Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 1190–1211
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Almost automorphic functions on time scales and almost automorphic solutions to shunting inhibitory cellular neural networks on time scales

Yongkun Li*, Bing Li, Xiaofang Meng

Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, People's Republic of China.

Communicated by J. J. Nieto

Abstract

In this paper, we first propose a new definition of almost automorphic functions on almost periodic time scales and study some of their basic properties. Then we prove a result ensuring the existence of an almost automorphic solution for both the linear nonhomogeneous dynamic equation on time scales and its associated homogeneous equation, assuming that the latter admits an exponential dichotomy. Finally, as an application of our results, we establish the existence and global exponential stability of almost automorphic solutions to a class of shunting inhibitory cellular neural networks with time-varying delays on time scales. Our results about the shunting inhibitory cellular neural networks with time-varying delays on time scales are new both for the case of differential equations (the time scale $\mathbb{T} = \mathbb{R}$) and difference equations (the time scale $\mathbb{T} = \mathbb{Z}$).

Keywords: Time scales, Almost automorphic functions, Dynamic equations, Exponential dichotomy.
2010 MSC: 34N05, 34K14, 43A60, 92B20.

1. Introduction

The theory of time scales was initiated by Hilger [17] in his Ph.D. thesis in 1988, which unifies the continuous and discrete cases. The theory of dynamic equations on time scales contains, links and extends the classical theory of differential and difference equations. In the recent years, there has been an increasing interest in studying the existence of periodic solutions and almost periodic solutions of various dynamic

*Corresponding author
Email addresses: yklie@ynu.edu.cn (Yongkun Li), bli123@126.com (Bing Li), xfmeng@126.com (Xiaofang Meng)

Received 2015-07-01

equations on time scales; we refer the reader to the papers [1, 2, 3, 8, 12, 14, 15, 16, 18, 22, 27, 29, 30, 33, 34, 35, 38, 40, 41, 43, 45, 46, 47, 48, 50].

The concept of almost automorphy was introduced in the literature by S. Bochner in 1955 in the context of differential geometry [5] (see also Bochner [6, 7]). Since then, this concept has been extended in various directions (see, e.g., [4, 11]).

Remark 1.1. Although every almost periodic function is almost automorphic, the converse is not true.

In order to study almost periodic, pseudo almost periodic and almost automorphic dynamic equations on time scales, the following concept of almost periodic time scales was proposed in [24].

Definition 1.2 ([24])**.** A time scale $\mathbb{T}$ is called almost periodic if

$$\Pi = \big\{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\big\} \neq \{0\}.$$

Based on Definition 1.2, almost periodic functions [24], pseudo almost periodic functions [25], almost automorphic functions [31, 39] and weighted piecewise pseudo almost automorphic functions [36] on time scales were defined successfully. For example, the authors of [39] proposed the following concept of almost automorphic functions on time scales.

Definition 1.3 ([39])**.** Let $\mathbb{T}$ be an almost periodic time scale and $\mathbb{X}$ be a Banach space. A bounded continuous function $f : \mathbb{T} \to \mathbb{X}$ is said to be almost automorphic if for every sequence $(s_n^{'}) \subset \Pi$ there exists a subsequence $(s_n) \subset (s_n^{'})$ such that

$$\lim_{n\to\infty} f(t + s_n) = \bar{f}(t)$$

is well defined for each $t \in \mathbb{T}$, and

$$\lim_{n\to\infty} \bar{f}(t - s_n) = f(t)$$

for each $t \in \mathbb{T}$.

A continuous function $f : \mathbb{T} \times \mathbb{X} \to \mathbb{X}$ is said to be almost automorphic if $f(t, x)$ is almost automorphic in $t \in \mathbb{T}$ uniformly in $x \in B$, where B is any bounded subset of $\mathbb{X}$.

The concept of almost periodic time scales in sense of Definition 1.2 is very restrictive, in the sense that it is a kind of periodic time scale (see [18]). This excludes many interesting time scales. Therefore, it is a challenging and important problem in theory and applications to find a new concept of almost periodic time scales. Recently, some new types of almost periodic time scales were introduced in [20, 28, 37]. However, there has been no definition of almost automorphic functions on these new almost periodic time scales yet.

Motivated by the above discussions, the main aim of this paper is to propose a new definition of almost automorphic functions on almost periodic time scales and study the existence of an almost automorphic solution for both the linear nonhomogeneous dynamic equation on time scales and its associated homogeneous equation. As an application of our results, the existence and global exponential stability of almost automorphic solutions to a class of shunting inhibitory cellular neural networks with time-varying delays on time scales is established.

The organization of this paper is as follows: In Section 2, we introduce some notations and definitions and state some preliminary results which are needed in later sections. In Section 3, we first introduce a new definition of almost automorphic functions on time scales, then we discuss some of their properties. Finally, we propose two open problems. In Section 4, we prove a result ensuring the existence of an almost automorphic solution for both the linear nonhomogeneous dynamic equation on time scales and its associated homogeneous equation, assuming that the associated homogeneous equation admits an exponential dichotomy. In Section 5, as an application of our results, we establish the existence and global exponential stability of almost automorphic solutions to a class of shunting inhibitory cellular neural networks with time-varying delays on time scales. In Section 6, we give examples to illustrate the feasibility and effectiveness of the results obtained in Section 5. Finally, we draw a conclusion in Section 7.

2. Preliminaries

In this section, we shall first recall some fundamental definitions and lemmas which are used in what follows.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$ for all $t \in \mathbb{T}$, while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T}, s < t\}$ for all $t \in \mathbb{T}$. Finally, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. A function $r : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)r(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and right-dense continuous functions $r : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{r \in \mathcal{R} : 1 + \mu(t)r(t) > 0, \ \forall t \in \mathbb{T}\}$. Let A be an $m \times n$-matrix-valued function on $\mathbb{T}$. We say that A is rd-continuous on $\mathbb{T}$ if each entry of A is rd-continuous on $\mathbb{T}$. We denote the class of all rd-continuous $m \times n$ matrix-valued functions on $\mathbb{T}$ by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}^{m\times n})$. An $n \times n$-matrix-valued function A on a time scale $\mathbb{T}$ is called regressive (with respect to $\mathbb{T}$) provided $(1 + \mu(t)A(t))$ is invertible for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$. For more knowledge of time scales, one can refer to [9, 10].

Definition 2.1 ([23])**.** Let $x \in \mathbb{R}^n$ and $A(t)$ be an $n \times n$ rd-continuous matrix on $\mathbb{T}$. The linear system

$$x^{\Delta}(t) = A(t)x(t), \ t \in \mathbb{T} \tag{2.1}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constants k, α, a projection P and a fundamental solution matrix $X(t)$ of (2.1), satisfying

$$|X(t)PX^{-1}(\sigma(s))| \le ke_{\ominus\alpha}(t, \sigma(s)), \ s, t \in \mathbb{T}, t \ge \sigma(s),$$
$$|X(t)(I - P)X^{-1}(\sigma(s))| \le ke_{\ominus\alpha}(\sigma(s), t), \ s, t \in \mathbb{T}, t \le \sigma(s),$$

where $|\cdot|$ is a matrix norm on $\mathbb{T}$, that is, if $A = (a_{ij})_{n\times n}$ then we can take $|A| = (\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}})$.

Definition 2.2 ([20])**.** Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales. We define

$$\operatorname{dist}(\mathbb{T}_1, \mathbb{T}_2) = \max\{\sup_{t\in\mathbb{T}_1}\{\operatorname{dist}(t, \mathbb{T}_2)\}, \sup_{t\in\mathbb{T}_2}\{\operatorname{dist}(t, \mathbb{T}_1)\}\}$$

where $\operatorname{dist}(t, \mathbb{T}_2) = \inf\limits_{s\in\mathbb{T}_2}\{|t - s|\}$, $\operatorname{dist}(t, \mathbb{T}_1) = \inf\limits_{s\in\mathbb{T}_1}\{|t - s|\}$. Let $\tau \in \mathbb{R}$ and $\mathbb{T}$ be a time scale. We define

$$\operatorname{dist}(\mathbb{T}, \mathbb{T}_\tau) = \max\{\sup_{t\in\mathbb{T}}\{\operatorname{dist}(t, \mathbb{T}_\tau)\}, \sup_{t\in\mathbb{T}_\tau}\{\operatorname{dist}(t, \mathbb{T})\}\},$$

where $\mathbb{T}_\tau := \mathbb{T} \cap \{\mathbb{T} - \tau\} = \mathbb{T} \cap \{t - \tau : \forall t \in \mathbb{T}\}$, $\operatorname{dist}(t, \mathbb{T}_\tau) = \inf\limits_{s\in\mathbb{T}_\tau}\{|t - s|\}$, $\operatorname{dist}(t, \mathbb{T}) = \inf\limits_{s\in\mathbb{T}}\{|t - s|\}$.

