3) There is a constant $\beta \ge 1$ such that $\|\sum_{j=1}^{\infty} x_j\| \le \sum_{j=1}^{\infty} \beta\|x_j\|$ for all $x_1, x_2, \dots \in X$ with $\sum_{j=1}^{\infty} x_j \in X$.

The pair $(X, \|\cdot\|)$ is called a *generalized quasi-normed space* if $\|\cdot\|$ is a generalized quasi-norm on X. The smallest possible C is called the *modulus of concavity* of $\|\cdot\|$.

A *generalized quasi-Banach space* is a complete generalized quasi-normed space.

In this paper, we show that the Hyers-Ulam stability of the functional inequalities (1.5) and (1.6) in generalized quasi-Banach spaces.

Throughout this paper, assume that X is a generalized quasi-normed vector space with generalized quasi-norm $\|\cdot\|$ and that $(Y, \|\cdot\|)$ is a generalized quasi-Banach space. Let β be the modulus of concavity of $\|\cdot\|$.

2. Hyers-Ulam stability of the functional inequality (1.5)

Throughout this section, assume that a, b, c and K are the nonzero scalars.

Proposition 2.1. *Let $f, g, h, p : X \to Y$ be mappings such that $g(0) = h(0) = p(0) = 0$ and*

$$\|af(x) + bg(y) + ch(z)\| \leq \left\| Kp\left(\frac{ax + by + cz}{K}\right)\right\| \tag{2.1}$$

for all $x, y, z \in X$. Then the mappings f, g and h are additive, for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (2.1), we get

$$\|af(0)\| \leq \|Kp(0)\| = 0.$$

So $f(0) = 0$.

Letting $(x, y, z) = \left(x, 0, -\frac{a}{c}x\right)$ in (2.1), we get

$$\left\|af(x) + ch(-\frac{a}{c}x)\right\| \leq \|Kp(0)\| = 0 \tag{2.2}$$

for all $x \in X$.

Replacing (x, y, z) by $(x, -\frac{a}{b}x, 0)$ in (2.1), we get

$$\left\|af(x) + bg\left(-\frac{a}{b}x\right)\right\| \leq \|Kp(0)\| = 0 \tag{2.3}$$

for all $x \in X$.

Replacing (x, y, z) by $(x, y, -\frac{ax+by}{c})$ in (2.1), we get

$$\left\|af(x) + bg(y) + ch(-\frac{ax + by}{c})\right\| \leq \|Kp(0)\| = 0 \tag{2.4}$$

for all $x, y \in X$.

By (2.2),(2.3) and (2.4), we get

$$f(x) - f(-\frac{b}{a}y) - f(x + \frac{b}{a}y) = 0 \tag{2.5}$$

for all $x, y \in X$.

Letting $x = 0$ in (2.5), we have $f(y) = -f(-y)$, and hence

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in X$. Since f is additive, it is clear that g and h are additive. And $f(x) = \frac{c}{a}h(\frac{a}{c}x)$, $g(x) = \frac{c}{b}h(\frac{b}{c}x)$, as desired. □

Next, we show that the Hyers-Ulam stability of the functional inequality (1.5).

Theorem 2.2. *Assume that mappings $f, g, h, p : X \to Y$ with $g(0) = h(0) = p(0) = 0$ satisfy the inequality*

$$\|af(x) + bg(y) + ch(z)\| \leq \left\| Kp\left(\frac{ax + by + cz}{K}\right)\right\| + \phi(x, y, z), \tag{2.6}$$

where $\phi : X^3 \to [0, \infty)$ satisfies $\phi(0, 0, 0) = 0$ and

$$\widetilde{\phi}(x, y, z) := \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j \phi\left(2^j x, 2^j y, 2^j z\right) < \infty$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\begin{aligned} \|f(x) - A(x)\| &\leq \frac{\beta^2}{2|a|}\left\{\widetilde{\phi}\left(x, -\frac{a}{b}x, 0\right) + \widetilde{\phi}\left(x, 0, -\frac{a}{c}x\right) + \widetilde{\phi}\left(2x, -\frac{a}{b}x, -\frac{a}{c}x\right)\right\} \\ \left\|g(x) - \frac{a}{b}A\left(\frac{b}{a}x\right)\right\| &\leq \frac{\beta^2}{2|b|}\left\{\widetilde{\phi}\left(-\frac{b}{a}x, x, 0\right) + \widetilde{\phi}\left(0, x, -\frac{b}{c}x\right) + \widetilde{\phi}\left(-\frac{b}{a}x, 2x, -\frac{b}{c}x\right)\right\} \\ \left\|h(x) - \frac{a}{c}A\left(\frac{c}{a}x\right)\right\| &\leq \frac{\beta^2}{2|c|}\left\{\widetilde{\phi}\left(0, -\frac{c}{b}x, x\right) + \widetilde{\phi}\left(-\frac{c}{a}x, 0, x\right) + \widetilde{\phi}\left(-\frac{c}{a}x, -\frac{c}{b}x, 2x\right)\right\} \end{aligned} \tag{2.7}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (2.6), we get $\|af(0)\| \leq \|Kp(0)\| + \phi(0, 0, 0) = \|Kp(0)\|$. So $f(0) = 0$.

Letting $(x, y, z) = (x, -\frac{a}{b}x, 0)$ in (2.6), we get

$$\left\|af(x) + bg\left(-\frac{a}{b}x\right)\right\| \leq \phi\left(x, -\frac{a}{b}x, 0\right) \tag{2.8}$$

for all $x \in X$.

Replacing (x, y, z) by $(x, 0, -\frac{a}{c}x)$ in (2.6), we get

$$\left\|af(x) + ch\left(-\frac{a}{c}x\right)\right\| \leq \phi\left(x, 0, -\frac{a}{c}x\right) \tag{2.9}$$

for all $x \in X$.

Replacing (x, y, z) by $(2x, -\frac{a}{b}x, -\frac{a}{c}x)$ in (2.6), we get

$$\left\|af(2x) + bg(-\frac{a}{b}x) + ch\left(-\frac{a}{c}x\right)\right\| \leq \phi\left(2x, -\frac{a}{b}x, -\frac{a}{c}x\right) \tag{2.10}$$

for all $x \in X$.

By (2.8),(2.9) and (2.10), it follows that

$$\|2f(x) - f(2x)\| \leq \frac{\beta}{|a|}\left(\phi\left(x, -\frac{a}{b}x, 0\right) + \phi\left(x, 0, -\frac{a}{c}x\right) + \phi\left(2x, -\frac{a}{b}x, -\frac{a}{c}x\right)\right) \tag{2.11}$$

for all $x \in X$. such that

$$\left\|f(x) - \frac{1}{2}f(2x)\right\| \leq \frac{\beta}{2|a|}\left(\phi\left(x, -\frac{a}{b}x, 0\right) + \phi\left(x, 0, -\frac{a}{c}x\right) + \phi\left(2x, -\frac{a}{b}x, -\frac{a}{c}x\right)\right) \tag{2.12}$$

for all $x \in X$.

It follows from (2.12) that

$$\begin{aligned} &\left\|(\frac{1}{2})^l f\left(2^l x\right) - (\frac{1}{2})^m f\left(2^m x\right)\right\| \\ &\leq \beta \sum_{j=l}^{m-1}\left\|(\frac{1}{2})^j f\left(2^j x\right) - (\frac{1}{2})^{j+1} f\left(2^{j+1} x\right)\right\| \\ &\leq \frac{\beta^2}{2|a|}\sum_{j=l}^{m-1}\left(\frac{1}{2}\right)^j\left[\phi\left(2^j x, -\frac{a}{b}2^j x, 0\right) + \phi\left(2^j x, 0, -\frac{a}{c}2^j x,\right) + \phi\left(2^{j+1}x, -\frac{a}{b}2^j x, -\frac{a}{c}2^j x,\right)\right] \end{aligned} \tag{2.13}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{(\frac{c}{a})^n f((\frac{a}{c})^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{(\frac{1}{2})^n f(2^n x)\}$ converges. We define the mapping $A : X \to Y$ by $A(x) = \lim_{n\to\infty}\{(\frac{1}{2})^n f(2^n x)\}$ for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$, we get