Definition 2.3 ([20])**.** A time scale $\mathbb{T}$ is called an almost periodic time scale if for every $\varepsilon > 0$ there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains a $\tau(\varepsilon)$ such that $\mathbb{T}_\tau \neq \emptyset$ and $\operatorname{dist}(\mathbb{T}, \mathbb{T}_\tau) < \varepsilon$, that is, for any $\varepsilon > 0$, the set $\Pi(\mathbb{T}, \varepsilon) = \{\tau \in \mathbb{R}, \operatorname{dist}(\mathbb{T}, \mathbb{T}_\tau) < \varepsilon\}$ is relatively dense. τ is called the ε-translation number of $\mathbb{T}$.

Obviously, if $\mathbb{T}$ is an almost periodic time scale, then $\inf \mathbb{T} = -\infty$ and $\sup \mathbb{T} = +\infty$; if $\mathbb{T}$ is a periodic time scale (see [20]), then $\operatorname{dist}(\mathbb{T}, \mathbb{T}_\tau) = 0$, that is $\mathbb{T} = \mathbb{T}_\tau$.

Remark 2.4. One can easily see that if a time scale is an almost periodic time scale under Definition 1.2, then it is also an almost periodic time scale under Definition 2.3.

Lemma 2.5 ([20]). *Let $\mathbb{T}$ be an almost periodic time scale under Definition 2.3. Then*

(*i*) *if $\tau \in \Pi(\mathbb{T}, \varepsilon)$, then $t + \tau \in \mathbb{T}$ for all $t \in \mathbb{T}_\tau$;*

(*ii*) *if $\varepsilon_1 < \varepsilon_2$, then $\Pi(\mathbb{T}, \varepsilon_1) \subset \Pi(\mathbb{T}, \varepsilon_2)$;*

(*iii*) *if $\tau \in \Pi(\mathbb{T}, \varepsilon)$, then $-\tau \in \Pi(\mathbb{T}, \varepsilon)$ and $\mathrm{dist}(\mathbb{T}_\tau, \mathbb{T}) = \mathrm{dist}(\mathbb{T}_{-\tau}, \mathbb{T})$;*

(*iv*) *if $\tau_1, \tau_2 \in \Pi(\mathbb{T}, \varepsilon)$, then $\tau_1 + \tau_2 \in \Pi(\mathbb{T}, 2\varepsilon)$.*

3. Almost periodic time scales and almost automorphic functions

In this section, we first introduce a new concept of almost periodic time scales and a new definition of almost automorphic functions on time scales, then we discuss some of their properties. The following definition is a slightly modified version of Definition 2 in [28].

Definition 3.1. A time scale $\mathbb{T}$ is called almost periodic if

(*i*) $\Pi := \{\tau \in \mathbb{R} : \mathbb{T}_\tau \neq \emptyset\} \neq \{0\}$ and $\widetilde{\mathbb{T}} \neq \emptyset$,

(*ii*) if $\tau_1, \tau_2 \in \Pi$, then $\tau_1 \pm \tau_2 \in \Pi$,

where $\mathbb{T}_\tau = \mathbb{T} \cap \{\mathbb{T} - \tau\} = \mathbb{T} \cap \{t - \tau : \forall t \in \mathbb{T}\}$ and $\widetilde{\mathbb{T}} = \bigcap\limits_{\tau \in \Pi} \mathbb{T}_\tau$.

It is obvious that if $\tau \in \Pi$, then $\pm\tau \in \Pi$ and $t \pm \tau \in \mathbb{T}$ for all $t \in \widetilde{\mathbb{T}}$.

Remark 3.2. Noticing the fact that $\Pi = \{\tau \in \mathbb{R} : \mathbb{T}_\tau \neq \emptyset\} \supset \Pi(\mathbb{T}, \varepsilon)$, from Lemma 2.5, one can easily see that if a time scale is an almost periodic time scale under Definition 2.3, then it is also an almost periodic time scale under Definition 3.1.

Definition 3.3. Let $\mathbb{T}$ be an almost periodic time scale and $\mathbb{X}$ be a Banach space. A bounded rd-continuous function $f : \mathbb{T} \to \mathbb{X}$ is said to be almost automorphic if for every sequence $(s_n^{'}) \subset \Pi$ there exists a subsequence $(s_n) \subset (s_n^{'})$ such that

$$\lim_{n\to\infty} f(t + s_n) = \bar{f}(t)$$

is well defined for each $t \in \widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty} \bar{f}(t - s_n) = f(t)$$

for each $t \in \widetilde{\mathbb{T}}$.

Denote by $AA(\mathbb{T}, \mathbb{X})$ the set of all almost automorphic functions on the time scale $\mathbb{T}$.

Remark 3.4. From Definition 3.3, one can easily see that if a function $f : \mathbb{T} \to \mathbb{X}$ is almost automorphic under Definition 1.3, then it is also almost automorphic under Definition 3.3.

Lemma 3.5. *Let $\mathbb{T}$ be an almost periodic time scale and suppose $f, f_1, f_2 \in AA(\mathbb{T}, \mathbb{X})$. The following assertions hold:*

(*i*) *$f_1 + f_2 \in AA(\mathbb{T}, \mathbb{X})$;*

(*ii*) *$\alpha f \in AA(\mathbb{T}, \mathbb{X})$ for any constant $\alpha \in \mathbb{R}$;*

(*iii*) *$f_c(t) \equiv f(c + t) \in AA(\mathbb{T}, \mathbb{X})$ for each fixed $c \in \widetilde{\mathbb{T}}$.*

Proof. (*i*) Let $f_1, f_2 \in AA(\mathbb{T},\mathbb{X})$. Then for every sequence $(s_n^{'}) \subset \Pi$ there exists a subsequence $(s_n) \subset (s_n^{'})$ such that

$$\lim_{n\to\infty} f_1(t+s_n) = \bar{f}_1(t) \quad \text{and} \quad \lim_{n\to\infty} f_2(t+s_n) = \bar{f}_2(t)$$

are well defined for each $t \in \widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty} \bar{f}_1(t-s_n) = f_1(t) \quad \text{and} \quad \lim_{n\to\infty} \bar{f}_2(t-s_n) = f_2(t)$$

for each $t \in \widetilde{\mathbb{T}}$. Therefore, we obtain that

$$\lim_{n\to\infty} (f_1+f_2)(t+s_n) = \lim_{n\to\infty} \big(f_1(t+s_n) + f_2(t+s_n)\big) = \bar{f}_1(t) + \bar{f}_2(t)$$

is well defined for each $t \in \widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty} (\bar{f}_1+\bar{f}_2)(t-s_n) = \lim_{n\to\infty} \big(\bar{f}_1(t-s_n) + \bar{f}_2(t-s_n)\big) = f_1(t) + f_2(t)$$

for each $t \in \widetilde{\mathbb{T}}$.

(*ii*) Since $f \in AA(\mathbb{T},\mathbb{X})$, it follows that for every sequence $(s_n^{'}) \subset \Pi$ there exists a subsequence $(s_n) \subset (s_n^{'})$ such that

$$\lim_{n\to\infty} (\alpha f)(t+s_n) = \lim_{n\to\infty} \alpha f(t+s_n) = \alpha \bar{f}(t) = (\alpha \bar{f})(t)$$

is well defined for each $t \in \widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty} (\alpha \bar{f})(t-s_n) = \lim_{n\to\infty} \alpha \bar{f}(t-s_n) = \alpha f(t) = (\alpha f)(t)$$

for each $t \in \widetilde{\mathbb{T}}$.

(*iii*) It follows from $f \in AA(\mathbb{T},\mathbb{X})$, $c \in \widetilde{\mathbb{T}}$ that for every sequence $(s_n^{'}) \subset \Pi$ there exists a subsequence $(s_n) \subset (s_n^{'})$ such that

$$\lim_{n\to\infty} f_c(t+s_n) = \lim_{n\to\infty} f((c+t)+s_n) = \bar{f}(c+t) = \bar{f}_c(t)$$

is well defined for each $t \in \widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty} \bar{f}_c(t-s_n) = \lim_{n\to\infty} \bar{f}((c+t)-s_n) = f(c+t) = f_c(t)$$

for each $t \in \widetilde{\mathbb{T}}$. The proof is completed. □

Lemma 3.6. *Let $\mathbb{T}$ be an almost periodic time scale. If the functions $f, \phi : \mathbb{T} \to \mathbb{X}$ are almost automorphic, then the function $\phi f : \mathbb{T} \to \mathbb{X}$ defined by $(\phi f)(t) = \phi(t) f(t)$ is also almost automorphic.*

Proof. Both functions ϕ and f are bounded since they are almost automorphic. We denote $K_1 = \sup_{t\in\mathbb{T}} \|\phi(t)\|$.