$$\begin{aligned} \|f(x) - A(x)\| &\le \frac{\beta^2}{2|a|}\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^j \\ &\quad \left[\phi\left(2^j x, -\frac{a}{b}2^j x, 0\right) + \phi\left(2^j x, 0, -\frac{a}{c}2^j x,\right) + \phi\left(2^{j+1}x, -\frac{a}{b}2^j x, -\frac{a}{c}2^j x,\right)\right] \\ &= \frac{\beta^2}{2|a|}\left\{\widetilde{\phi}\left(x, -\frac{a}{b}x, 0\right) + \widetilde{\phi}\left(x, 0, -\frac{a}{c}x\right) + \widetilde{\phi}\left(2x, -\frac{a}{b}x, -\frac{a}{c}x\right)\right\} \end{aligned} \tag{2.14}$$

for all $x \in X$.

Similarly, there exists a mapping $B : X \to Y$ such that $B(x) = \lim_{n\to\infty}\frac{1}{2^n}g(2^n x)$ and

$$\|g(x) - B(x)\| \le \frac{\beta^2}{2|b|}\left\{\widetilde{\phi}\left(-\frac{b}{a}x, x, 0\right) + \widetilde{\phi}\left(0, x, -\frac{b}{c}x\right) + \widetilde{\phi}\left(-\frac{b}{a}x, 2x, -\frac{b}{c}x\right)\right\} \tag{2.15}$$

for all $x \in X$.

We also obtain a mapping $C : X \to Y$ such that $C(x) := \lim_{n\to\infty}\frac{1}{2^n}h(2^n x)$, and

$$\|h(x) - C(x)\| \le \frac{\beta^2}{2|K|}\left\{\widetilde{\phi}\left(-\frac{c}{a}x, 0, x\right) + \widetilde{\phi}\left(0, -\frac{c}{b}x, x\right) + \widetilde{\phi}\left(-\frac{c}{a}x, -\frac{c}{b}x, 2x\right)\right\}$$

for all $x \in X$.

Next, we show that A is an additive mapping.

$$\begin{aligned} \|A(x) + A(y) - A(x+y)\| &= \lim_{n\to\infty}(\frac{1}{2})^n \left\|f(2^n x) + f(2^n y) - f(2^n(x+y))\right\| \\ &\le \beta\frac{1}{|a|}\lim_{n\to\infty}(\frac{1}{2})^n\left[\left\|af(2^n x) + bg\left(-\frac{a}{b}2^n x\right)\right\|\right. \\ &\quad + \left\|af(2^n y) + ch\left(-\frac{a}{c}2^n y\right)\right\| \\ &\quad \left.+ \left\|af(2^n(x+y)) + bg\left(-\frac{a}{b}2^n x\right) + ch\left(-\frac{a}{c}2^n y\right)\right\|\right] \\ &\le \beta\frac{1}{|a|}\lim_{n\to\infty}(\frac{1}{2})^n\left[\phi\left(2^n x, -\frac{a}{b}(2^n x), 0\right) + \phi\left(2^n y, 0, -\frac{a}{c}(2^n y)\right)\right. \\ &\quad \left.+\phi\left(2^n x + 2^n y, -\frac{a}{b}2^n x, -\frac{a}{c}2^n y\right)\right] \\ &= 0 \end{aligned}$$

for all $x, y \in X$. Thus the mapping $A : X \to Y$ is additive.

Now, we prove the uniqueness of A. Assume that $T : X \to Y$ is another additive mapping satisfying (2.7). We obtain

$$\begin{aligned} \|A(x) - T(x)\| &= \frac{1}{2^n}\left\|A(2^n x) - T(2^n x)\right\| \\ &\le \beta\cdot(\frac{1}{2})^n\left[\left\|A(2^n x) - f(2^n x)\right\|\right. \\ &\quad \left.+ \left\|T(2^n x) - f(2^n x)\right\|\right] \\ &\le \frac{\beta^3}{|a|}\left[\widetilde{\phi}\left(2^n x, -\frac{a}{b}2^n x, 0\right) + \widetilde{\phi}\left(2^n x, 0, \frac{a}{c}2^n x\right) + \widetilde{\phi}\left(2^n x, -\frac{a}{b}2^n x, -\frac{a}{c}2^n x\right)\right], \end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. Then we can conclude that $A(x) = T(x)$ for all $x \in X$.