Given a sequence $(s_n^{'}) \subset \Pi$, there exists a subsequence $(s_n^{''}) \subset (s_n^{'})$ such that $\lim_{n\to\infty} \phi(t+s_n^{''}) = \bar{\phi}(t)$ is well defined for each $t \in \widetilde{\mathbb{T}}$ and $\lim_{n\to\infty} \bar{\phi}(t-s_n^{''}) = \phi(t)$ for each $t \in \widetilde{\mathbb{T}}$. Since f is almost automorphic, there exists a subsequence $(s_n) \subset (s_n^{''})$ such that $\lim_{n\to\infty} f(t+s_n) = \bar{f}(t)$ is well defined for each $t \in \widetilde{\mathbb{T}}$ and $\lim_{n\to\infty} \bar{f}(t-s_n) = f(t)$ for each $t \in \widetilde{\mathbb{T}}$. Now, we have

$$\begin{aligned}
\|\phi(t+s_n)f(t+s_n) - \bar{\phi}(t)\bar{f}(t)\| &\le \|\phi(t+s_n)f(t+s_n) - \phi(t+s_n)\bar{f}(t)\| + \|\phi(t+s_n)\bar{f}(t) - \bar{\phi}(t)\bar{f}(t)\| \\
&\le K_1\|f(t+s_n) - \bar{f}(t)\| + K_2\|\phi(t+s_n) - \bar{\phi}(t)\| \\
&\le (K_1+K_2)\varepsilon
\end{aligned}$$

for n sufficiently large, where $K_2 = \sup_{t\in\mathbb{T}} \|\bar{f}(t)\| < \infty$. Thus, we obtain

$$\lim_{n\to\infty} \phi(t+s_n)f(t+s_n) = \bar{\phi}(t)\bar{f}(t)$$

for each $t \in \widetilde{\mathbb{T}}$.

It is also easy to check that

$$\lim_{n\to\infty} \bar{\phi}(t-s_n)\bar{f}(t-s_n) = \phi(t)f(t)$$

for each $t \in \widetilde{\mathbb{T}}$. The proof is now complete. □

Lemma 3.7. *Let $\mathbb{T}$ be an almost periodic time scale and (f_n) be a sequence of almost automorphic functions such that $\lim_{n\to\infty} f_n(t) = f(t)$ converges uniformly for each $t \in \widetilde{\mathbb{T}}$. Then f is an almost automorphic function.*

Proof. Consider a sequence $(s_n^{'}) \subset \Pi$. As in the standard case of almost automorphic functions, the approach follows across the diagonal procedure. Since $f_1 \in AA(\mathbb{T}, \mathbb{X})$, there exists a subsequence $(s_n^{(1)}) \subset (s_n^{'})$ such that

$$\lim_{n\to\infty} f_1(t+s_n^{(1)}) = \bar{f}_1(t)$$

is well defined for each $t \in \widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty} \bar{f}_1(t-s_n^{(1)}) = f_1(t)$$

for each $t \in \widetilde{\mathbb{T}}$. Since $f_2 \in AA(\mathbb{T}, \mathbb{X})$, there exists a subsequence $(s_n^{(2)}) \subset (s_n^{(1)})$ such that

$$\lim_{n\to\infty} f_2(t+s_n^{(2)}) = \bar{f}_2(t)$$

is well defined for each $t \in \widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty} \bar{f}_2(t-s_n^{(2)}) = f_2(t)$$

for each $t \in \widetilde{\mathbb{T}}$. Following this procedure, we can construct a subsequence $(s_n^{(n)}) \subset (s_n^{'})$ such that

$$\lim_{n\to\infty} f_i(t+s_n^{(n)}) = \bar{f}_i(t) \tag{3.1}$$

for each $t \in \widetilde{\mathbb{T}}$ and all $i = 1, 2, \ldots$.

Note that

$$\|\bar{f}_i(t) - \bar{f}_j(t)\| \le \|\bar{f}_i(t) - f_i(t+s_n^{(n)})\| + \|f_i(t+s_n^{(n)}) - f_j(t+s_n^{(n)})\| + \|f_j(t+s_n^{(n)}) - \bar{f}_j(t)\|. \tag{3.2}$$

By the uniform convergence of (f_n), for any $\varepsilon > 0$ we can find $N = N(\varepsilon) \in \mathbb{N}$ sufficiently large such that for all $i, j > N$, we have

$$\|f_i(t+s_n^{(n)}) - f_j(t+s_n^{(n)})\| < \varepsilon \tag{3.3}$$

for all $t \in \widetilde{\mathbb{T}}$ and all $n \ge 1$.

Hence, taking i, j sufficiently large in (3.2) and using (3.1) and (3.3), we can conclude that $(\bar{f}_i(t))$ is a Cauchy sequence. Since $\mathbb{X}$ is a Banach space, it follows that $(\bar{f}_i(t))$ is a sequence which converges pointwise on $\mathbb{X}$. Let $\bar{f}(t)$ be the limit of $(\bar{f}_i(t))$. Then for each $i = 1, 2, \ldots$ we have

$$\|f(t+s_n^{(n)}) - \bar{f}(t)\| \le \|f(t+s_n^{(n)}) - f_i(t+s_n^{(n)})\| + \|f_i(t+s_n^{(n)}) - \bar{f}_i(t)\| + \|\bar{f}_i(t) - \bar{f}(t)\|. \tag{3.4}$$

Thus, for i sufficiently large, by (3.4) and using the almost automorphicity of f_i and the convergence of the functions f_i and $\bar{f}_i$, we obtain

$$\lim_{n\to\infty} f(t+s_n^{(n)}) = \bar{f}(t)$$

for each $t \in \widetilde{\mathbb{T}}$. Similarly, we can get

$$\lim_{n\to\infty} \bar{f}(t-s_n^{(n)}) = f(t)$$

for each $t \in \widetilde{\mathbb{T}}$. This completes the proof. □

Lemma 3.8. *Let $\mathbb{T}$ be an almost periodic time scale and $\mathbb{X}, \mathbb{Y}$ be Banach spaces. If $f : \mathbb{T} \to \mathbb{X}$ is an almost automorphic function and $\varphi : \mathbb{X} \to \mathbb{Y}$ is a continuous function, then $\varphi \circ f : \mathbb{T} \to \mathbb{Y}$ is an almost automorphic function.*

Proof. Since $f \in AA(\mathbb{T}, \mathbb{X})$, for every sequence $(s_n^{'}) \subset \Pi$ there exists a subsequence $(s_n) \subset (s_n^{'})$ such that $\lim_{n\to\infty} f(t + s_n) = \bar{f}(t)$ is well defined for each $t \in \widetilde{\mathbb{T}}$ and $\lim_{n\to\infty} \bar{f}(t - s_n) = f(t)$ for each $t \in \widetilde{\mathbb{T}}$.

By the continuity of the function φ, it follows that

$$\lim_{n\to\infty} \varphi(f(t + s_n)) = \varphi\big(\lim_{n\to\infty} f(t + s_n)\big) = (\varphi \circ \bar{f})(t).$$

On the other hand, we can get

$$\lim_{n\to\infty} \varphi(\bar{f}(t - s_n)) = \varphi\big(\lim_{n\to\infty} \bar{f}(t - s_n)\big) = (\varphi \circ f)(t)$$

for each $t \in \widetilde{\mathbb{T}}$. Thus, $\varphi \circ f \in AA(\mathbb{T}, \mathbb{Y})$. The proof is complete. □

Definition 3.9. Let $\mathbb{T}$ be an almost periodic time scale and $\mathbb{X}$ be a Banach space. A bounded rd-continuous function $f : \mathbb{T} \times \mathbb{X} \to \mathbb{X}$ is said to be almost automorphic if for any sequence $(s_n^{'}) \subset \Pi$ there exists a subsequence $(s_n) \subset (s_n^{'})$ such that

$$\lim_{n\to\infty} f(t + s_n, x) = \bar{f}(t, x)$$

is well defined for each $t \in \widetilde{\mathbb{T}}$, $t + s_n \in \mathbb{T}$, $x \in \mathbb{X}$, and

$$\lim_{n\to\infty} \bar{f}(t - s_n, x) = f(t, x)$$

for each $t \in \widetilde{\mathbb{T}}$, $x \in \mathbb{X}$.

Denote by $AA(\mathbb{T} \times \mathbb{X}, \mathbb{X})$ the set of all such functions.

Remark 3.10. From Definition 3.9, one can easily see that if $f : \mathbb{T} \times \mathbb{X} \to \mathbb{X}$ is an almost automorphic function under Definition 1.3, then it is also an almost automorphic function under Definition 3.9.

Lemma 3.11. *Let $f \in AA(\mathbb{T} \times \mathbb{X}, \mathbb{X})$ satisfy the Lipschitz condition in $x \in \mathbb{X}$ uniformly in $t \in \widetilde{\mathbb{T}}$. If $\varphi \in AA(\mathbb{T}, \mathbb{X})$, then $f(t, \varphi(t))$ is almost automorphic.*

Proof. For any sequence $(s_n^{'}) \subset \Pi$, there exists a subsequence $(s_n) \subset (s_n^{'})$ such that

$$\lim_{n\to\infty} f(t + s_n, x) = \bar{f}(t, x)$$

is well defined for each $t \in \widetilde{\mathbb{T}}$, $x \in \mathbb{X}$, and

$$\lim_{n\to\infty} \bar{f}(t - s_n, x) = f(t, x)$$

for each $t \in \widetilde{\mathbb{T}}$, $x \in \mathbb{X}$. Since $\varphi \in AA(\mathbb{T}, \mathbb{X})$, there exists a subsequence $(\tau_n) \subset (s_n)$ such that

$$\lim_{n\to\infty} \varphi(t + \tau_n) = \bar{\varphi}(t)$$

is well defined for each $t \in \widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty} \bar{\varphi}(t - \tau_n) = \varphi(t)$$

for each $t \in \widetilde{\mathbb{T}}$. Since f satisfies the Lipschitz condition in $x \in \mathbb{X}$ uniformly in $t \in \widetilde{\mathbb{T}}$, there exists a positive constant L such that

$$\|f(t + \tau_n, \varphi(t + \tau_n)) - \bar{f}(t, \bar{\varphi}(t))\| \leq \|f(t + \tau_n, \varphi(t + \tau_n)) - f(t + \tau_n, \bar{\varphi}(t))\| + \|f(t + \tau_n, \bar{\varphi}(t)) - \bar{f}(t, \bar{\varphi}(t))\|$$

$$\leq L\|\varphi(t+\tau_n)-\bar{\varphi}(t)\|+\|f(t+\tau_n,\bar{\varphi}(t))-\bar{f}(t,\bar{\varphi}(t))\|\to 0,\ \ n\to\infty$$

and

$$\begin{aligned}\|\bar{f}(t-\tau_n,\bar{\varphi}(t-\tau_n))-f(t,\varphi(t))\| &\leq \|\bar{f}(t-\tau_n,\bar{\varphi}(t-\tau_n))-\bar{f}(t-\tau_n,\varphi(t))\|+\|\bar{f}(t-\tau_n,\varphi(t))-f(t,\varphi(t))\|\\ &\leq L\|\bar{\varphi}(t-\tau_n)-\varphi(t)\|+\|\bar{f}(t-\tau_n,\varphi(t))-f(t,\varphi(t))\|\to 0,\ \ n\to\infty.\end{aligned}$$

Hence, for any sequence $(s_n^{'})\subset\Pi$ there exists a subsequence $(\tau_n)\subset(s_n^{'})$ such that

$$\lim_{n\to\infty} f(t+\tau_n,\varphi(t+\tau_n))=\bar{f}(t,\bar{\varphi}(t))$$

is well defined for each $t\in\widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty} \bar{f}(t-\tau_n,\bar{\varphi}(t-\tau_n))=f(t,\varphi(t))$$

for each $t\in\widetilde{\mathbb{T}}$, that is, $f(t,\varphi(t))$ is almost automorphic. This completes the proof. □

Definition 3.12. Let $\mathbb{T}$ be an almost periodic time scale. The graininess function $\mu:\mathbb{T}\to\mathbb{R}_+$ is said to be almost automorphic if for every sequence $(s_n^{'})\subset\Pi$, there exists a subsequence $(s_n)\subset(s_n^{'})$ such that

$$\lim_{n\to\infty}\mu(t+s_n)=\bar{\mu}(t)$$

is well defined for each $t\in\widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty}\bar{\mu}(t-s_n)=\mu(t)$$

for each $t\in\widetilde{\mathbb{T}}$.

In the following, in order to make the graininess function μ have a better property, we will use Definition 2.3 as the definition of almost periodic time scales.

From Corollary 15, Theorem 19 in [20] and Definition 3.12, we can obtain

Lemma 3.13. *Let $\mathbb{T}$ be an almost periodic time scale under Definition 2.3. Then the graininess function μ is almost automorphic.*

Open problem 1. *Let $\mathbb{T}$ be an almost periodic time scale under Definition 3.1. Does it follow that the graininess function μ is almost automorphic?*

Open problem 2. *Let the graininess function μ of $\mathbb{T}$ be almost automorphic. Does it follow that $\mathbb{T}$ is an almost periodic time scale under Definition 3.1?*

4. Automorphic solutions to linear dynamic equations

Consider the linear nonhomogeneous dynamic equation on time scales

$$x^{\Delta}(t)=A(t)x(t)+f(t),\ \ t\in\mathbb{T},\tag{4.1}$$

where $A:\mathbb{T}\to\mathbb{R}^{n\times n}$ and $f:\mathbb{T}\to\mathbb{R}^n$, and its associated homogeneous equation

$$x^{\Delta}(t)=A(t)x(t),\ \ t\in\mathbb{T}.\tag{4.2}$$

Throughout this section, we restrict our discussions to almost periodic time scales under Definition 2.3 and we assume that $A(t)$ is almost automorphic on $\mathbb{T}$, which means that each entry of the matrix $A(t)$ is almost automorphic.

Lemma 4.1. *Let $\mathbb{T}$ be an almost periodic time scale, $A(t) \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ be almost automorphic and nonsingular on $\mathbb{T}$ and the sets $\{A^{-1}(t)\}_{t\in\mathbb{T}}$ and $\{(I+\mu(t)A(t))^{-1}\}_{t\in\mathbb{T}}$ be bounded on $\mathbb{T}$. Then, $A^{-1}(t)$ and $(I+\mu(t)A(t))^{-1}$ are almost automorphic on $\mathbb{T}$. Moreover, suppose that $f \in AA(\mathbb{T}, \mathbb{R}^n)$ and (4.2) admits an exponential dichotomy. Then (4.1) has a solution $x(t) \in AA(\mathbb{T}, \mathbb{R}^n)$, and $x(t)$ is given by*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))f(s)\Delta s, \tag{4.3}$$

where $X(t)$ is the fundamental solution matrix of (4.2).

Proof. We divide the proof into several steps.

Step 1. $A^{-1}(t)$ is almost automorphic on $\mathbb{T}$.

Consider a sequence $(s_n^{'}) \subset \Pi$. Since $A(t)$ is almost automorphic on time scales, there exists a subsequence $(s_n) \subset (s_n^{'})$ such that

$$\lim_{n\to\infty} A(t+s_n) = \bar{A}(t)$$

is well defined for each $t \in \widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty} \bar{A}(t-s_n) = A(t)$$

for each $t \in \widetilde{\mathbb{T}}$.

Given $t \in \widetilde{\mathbb{T}}$, define $A_n = A(t+s_n)$, $n \in \mathbb{N}$. By hypothesis, the set $\{A_n^{-1}\}_{n\in\mathbb{N}}$ is bounded, that is, there exists a positive constant M such that $|A_n^{-1}| < M$. From the identity $A_n^{-1} - A_m^{-1} = A_n^{-1}(A_m - A_n)A_m^{-1}$, it follows that $\{A_n\}$ and $\{A_n^{-1}\}$ are Cauchy sequences. Hence, for each $t \in \widetilde{\mathbb{T}}$ fixed, there exists a matrix S such that

$$A_n^{-1}(t) \to S(t),\ n \to \infty.$$

Then we have

$$\lim_{n\to\infty} A_n A_n^{-1} = \bar{A}\bar{A}^{-1} = I,$$

where I denotes the identity matrix, and we obtain that $\bar{A}(t)$ is invertible and $\bar{A}^{-1}(t) = S(t)$ for each $t \in \widetilde{\mathbb{T}}$. Since the map $A \to A^{-1}$ is continuous on the set of nonsingular matrices, we infer that

$$\lim_{n\to\infty} A^{-1}(t+s_n) = \bar{A}^{-1}(t)$$

is well defined for each $t \in \widetilde{\mathbb{T}}$. Similarly, we can get that

$$\lim_{n\to\infty} \bar{A}^{-1}(t-s_n) = A^{-1}(t)$$

for each $t \in \widetilde{\mathbb{T}}$.

Step 2. $(I+\mu(t)A(t))^{-1}$ is almost automorphic on $\mathbb{T}$.

Since $A(t)$ and $\mu(t)$ are almost automorphic functions, for every sequence $(s_n^{'}) \subset \Pi$ there exists a subsequence $(s_n) \subset (s_n^{'})$ such that

$$\lim_{n\to\infty} A(t+s_n) = \bar{A}(t) \quad \text{and} \quad \lim_{n\to\infty} \mu(t+s_n) = \bar{\mu}(t)$$

are well defined for each $t \in \widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty} \bar{A}(t-s_n) = A(t) \quad \text{and} \quad \lim_{n\to\infty} \bar{\mu}(t-s_n) = \mu(t)$$

for each $t \in \widetilde{\mathbb{T}}$. Therefore, we have

$$\lim_{n\to\infty} \big(I + A(t+s_n)\mu(t+s_n)\big) = I + \bar{A}(t)\bar{\mu}(t)$$

for each $t \in \widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty} \big(I + \bar{A}(t-s_n)\bar{\mu}(t-s_n)\big) = I + A(t)\mu(t)$$

for each $t \in \widetilde{\mathbb{T}}$. Hence, $(I + A(t)\mu(t))$ is almost automorphic on $\mathbb{T}$. In addition, since $A(t)$ is a regressive matrix, $(I+A(t)\mu(t))$ is nonsingular on $\mathbb{T}$. By hypothesis, the set $\{(I+A(t)\mu(t))^{-1}\}_{t\in\mathbb{T}}$ is bounded. Similarly as in the proof of Step 1, we obtain that $(I + A(t)\mu(t))^{-1}$ is almost automorphic on $\mathbb{T}$.