Replacing (x,y,z) by $(2^n x, -\frac{a}{b}2^n x, 0)$ in (2.6), we get

$$\frac{1}{2^n}\left\|af(2^n x)+bg\left(-\frac{a}{b}2^n x\right)\right\| \le \frac{1}{2^n}\phi\left(2^n x, -\frac{a}{b}2^n x, 0\right),$$

and so

$$aA(x)+bB\left(-\frac{a}{b}x\right)=0$$

for all $x\in X$. Similarly $aA(x)+cC\left(-\frac{a}{c}x\right)=0$ for all $x\in X$. And $aA(x)+bB(y)+cC\left(-\frac{ax+by}{c}\right)=0$. Hence

$$aA(x)-aA\left(-\frac{b}{a}y\right)-aA\left(x+\frac{b}{a}y\right)=0 \tag{2.16}$$

for all $x,y\in X$.

Letting $x=y=0$ in (2.16), we have $A(0)=0$. Letting $x=0$ in (2.16), $A(-y)=-A(y)$, such that $B(x)=\frac{a}{b}A\left(\frac{b}{a}x\right)$ and $C(x)=\frac{a}{c}A\left(\frac{c}{a}x\right)$. Therefore the inequalities (2.7) hold. □

Corollary 2.3. *Let q and θ be positive real numbers with $0<q<1$. Let $f,g,h,p: X\to Y$ be mappings with $g(0)=h(0)=p(0)=0$ satisfying*

$$\|af(x)+bg(y)+ch(z)\| \le \left\|Kp\left(\frac{ax+by+cz}{K}\right)\right\| + \theta(\|x\|^q+\|y\|^q+\|z\|^q)$$

for all $x,y,z\in X$. Then there exists a unique additive mapping $A: X\to Y$ such that

$$\begin{aligned}
\|f(x)-A(x)\| &\le \frac{\beta^2\theta}{|a|}\frac{2^{1-q}}{2^{1-q}-1}\left(1+2^{q-1}+\frac{|a|^q}{|b|^q}+\frac{|a|^q}{|c|^q}\right)\|x\|^q\\
\left\|g(x)-\frac{a}{b}A\left(\frac{b}{a}x\right)\right\| &\le \frac{\beta^2\theta}{|b|}\frac{2^{1-q}}{2^{1-q}-1}\left(1+2^{q-1}+\frac{|b|^q}{|a|^q}+\frac{|b|^q}{|c|^q}\right)\|x\|^q\\
\left\|h(x)-\frac{a}{K}A\left(\frac{K}{a}x\right)\right\| &\le \frac{\beta^2\theta}{|c|}\frac{2^{1-q}}{2^{1-q}-1}\left(1+2^{q-1}+\frac{|c|^q}{|b|^q}+\frac{|c|^q}{|a|^q}\right)\|x\|^q
\end{aligned}$$

for all $x\in X$.

3. Hyers-Ulam stability of the functional inequality (1.6)

Throughout this section, assume that K,a,b are nonzero real numbers with $|a|\ge K$.

Proposition 3.1. *Let $f,g,h,p: X\to Y$ be mappings with $p(0)=0$ such that*

$$\|af(x)+bg(y)+Kh(z)\| \le \left\|Kp\left(\frac{ax+by}{K}+z\right)\right\| \tag{3.1}$$

for all $x,y,z\in X$. Then the mappings $f: X\to Y$ is additive.

Proof. Letting $x=y=z=0$ in (3.1), we get

$$\|af(0)\| \le \|Kp(0)\| = 0.$$

So $f(0)=0$.

Letting $y=-\frac{a}{b}x$ and $z=0$ in (3.1), we get

$$\left\|af(x)+bg\left(-\frac{a}{b}x\right)\right\| \le \|Kp(0)\| = 0 \tag{3.2}$$

for all $x \in X$. So $f(x) = -\frac{b}{a} g(-\frac{a}{b} x)$ for all $x \in X$.

Replacing x by $-x$ and letting $y = 0$ and $z = \frac{a}{K} x$ in (3.1), we get

$$\left\| af(-x) + Kh\left(\frac{a}{K}x\right) \right\| \leq \|Kp(0)\| = 0 \tag{3.3}$$

for all $x \in X$. So $f(-x) = -\frac{K}{a} h(\frac{a}{K} x)$ for all $x \in X$.

Thus we get

$$\|f(x) + f(-x)\| = \frac{1}{|a|} \left\| af(0) + bg\left(-\frac{a}{b}x\right) + Kh\left(\frac{a}{K}x\right) \right\| \leq \frac{1}{|a|} |K| \|p(0)\| = 0$$

for all $x \in X$. So $f(-x) = -f(x)$ for all $x \in X$. Similarly, we can show that $g(-x) = -g(x)$ and $h(-x) = -h(x)$.

Letting $z = \frac{-x-y}{K}$ in (3.1), we get

$$\begin{aligned} \left\| af(x) + bg(y) - Kh\left(\frac{ax+by}{K}\right) \right\| &= \left\| af(x) + bg(y) + Kh\left(\frac{-ax-by}{K}\right) \right\| \\ &\leq \|Kp(0)\| = 0 \end{aligned}$$

for all $x, y \in X$. By (3.2) and (3.3),

$$af(x) - af(-\frac{b}{a}y) - af(x + \frac{b}{a}y) = 0 \tag{3.4}$$

for all $x, y \in X$. Thus

$$f(x) + f(y) - f(x+y) = 0$$

for all $x, y \in X$, as desired. □

Theorem 3.2. *Assume that mappings $f, g, h, p : X \to Y$ with $g(0) = h(0) = p(0) = 0$ satisfy the inequality*

$$\|af(x) + bg(y) + Kh(z)\| \leq \left\| Kp\left(\frac{ax+by}{K} + z\right) \right\| + \phi(x, y, z), \tag{3.5}$$

where $\phi : X^3 \to [0, \infty)$ satisfies $\phi(0,0,0) = 0$ and

$$\widetilde{\phi}(x, y, z) := \sum_{j=1}^{\infty} \frac{1}{2^j} \phi\left(2^j x, 2^j y, 2^j z\right) < \infty$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\begin{aligned} &\|f(x) - A(x)\| \leq \frac{\beta^2}{2|a|} \left\{ \widetilde{\phi}\left(x, -\frac{a}{b}x, 0\right) + \widetilde{\phi}\left(x, 0, -\frac{a}{K}x\right) + \widetilde{\phi}\left(2x, -\frac{a}{b}x, -\frac{a}{K}x\right) \right\} \\ &\left\| g(x) - \frac{a}{b} A\left(\frac{b}{a}x\right) \right\| \leq \frac{\beta^2}{2|b|} \left\{ \widetilde{\phi}\left(-\frac{b}{a}x, x, 0\right) + \widetilde{\phi}\left(0, x, -\frac{b}{K}x\right) + \widetilde{\phi}\left(-\frac{b}{a}x, 2x, -\frac{b}{K}x\right) \right\} \\ &\left\| h(x) - \frac{a}{K} A\left(\frac{K}{a}x\right) \right\| \leq \frac{\beta^2}{2|K|} \left\{ \widetilde{\phi}\left(-\frac{K}{a}x, 0, x\right) + \widetilde{\phi}\left(0, -\frac{K}{b}x, x\right) + \widetilde{\phi}\left(-\frac{K}{a}x, -\frac{K}{b}x, 2x\right) \right\} \end{aligned} \tag{3.6}$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (3.5), we get $\|af(0)\| \leq \|Kp(0)\| + \phi(0,0,0) = 0$. So $f(0) = 0$.

Letting $x = x, y = -\frac{ax}{b}, z = 0$ in (3.5), we obtain

$$\left\| af(x) + bg\left(-\frac{a}{b}x\right) \right\| \leq \phi\left(x, -\frac{a}{b}x, 0\right)$$

for all $x \in X$.

Letting $y = 0, z = -\frac{ax}{K}$ in (3.5), we obtain

$$\left\|af(x) + Kh\left(-\frac{a}{K}x\right)\right\| \le \phi\left(x, 0, -\frac{a}{K}x\right)$$

for all $x \in X$.