Step 3. The system (4.1) has an automorphic solution.

Since the linear system (4.2) admits an exponential dichotomy, the system (4.1) has a bounded solution $x(t)$ given by

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))f(s)\Delta s.$$

Since $A(t)$, $\mu(t)$ and $f(t)$ are almost automorphic functions, for every sequence $(s_n^{'}) \subset \Pi$ there exists a subsequence $(s_n) \subset (s_n^{'})$ such that

$$\lim_{n\to\infty} A(t+s_n) = \bar{A}(t), \quad \lim_{n\to\infty} \mu(t+s_n) = \bar{\mu}(t) \quad \text{and} \quad \lim_{n\to\infty} f(t+s_n) = \bar{f}(t)$$

are well defined for each $t \in \widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty} \bar{A}(t-s_n) = A(t), \quad \lim_{n\to\infty} \bar{\mu}(t-s_n) = \mu(t) \quad \text{and} \quad \lim_{n\to\infty} \bar{f}(t-s_n) = f(t)$$

for each $t \in \widetilde{\mathbb{T}}$.

We let

$$B(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s$$

and

$$\bar{B}(t) = \int_{-\infty}^{t} M(t,s)\bar{f}(s)\Delta s,$$

where $M(t,s) = \lim_{n\to\infty} X(t+s_n)PX^{-1}(\sigma(s+s_n))$.

Then, we obtain

$$\begin{aligned}
\|B(t+s_n) - \bar{B}(t)\| &= \Big\| \int_{-\infty}^{t+s_n} X(t+s_n)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{-\infty}^{t} M(t,s)\bar{f}(s)\Delta s \Big\| \\
&= \Big\| \int_{-\infty}^{t} X(t+s_n)PX^{-1}(\sigma(s+s_n))f(s+s_n)\Delta s - \int_{-\infty}^{t} M(t,s)\bar{f}(s)\Delta s \Big\| \\
&\le \Big\| \int_{-\infty}^{t} X(t+s_n)PX^{-1}(\sigma(s+s_n))f(s+s_n)\Delta s \\
&\quad - \int_{-\infty}^{t} X(t+s_n)PX^{-1}(\sigma(s+s_n))\bar{f}(s)\Delta s \Big\| \\
&\quad + \Big\| \int_{-\infty}^{t} X(t+s_n)PX^{-1}(\sigma(s+s_n))\bar{f}(s)\Delta s - \int_{-\infty}^{t} M(t,s)\bar{f}(s)\Delta s \Big\| \\
&= \Big\| \int_{-\infty}^{t} X(t+s_n)PX^{-1}(\sigma(s+s_n))\big(f(s+s_n) - \bar{f}(s)\big)\Delta s \Big\| \\
&\quad + \Big\| \int_{-\infty}^{t} \big(X(t+s_n)PX^{-1}(\sigma(s+s_n)) - M(t,s)\big)\bar{f}(s)\Delta s \Big\|.
\end{aligned} \tag{4.4}$$

In addition, since the function f is almost automorphic, we have that $\bar{f}$ is a bounded function. Passing to the limit in (4.4), we get

$$\lim_{n\to\infty} B(t+s_n) = \bar{B}(t) \tag{4.5}$$

for each $t \in \widetilde{\mathbb{T}}$. Analogously, one can prove that

$$\lim_{n\to\infty} \bar{B}(t - s_n) = B(t) \tag{4.6}$$

for each $t \in \widetilde{\mathbb{T}}$.

On the other hand, let

$$C(t) = \int_t^{+\infty} X(t)(I - P)X^{-1}(\sigma(s))f(s)\Delta s$$

and

$$\bar{C}(t) = \int_{-\infty}^{t} N(t, s)\bar{f}(s)\Delta s,$$

where $N(t, s) = \lim_{n\to\infty} X(t + s_n)(I - P)X^{-1}(\sigma(s + s_n))$. Then, similar to the proofs of (4.5) and (4.6), we can show that given a sequence $(s_n^{'}) \subset \Pi$, there exists a subsequence $(s_n) \subset (s_n^{'})$ such that

$$\lim_{n\to\infty} C(t + s_n) = \bar{C}(t)$$

for each $t \in \widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty} \bar{C}(t - s_n) = C(t)$$

for each $t \in \widetilde{\mathbb{T}}$.

Finally, we define $\bar{x}(t) = \bar{B}(t) - \bar{C}(t)$. Then, by the definition of x from (4.3), one can prove that given a sequence $(s_n^{'}) \subset \Pi$, there exists a subsequence $(s_n) \subset (s_n^{'})$ such that

$$\lim_{n\to\infty} x(t + s_n) = \bar{x}(t)$$

is well defined for each $t \in \widetilde{\mathbb{T}}$, and

$$\lim_{n\to\infty} \bar{x}(t - s_n) = x(t)$$

for each $t \in \widetilde{\mathbb{T}}$. Hence, $x(t)$ is an almost automorphic solution of system (4.1). This completes the proof. □

Similar to the proof of Lemma 2.15 in [23], one can easily show that

Lemma 4.2. *Let $c_i(t)$ be almost automorphic on $\mathbb{T}$, where $c_i(t) > 0, -c_i(t) \in \mathcal{R}^+, t \in \widetilde{\mathbb{T}}, i = 1, 2, \ldots, n$ and $\min_{1\le i\le n}\{\inf_{t\in\widetilde{\mathbb{T}}} c_i(t)\} = \widetilde{m} > 0$. Then the linear system*

$$x^{\Delta}(t) = \mathrm{diag}(-c_1(t), -c_2(t), \ldots, -c_n(t))x(t)$$

admits an exponential dichotomy on $\mathbb{T}$.

5. An application

In the last forty years, shunting inhibitory cellular neural networks (SICNNs) have been extensively applied in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing. Hence, they have been the object of intensive analysis by numerous authors in recent years. Many important results on the dynamical behaviors of SICNNs have been established and successfully applied to signal processing, pattern recognition, associative memories, and so on. We refer the reader to [13, 19, 21, 23, 26, 32, 42, 44, 49] and the references cited therein. However, to the best of our knowledge, there is no paper published on the existence of almost automorphic solutions to SICNNs governed by differential or difference equations. So, our main purpose in this section is to study the existence of almost automorphic solutions to the following SICNN with time-varying delays on time scales

$$\begin{aligned} x_{ij}^{\Delta}(t) &= -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(t)f(x_{kl}(t-\tau_{kl}(t)))x_{ij}(t) \\ &\quad - \sum_{C_{kl}\in N_p(i,j)} C_{ij}^{kl}(t)\int_{t-\delta_{kl}(t)}^{t} g(x_{kl}(u))\Delta u x_{ij}(t) + L_{ij}(t), \end{aligned} \tag{5.1}$$

where $t\in\mathbb{T}$, $\mathbb{T}$ is an almost periodic time scale in the sense of Definition 2.3, $i=1,2,\dots,m$, $j=1,2,\dots,n$, C_{ij} denotes the cell at the (i,j) position of the lattice, the r-neighborhood $N_r(i,j)$ of C_{ij} is given by

$$N_r(i,j)=\{C_{kl}:\max(|k-i|,|l-j|)\le r, 1\le k\le m, 1\le l\le n\},$$

$N_p(i,j)$ is similarly specified, x_{ij} is the activity of the cell C_{ij}, $L_{ij}(t)$ is the external input to C_{ij}, $a_{ij}(t)>0$ represents the passive decay rate of the cell activity, $B_{ij}^{kl}(t)\ge 0$ and $C_{ij}^{kl}(t)\ge 0$ are the connections or coupling strengths of postsynaptic activity of the cells in $N_r(i,j)$ and $N_p(i,j)$ transmitted to cell C_{ij} depending upon variable delays and continuously distributed delays, respectively, and the activity functions $f(\cdot)$ and $g(\cdot)$ are continuous functions representing the output or firing rate of the cell C_{kl}, $\tau_{kl}(t)$ and $\delta_{kl}(t)$ are transmission delays at time t and satisfy $t-\tau_{kl}(t)\in\mathbb{T}$ and $t-\delta_{kl}(t)\in\mathbb{T}$ for $t\in\mathbb{T}$, $k=1,2,\dots,m$, $l=1,2,\dots,n$.

The initial condition associated with system (5.1) is of the form

$$x_{ij}(s)=\varphi_{ij}(s),\quad s\in[-\theta,0]_{\mathbb{T}},\quad i=1,2,\dots,m, j=1,2,\dots,n,$$

where $\theta=\max\big\{\max\limits_{1\le k\le m,1\le l\le n}\overline{\tau_{kl}}, \max\limits_{1\le k\le m,1\le l\le n}\overline{\delta_{kl}}\big\}$, and $\varphi_{ij}(\cdot)$ denotes a real-value bounded rd-continuous function defined on $[-\theta,0]_{\mathbb{T}}$.

Throughout this section, we let $[a,b]_{\mathbb{T}}=\{t|t\in[a,b]\cap\mathbb{T}\}$ and we restrict our discussions to almost periodic time scales in the sense of Definition 2.3. For convenience, for an almost automorphic function $f:\mathbb{T}\to\mathbb{R}$, denote $\underline{f}=\inf\limits_{t\in\mathbb{T}}f(t)$, $\overline{f}=\sup\limits_{t\in\mathbb{T}}f(t)$. We denote by $\mathbb{R}$ the set of real numbers, and by $\mathbb{R}^+$ the set of positive real numbers.

Set

$$x=\{x_{ij}(t)\}=(x_{11}(t),\dots,x_{1n}(t),\dots,x_{i1}(t),\dots,x_{in}(t),\dots,x_{m1}(t)\dots,x_{mn}(t))\in\mathbb{R}^{m\times n}.$$

For all $x=\{x_{ij}(t)\}\in\mathbb{R}^{m\times n}$, we define the norm $\|x(t)\|=\max_{i,j}|x_{ij}(t)|$ and $\|x\|=\sup\limits_{t\in\mathbb{T}}\|x(t)\|$.

Let $\mathbb{B}=\{\varphi|\varphi=\{\varphi_{ij}(t)\}=(\varphi_{11}(t),\dots,\varphi_{1n}(t),\dots,\varphi_{i1}(t),\dots,\varphi_{in}(t),\dots,\varphi_{m1}(t)\dots,\varphi_{mn}(t))\}$, where φ is an almost automorphic function on $\mathbb{T}$. For all $\varphi\in\mathbb{B}$, if we define the norm $\|\varphi\|_{\mathbb{B}}=\sup\limits_{t\in\mathbb{T}}\|\varphi(t)\|$, then $\mathbb{B}$ is a Banach space.

Definition 5.1. The almost automorphic solution $x^*(t)=\{x_{ij}^*(t)\}$ of system (5.1) with initial value $\varphi^*=\{\varphi_{ij}^*(t)\}$ is said to be globally exponentially stable if there exist positive constants λ with $\ominus\lambda\in\mathcal{R}^+$ and $M>1$ such that every solution $x(t)=\{x_{ij}(t)\}$ of system (5.1) with initial value $\varphi=\{\varphi_{ij}(t)\}$ satisfies

$$\|x-x^*\|_{\mathbb{B}}\le Me_{\ominus\lambda}(t,t_0)\|\varphi-\varphi^*\|_{\mathbb{B}},\quad \forall t\in[t_0,+\infty)_{\mathbb{T}},\ t_0\in\mathbb{T}.$$

Theorem 5.2. *Suppose that*

(H_1) *for* $ij\in\Lambda=\{11,12,\dots,1n,\dots,m1,m2,\dots,mn\}$, $-a_{ij}\in\mathcal{R}^+$, *where* $\mathcal{R}^+$ *denotes the set of positively regressive functions from* $\mathbb{T}$ *to* $\mathbb{R}$, $a_{ij},B_{ij}^{kl},C_{ij}^{kl},\tau_{kl},L_{ij}\in\mathbb{B}$;

(H_2) *functions* $f,g\in C(\mathbb{R},\mathbb{R})$ *and there exist positive constants* L^f,L^g,M^f,M^g *such that for all* $u,v\in\mathbb{R}$,

$$|f(u)-f(v)|\le L^f|u-v|,\quad f(0)=0,\quad |f(u)|\le M^f,$$

$$|g(u)-g(v)|\le L^g|u-v|,\quad g(0)=0,\quad |g(u)|\le M^g;$$

(H_3) *there exists a constant* ρ *such that*

$$\max_{i,j}\left\{\frac{\sum\limits_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}M^f\rho^2+\sum\limits_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}M^g\overline{\delta_{kl}}\rho^2+\overline{L_{ij}}}{\underline{a_{ij}}}\right\}<\rho$$

and

$$\max_{i,j}\left\{\frac{\sum\limits_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}(M^f+L^f)\rho+\sum\limits_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}(M^g+L^g)\overline{\delta_{kl}}\rho}{\underline{a_{ij}}}\right\}<1.$$

Then, system (5.1) *has a unique almost automorphic solution in* $\mathbb{E}=\{\varphi\in\mathbb{B}:\|\varphi\|_{\mathbb{B}}\le\rho\}$.

Proof. For any given $\varphi\in\mathbb{B}$, we consider the following system

$$\begin{aligned}x_{ij}^{\Delta}(t) &= -a_{ij}(t)x_{ij}(t)-\sum_{C_{kl}\in N_r(i,j)}B_{ij}^{kl}(t)f(\varphi_{kl}(t-\tau_{kl}(t)))\varphi_{ij}(t)\\ &\quad-\sum_{C_{kl}\in N_p(i,j)}C_{ij}^{kl}(t)\int_{t-\delta_{kl}(t)}^{t}g(\varphi_{kl}(u))\Delta u\varphi_{ij}(t)+L_{ij}(t),\ ij=11,12,\dots,mn.\end{aligned}\tag{5.2}$$

Since $\min\limits_{1\le i\le m;1\le j\le n}\{\inf\limits_{t\in\mathbb{T}}a_{ij}(t)\}>0$ and $-a_{ij}\in\mathcal{R}^+$, it follows from Lemma 4.2 that the linear system

$$x_{ij}^{\Delta}(t)=-a_{ij}(t)x_{ij}(t),\quad ij=11,12,\dots,mn$$

admits an exponential dichotomy on $\mathbb{T}$. Thus, by Lemma 4.1, we obtain that system (5.2) has exactly one almost automorphic solution as follows

$$\begin{aligned}x^{\varphi}(t):=\{x_{ij}^{\varphi}(t)\}=\bigg\{\int_{-\infty}^{t}e_{-a_{ij}}(t,\sigma(s))\Big(-\sum_{C_{kl}\in N_r(i,j)}B_{ij}^{kl}(s)f(\varphi_{kl}(s-\tau_{kl}(s)))\varphi_{ij}(s)\\ -\sum_{C_{kl}\in N_p(i,j)}C_{ij}^{kl}(s)\int_{t-\delta_{kl}(s)}^{s}g(\varphi_{kl}(u))\Delta u\varphi_{ij}(s)+L_{ij}(s)\Big)\Delta s\bigg\}.\end{aligned}$$

Now, we define the operator $T:\mathbb{B}\to\mathbb{B}$ by setting

$$T(\varphi(t))=x^{\varphi}(t),\quad\forall\varphi\in\mathbb{B}.$$

First, we will check that for any $\varphi\in\mathbb{E}$, $T\varphi\in\mathbb{E}$. For any $\varphi\in\mathbb{E}$, we have

$$\begin{aligned}\|T(\varphi)\|_{\mathbb{B}} &= \sup_{t\in\mathbb{T}}\max_{i,j}\bigg\{\bigg|\int_{-\infty}^{t}e_{-a_{ij}}(t,\sigma(s))\Big(-\sum_{C_{kl}\in N_r(i,j)}B_{ij}^{kl}(s)f(\varphi_{kl}(s-\tau_{kl}(s)))\varphi_{ij}(s)\\ &\quad-\sum_{C_{kl}\in N_p(i,j)}C_{ij}^{kl}(s)\int_{t-\delta_{kl}(s)}^{s}g(\varphi_{kl}(u))\Delta u\varphi_{ij}(s)+L_{ij}(s)\Big)\Delta s\bigg|\bigg\}\\ &\le \sup_{t\in\mathbb{T}}\max_{i,j}\bigg\{\bigg|\int_{-\infty}^{t}e_{-\underline{a_{ij}}}(t,\sigma(s))\Big(\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}f(\varphi_{kl}(s-\tau_{kl}(s)))\varphi_{ij}(s)\\ &\quad+\sum_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}\int_{t-\delta_{kl}(s)}^{s}g(\varphi_{kl}(u))\Delta u\varphi_{ij}(s)\Big)\Delta s\bigg|\bigg\}+\max_{i,j}\frac{\overline{L_{ij}}}{\underline{a_{ij}}}\\ &\le \sup_{t\in\mathbb{T}}\max_{i,j}\bigg\{\int_{-\infty}^{t}e_{-\underline{a_{ij}}}(t,\sigma(s))\Big(\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}M^f|\varphi_{kl}(s-\tau_{kl}(s))||\varphi_{ij}(s)|\end{aligned}$$

$$
\begin{aligned}
&+\sum_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}M^g\int_{t-\delta_{kl}(s)}^{s}|\varphi_{kl}(u)|\Delta u|\varphi_{ij}(s)|\Big)\Delta s\Big\}+\max_{i,j}\frac{\overline{L_{ij}}}{\underline{a_{ij}}}\\
\leq\ &\sup_{t\in\mathbb{T}}\max_{i,j}\Big\{\int_{-\infty}^{t}e_{-\underline{a_{ij}}}(t,\sigma(s))\Big(\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}M^f\rho^2\\
&+\sum_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}M^g\overline{\delta_{kl}}\rho^2\Big)\Delta s\Big\}+\max_{i,j}\frac{\overline{L_{ij}}}{\underline{a_{ij}}}\\
\leq\ &\max_{i,j}\left\{\frac{\sum\limits_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}M^f\rho^2+\sum\limits_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}M^g\overline{\delta_{kl}}\rho^2+\overline{L_{ij}}}{\underline{a_{ij}}}\right\}\leq\rho.
\end{aligned}
$$