Letting $x = 2x, y = -\frac{ax}{b}, z = -\frac{a}{K}x$ in (3.5), we get

$$\left\|af\left(2x\right) + bg\left(-\frac{a}{b}x\right) + Kh\left(-\frac{a}{K}x\right)\right\| \le \phi\left(2x, -\frac{a}{b}x, -\frac{a}{K}x\right)$$

for all $x \in X$. So

$$\begin{aligned}\left\|f(x) - \frac{1}{2}f\left(2x\right)\right\| &\le \frac{\beta}{2|a|}\left[\left\|af(x) + bg\left(-\frac{a}{b}x\right)\right\| + \left\|af\left(x\right) + Kh\left(-\frac{a}{K}x\right)\right\|\right.\\ &\quad \left. + \left\|af(2x) + bg\left(-\frac{a}{b}x\right) + Kh\left(-\frac{a}{K}x\right)\right\|\right]\\ &\le \frac{\beta}{2|a|}\left[\phi\left(x, -\frac{a}{b}x, 0\right) + \phi\left(x, , 0 - \frac{a}{K}x\right) + \phi\left(2x, -\frac{a}{b}x, -\frac{a}{K}x\right)\right]\end{aligned} \tag{3.7}$$

for all $x \in X$. It follows from (3.7) that

$$\begin{aligned}&\left\|\frac{1}{2^l}f\left(2^l x\right) - \frac{1}{2^m}f\left(2^m x\right)\right\|\\ &\le \beta\sum_{j=l}^{m-1}\left\|\frac{1}{2^j}f\left(2^j x\right) - \frac{1}{2^{j+1}}f\left(2^{j+1}x\right)\right\|\\ &\le \frac{\beta^2}{2|a|}\sum_{j=l}^{m-1}\frac{1}{2^j}\left[\phi\left(2^j x, -\frac{a}{b}2^j x, 0\right) + \phi\left(2^j x, , 0 - \frac{a}{K}2^j x\right) + \phi\left(2^{j+1}x, -\frac{a}{b}2^j x, -\frac{a}{K}2^j x\right)\right]\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\left\{\frac{1}{2^n}f(2^n x)\right\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\left\{\frac{1}{2^n})f(2^n x)\right\}$ converges. So we may define the mapping $A : X \to Y$ by $A(x) = \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$ for all $x \in X$.

Moreover, by letting $l = 0$ and passing the limit $m \to \infty$, we get the first formula of (3.6).

Similarly, there exists a mapping $B : X \to Y$ such that $B(x) = \lim_{n\to\infty}\frac{1}{2^n}g(2^n x)$ and

$$\|g(x) - B(x)\| \le \frac{\beta^2}{2|b|}\left\{\widetilde{\phi}\left(-\frac{b}{a}x, x, 0\right) + \widetilde{\phi}\left(0, x, -\frac{b}{K}x\right) + \widetilde{\phi}\left(-\frac{b}{a}x, 2x, -\frac{b}{K}x\right)\right\} \tag{3.8}$$

for all $x \in X$.

We also obtain a mapping $C : X \to Y$ such that $C(x) := \lim_{n\to\infty}\frac{1}{2^n}h(2^n x)$, and

$$\|h(x) - C(x)\| \le \frac{\beta^2}{2|K|}\left\{\widetilde{\phi}\left(-\frac{K}{a}x, 0, x\right) + \widetilde{\phi}\left(0, -\frac{K}{b}x, x\right) + \widetilde{\phi}\left(-\frac{K}{a}x, -\frac{K}{b}x, 2x\right)\right\}$$

for all $x \in X$.

Now, we show that A is additive.

$$\begin{aligned}\|A(x) + A(y) - A(x+y)\| &= \lim_{n\to\infty}\frac{1}{2^n}\left\|f\left(2^n x\right) + f\left(2^n y\right) - f\left(2^n(x+y)\right)\right\|\\ &\le \frac{\beta}{|a|}\lim_{n\to\infty}\frac{1}{2^n}\left[\left\|af\left(2^n x\right) + bg\left(-\frac{a}{b}2^n x\right)\right\|\right.\\ &\quad + \left\|af\left(2^n y\right) + Kh\left(-\frac{a}{K}2^n y\right)\right\|\end{aligned}$$

$$\begin{aligned}
&+ \left\|af\left(2^n(x+y)\right) + bg\left(-\frac{a}{b}2^n x\right) + Kh\left(-\frac{a}{K}2^n y\right)\right\|\Big] \\
&\leq \frac{\beta}{|a|} \lim_{n\to\infty} \frac{1}{2^n}\Big[\phi\left(2^n x, -\frac{a}{b}(2^n x), 0\right) + \phi\left(2^n y, 0, -\frac{a}{K}(2^n y)\right) \\
&+\phi\left(2^n x + 2^n y, -\frac{a}{b}2^n y, -\frac{a}{K}2^n y\right)\Big] \\
&= 0
\end{aligned}$$

for all $x, y \in X$. So the mapping $A : X \to Y$ is an additive mapping.

Now, we show that the uniqueness of A. Assume that $T : X \to Y$ is another additive mapping satisfying (3.6). Then we get

$$\begin{aligned}
\|A(x) - T(x)\| &= \lim_{n\to\infty} \frac{1}{2^n}\|A(2^n x) - T(2^n x)\| \\
&\leq \beta \lim_{n\to\infty} \frac{1}{2^n}\left[\|A(2^n x) - f(2^n x)\| + \|T(2^n x) - f(2^n x)\|\right] \\
&\leq \beta\frac{\beta^2}{|a|} \lim_{n\to\infty}\left[\widetilde{\phi}\left(x, -\frac{a}{b}x, 0\right) + \widetilde{\phi}\left(x, 0, -\frac{a}{K}x\right) + \widetilde{\phi}\left(2x, -\frac{a}{b}x, -\frac{a}{K}x\right)\right] \\
&= 0
\end{aligned}$$

for all $x \in X$. Thus we may conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A. So the mapping $A : X \to Y$ is a unique additive mapping satisfying (3.6).

Replacing (x, y, z) by $(2^n x, -\frac{a}{b}2^n x, 0)$ in (3.5), we get

$$\frac{1}{2^n}\left\|af(2^n x) + bg\left(-\frac{a}{b}2^n x\right)\right\| \leq \frac{1}{2^n}\phi\left(2^n x, -\frac{a}{b}2^n x, 0\right),$$

and so

$$aA(x) + bB\left(-\frac{a}{b}x\right) = 0$$

for all $x \in X$. Similarly $aA(x) + KC\left(-\frac{a}{K}x\right) = 0$ for all $x \in X$. And $aA(x) + bB(y) + KC\left(-\frac{ax+by}{K}\right) = 0$. Hence

$$aA(x) - aA\left(-\frac{b}{a}y\right) - aA\left(x + \frac{b}{a}y\right) = 0 \tag{3.9}$$

for all $x, y \in X$.

Letting $x = y = 0$ in (3.9), we have $A(0) = 0$. Letting $x = 0$ in (3.9), $A(-y) = -A(y)$, such that $B(x) = \frac{a}{b}A\left(\frac{b}{a}x\right)$ and $C(x) = \frac{a}{K}A\left(\frac{K}{a}x\right)$. □

Corollary 3.3. *Let q, θ and K be positive real numbers with $q > 1$. Let $f, h, g, p : X \to Y$ be mappings with $h(0) = g(0) = p(0)$ satisfying*

$$\|af(x) + bg(y) + Kh(z)\| \leq \left\|Kp\left(\frac{ax + by}{K} + z\right)\right\| + \theta(\|x\|^q + \|y\|^q + \|z\|^q)$$

for all $x, y, z \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\begin{aligned}
\|f(x) - A(x)\| &\leq \frac{\beta^2\theta}{|a|}\frac{2}{2^q - 1}\left(1 + 2^{q-1} + \frac{|a|^q}{|b|^q} + \frac{|a|^q}{|c|^q}\right)\|x\|^q \\
\left\|g(x) - \frac{a}{b}A\left(\frac{b}{a}x\right)\right\| &\leq \frac{\beta^2\theta}{|b|}\frac{2}{2^q - 1}\left(1 + 2^{q-1} + \frac{|b|^q}{|a|^q} + \frac{|b|^q}{|c|^q}\right)\|x\|^q \\
\left\|h(x) - \frac{a}{K}A\left(\frac{K}{a}x\right)\right\| &\leq \frac{\beta^2\theta}{|K|}\frac{2}{2^q - 1}\left(1 + 2^{q-1} + \frac{|K|^q}{|b|^q} + \frac{|K|^q}{|a|^q}\right)\|x\|^q
\end{aligned}$$

for all $x \in X$.

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