Therefore, we have that $\|T(\varphi)\|_{\mathbb{B}}\leq\rho$. This implies that T is a self-mapping from $\mathbb{E}$ to $\mathbb{E}$. Next, we prove that T is a contraction mapping on $\mathbb{E}$. In fact, for any $\varphi,\psi\in\mathbb{E}$, we can get

$$
\begin{aligned}
\|T(\varphi)-T(\psi)\|_{\mathbb{B}}&=\sup_{t\in\mathbb{T}}\|T(\varphi)(t)-T(\psi)(t)\|\\
&=\sup_{t\in\mathbb{T}}\max_{i,j}\Big\{\Big|\int_{-\infty}^{t}e_{-a_{ij}}(t,\sigma(s))\Big(\sum_{C_{kl}\in N_r(i,j)}B_{ij}^{kl}(s)\big[f(\varphi_{kl}(s-\tau_{kl}(s)))\varphi_{ij}(s)\\
&\quad-f(\psi_{kl}(s-\tau_{kl}(s)))\psi_{ij}(s)\big]+\sum_{C_{kl}\in N_p(i,j)}C_{ij}^{kl}(s)\Big[\int_{t-\delta_{kl}(s)}^{s}g(\varphi_{kl}(u))\Delta u\varphi_{ij}(s)\\
&\quad-\int_{t-\delta_{kl}(s)}^{s}g(\psi_{kl}(u))\Delta u\psi_{ij}(s)\Big]\Big)\Delta s\Big|\Big\}\\
&\leq\sup_{t\in\mathbb{T}}\max_{i,j}\Big\{\int_{-\infty}^{t}e_{-\underline{a_{ij}}}(t,\sigma(s))\Big(\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}|f(\varphi_{kl}(s-\tau_{kl}(s)))||\varphi_{ij}(s)-\psi_{ij}(s)|\\
&\quad+\sum_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}\Big|\int_{t-\delta_{kl}(s)}^{s}g(\varphi_{kl}(u))\Delta u\Big||\varphi_{ij}(s)-\psi_{ij}(s)|\Big)\Delta s\Big\}\\
&\quad+\sup_{t\in\mathbb{T}}\max_{i,j}\Big\{\int_{-\infty}^{t}e_{-\underline{a_{ij}}}(t,\sigma(s))\Big(\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}|f(\varphi_{kl}(s-\tau_{kl}(s)))-f(\psi_{kl}(s-\tau_{kl}(s)))|\\
&\quad\times|\psi_{ij}(s)|+\sum_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}\Big|\int_{t-\delta_{kl}(s)}^{s}\big(g(\varphi_{kl}(u))-g(\psi_{kl}(u))\big)\Delta u\Big||\psi_{ij}(s)|\Big)\Big\}\\
&\leq\sup_{t\in\mathbb{T}}\max_{i,j}\Big\{\int_{-\infty}^{t}e_{-\underline{a_{ij}}}(t,\sigma(s))\Big(\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}M^f|\varphi_{kl}(s-\tau_{kl}(s))||\varphi_{ij}(s)-\psi_{ij}(s)|\\
&\quad+\sum_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}\int_{t-\delta_{kl}(s)}^{s}M^g|\varphi_{kl}(u)|\Delta u|\varphi_{ij}(s)-\psi_{ij}(s)|\Big)\Delta s\Big\}\\
&\quad+\sup_{t\in\mathbb{T}}\max_{i,j}\Big\{\int_{-\infty}^{t}e_{-\underline{a_{ij}}}(t,\sigma(s))\Big(\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}L^f|\varphi_{kl}(s-\tau_{kl}(s))-\psi_{kl}(s-\tau_{kl}(s))|\\
&\quad\times|\psi_{ij}(s)|+\sum_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}\int_{t-\delta_{kl}(s)}^{s}L^g|\varphi_{kl}(u)-\psi_{kl}(u)|\Delta u|\psi_{ij}(s)|\Big)\Big\}\\
&\leq\sup_{t\in\mathbb{T}}\max_{i,j}\Big\{\int_{-\infty}^{t}e_{-\underline{a_{ij}}}(t,\sigma(s))\Big(\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}M^f\rho+\sum_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}M^g\overline{\delta_{kl}}\rho\Big)\Delta s\Big\}\|\varphi-\psi\|_{\mathbb{B}}
\end{aligned}
$$

$$+\sup_{t\in\mathbb{T}}\max_{i,j}\left\{\int_{-\infty}^{t} e_{-\underline{a_{ij}}}(t,\sigma(s))\Big(\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}L^f\rho+\sum_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}L^g\overline{\delta_{kl}}\rho\Big)\Delta s\right\}\|\varphi-\psi\|_{\mathbb{B}}$$

$$\le \max_{i,j}\left\{\frac{\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}(M^f+L^f)\rho+\sum_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}(M^g+L^g)\overline{\delta_{kl}}\rho}{\underline{a_{ij}}}\right\}\|\varphi-\psi\|_{\mathbb{B}}.$$

By (H_3), we have that $\|T(\varphi)-T(\psi)\|_{\mathbb{B}}<\|\varphi-\psi\|_{\mathbb{B}}$. Hence, T is a contraction mapping from $\mathbb{E}$ to $\mathbb{E}$. Therefore, T has a fixed point in $\mathbb{E}$, that is, (5.1) has a unique almost automorphic solution in $\mathbb{E}$. This completes the proof. □

Theorem 5.3. *Assume that* (H_1)-(H_3) *hold. Then system* (5.1) *has a unique almost automorphic solution which is globally exponentially stable.*

Proof. From Theorem 5.2, we see that system (5.1) has at least one almost automorphic solution $x^*(t)=\{x^*_{ij}(t)\}$ with the initial condition $\varphi^*(t)=\{\varphi^*_{ij}(t)\}$. Suppose that $\{x(t)\}=\{(x^*_{ij}(t)\}$ is an arbitrary solution with the initial condition $\varphi(t)=\{\varphi_{ij}(t)\}$. Set $y(t)=x(t)-x^*(t)$. Then it follows from system (5.1) that

$$\begin{aligned}y_{ij}^{\Delta}(t)=-a_{ij}(t)y_{ij}(t)&-\sum_{C_{kl}\in N_r(i,j)}B_{ij}^{kl}(t)\big(f(x_{kl}(t-\tau_{kl}(t)))x_{ij}(t)\\&-f(x^*_{kl}(t-\tau_{kl}(t)))x^*_{ij}(t)\big)-\sum_{C_{kl}\in N_p(i,j)}C_{ij}^{kl}(t)\Big(\int_{t-\delta_{kl}(t)}^{t}g(x_{kl}(u))\Delta u x_{ij}(t)\\&-\int_{t-\delta_{kl}(t)}^{t}g(x^*_{kl}(u))\Delta u x^*_{ij}(t)\Big)\end{aligned}\tag{5.3}$$

for $i=1,2,\dots,m$, $j=1,2,\dots,n$, and the initial condition of (5.3) is

$$\psi_{ij}(s)=\varphi_{ij}(s)-\varphi^*_{ij}(s),\quad s\in[-\theta,0]_{\mathbb{T}},\quad ij=11,12,\dots,mn.$$

Then, it follows from (5.3) that for $i=1,2,\dots,m$, $j=1,2,\dots,n$ and $t\ge t_0$, we have

$$\begin{aligned}y_{ij}(t)\;=\;&y_{ij}(t_0)e_{-a_{ij}}(t,t_0)-\int_{t_0}^{t}e_{-a_{ij}}(t,\sigma(s))\Big\{\sum_{C_{kl}\in N_r(i,j)}B_{ij}^{kl}(s)\big(f(x_{kl}(s-\tau_{kl}(s)))x_{ij}(s)\\&-f(x^*_{kl}(s-\tau_{kl}(s)))x^*_{ij}(s)\big)+\sum_{C_{kl}\in N_p(i,j)}C_{ij}^{kl}(s)\Big(\int_{t-\delta_{kl}(s)}^{s}g(x_{kl}(u))\Delta u x_{ij}(s)\\&-\int_{s-\delta_{kl}(s)}^{s}g(x^*_{kl}(u))\Delta u x^*_{ij}(s)\Big)\Big\}\Delta s.\end{aligned}\tag{5.4}$$

Let S_{ij} be defined as follows:

$$S_{ij}(\omega)=\underline{a_{ij}}-\omega-\exp(\omega\sup_{s\in\mathbb{T}}\mu(s))W_{ij}(\omega),\ i=1,2,\dots,m,j=1,2,\dots,n,$$

where

$$\begin{aligned}W_{ij}(\omega)=\Big(&\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}M^f\rho+\sum_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}(M^g+L^g)\overline{\delta_{kl}}\rho\exp\left\{\omega\overline{\delta_{kl}}\right\}\\&+\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}L^f\rho\exp\left\{\omega\overline{\tau_{kl}}\right\}\Big),\quad i=1,2,\dots,m,j=1,2,\dots,n.\end{aligned}$$

By (H_3), we get

$$S_{ij}(0)=\underline{a_{ij}}-W_{ij}(0)>0,\ i=1,2,\dots,m,j=1,2,\dots,n.$$

Since S_{ij} is continuous on $[0,+\infty)$ and $S_{ij}(\omega) \to -\infty$, as $\omega \to +\infty$, there exist $\xi_{ij} > 0$ such that $S_{ij}(\xi_{ij}) = 0$ and $S_{ij}(\omega) > 0$ for $\omega \in (0,\xi_{ij})$, $i = 1,2,\dots,m$, $j = 1,2,\dots,n$. Take $a = \min\limits_{1\le i\le m,1\le j\le n}\{\xi_{ij}\}$. We have $S_{ij}(a) \ge 0, i = 1,2,\dots,m, j = 1,2,\dots,n$, so we can choose a positive constant $0 < \lambda < \min\{a, \min\limits_{1\le i\le m,1\le j\le n}\{\underline{a_{ij}}\}\}$ such that

$$S_{ij}(\lambda) > 0, \quad i = 1,2,\dots,m, j = 1,2,\dots,n,$$

which implies that

$$\frac{\exp(\lambda \sup\limits_{s\in\mathbb{T}}\mu(s))W_{ij}(\lambda)}{\underline{a_{ij}} - \lambda} < 1,\ i = 1,2,\dots,m, j = 1,2,\dots,n. \tag{5.5}$$

Let

$$M = \max_{1\le i\le m,1\le j\le n}\left\{\frac{\underline{a_{ij}}}{W_{ij}(0)}\right\}.$$

By (H_3) and (5.5), we have $M > 1$.

Moreover, we have $e_{\ominus\lambda}(t,t_0) > 1$, where $t \le t_0$. Hence, it is obvious that

$$\|y(t)\|_{\mathbb{B}} \le M e_{\ominus\lambda}(t,t_0)\|\psi\|_{\mathbb{B}}, \qquad \forall t \in [t_0-\theta,t_0]_{\mathbb{T}},$$

where $\lambda \in \mathcal{R}^+$. In the following, we will show that

$$\|y(t)\|_{\mathbb{B}} \le M e_{\ominus\lambda}(t,t_0)\|\psi\|_{\mathbb{B}}, \quad \forall t \in (t_0,+\infty)_{\mathbb{T}}. \tag{5.6}$$

To prove (5.6), we first show that for any $p > 1$, the following inequality holds:

$$\|y(t)\|_{\mathbb{B}} < pM e_{\ominus\lambda}(t,t_0)\|\psi\|_{\mathbb{B}}, \quad \forall t \in (t_0,+\infty)_{\mathbb{T}}, \tag{5.7}$$

which implies that, for $i = 1,2,\dots,m$, $j = 1,2,\dots,n$, we have

$$|y_{ij}(t)| < pM e_{\ominus\lambda}(t,t_0)\|\psi\|_{\mathbb{B}}, \quad \forall t \in (t_0,+\infty)_{\mathbb{T}}. \tag{5.8}$$

If (5.8) is not true, then there exists $t_1 \in (t_0,+\infty)_{\mathbb{T}}$ such that

$$|y_{ij}(t_1)| \ge pM e_{\ominus\lambda}(t_1,t_0)\|\psi\|_{\mathbb{B}} \ \text{ and } \ |y_{ij}(t)| < pM e_{\ominus\lambda}(t,t_0)\|\psi\|_{\mathbb{B}}, \quad \forall t \in (t_0,t_1)_{\mathbb{T}}.$$

Therefore, there must exist a constant $c \ge 1$ such that

$$|y_{ij}(t_1)| = cpM e_{\ominus\lambda}(t_1,t_0)\|\psi\|_{\mathbb{B}} \ \text{ and } \ |y_{ij}(t)| < cpM e_{\ominus\lambda}(t,t_0)\|\psi\|_{\mathbb{B}}, \quad \forall t \in (t_0,t_1)_{\mathbb{T}}.$$

Note that, in view of (5.4), we have that

$$\begin{aligned} |y_{ij}(t_1)| \ &= \ \bigg| y_{ij}(t_0)e_{-a_{ij}}(t_1,t_0) - \int_{t_0}^{t_1} e_{-a_{ij}}(t_1,\sigma(s))\bigg\{\sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(s)\big(f(x_{kl}(s-\tau_{kl}(s)))x_{ij}(s) \\ &\quad -f(x_{kl}^*(s-\tau_{kl}(s)))x_{ij}^*(s)\big) + \sum_{C_{kl}\in N_p(i,j)} C_{ij}^{kl}(s)\Big(\int_{s-\delta_{kl}(s)}^{s} g(x_{kl}(u))\Delta u x_{ij}(s) \\ &\quad - \int_{s-\delta_{kl}(s)}^{s} g(x_{kl}^*(u))\Delta u x_{ij}^*(s)\Big)\bigg\}\Delta s\bigg| \\ &\le \ e_{-a_{ij}}(t_1,t_0)\|\psi\|_{\mathbb{B}} + \int_{t_0}^{t_1} e_{-a_{ij}}(t_1,\sigma(s))\bigg\{\sum_{C_{kl}\in N_r(i,j)} \overline{B_{ij}^{kl}}\big|f(x_{kl}(s-\tau_{kl}(s)))x_{ij}(s) \end{aligned}$$

$$\begin{aligned}
&\quad -f(x^*_{kl}(s-\tau_{kl}(s)))x^*_{ij}(s)\big| + \sum_{C_{kl}\in N_p(i,j)} \overline{C^{kl}_{ij}}\Big|\int_{s-\delta_{kl}(s)}^{s} g(x_{kl}(u))\Delta u x_{ij}(s) \\
&\quad -\int_{s-\delta_{kl}(s)}^{s} g(x^*_{kl}(u))\Delta u x^*_{ij}(s)\Big|\Big\}\Delta s \\
&\leq e_{-a_{ij}}(t_1,t_0)\|\psi\|_{\mathbb{B}} + \int_{t_0}^{t_1} e_{-a_{ij}}(t_1,\sigma(s))\Big\{\sum_{C_{kl}\in N_r(i,j)} \overline{B^{kl}_{ij}} M^f |x_{kl}(s-\tau_{kl}(s))||y_{ij}(s)| \\
&\quad + \sum_{C_{kl}\in N_p(i,j)} \overline{C^{kl}_{ij}} \int_{s-\delta_{kl}(s)}^{s} M^g |x_{kl}(u)|\Delta u |y_{ij}(s)| \\
&\quad + \sum_{C_{kl}\in N_r(i,j)} \overline{B^{kl}_{ij}} L^f |y_{kl}(s-\tau_{kl}(s))||x^*_{ij}(s)| \\
&\quad + \sum_{C_{kl}\in N_p(i,j)} \overline{C^{kl}_{ij}} \int_{s-\delta_{kl}(s)}^{s} L^g |y_{kl}(u)|\Delta u\big||x^*_{ij}(s)|\Big)\Big\}\Delta s \\
&\leq e_{-a_{ij}}(t_1,t_0)\|\psi\|_{\mathbb{B}} + cpMe_{\ominus\lambda}(t_1,t_0)\|\psi\|_{\mathbb{B}}\Big|\int_{t_0}^{t_1} e_{-a_{ij}\oplus\lambda}(t_1,\sigma(s)) \\
&\quad \times\Big\{\sum_{C_{kl}\in N_r(i,j)} \overline{B^{kl}_{ij}} M^f \rho e_\lambda(\sigma(s),s) + \sum_{C_{kl}\in N_p(i,j)} \overline{C^{kl}_{ij}} M^g \overline{\delta_{kl}} \rho e_\lambda(\sigma(s),s-\delta_{kl}(s)) \\
&\quad + \sum_{C_{kl}\in N_r(i,j)} \overline{B^{kl}_{ij}} L^f \rho e_\lambda(\sigma(s),s-\tau_{kl}(s)) \\
&\quad + \sum_{C_{kl}\in N_p(i,j)} \overline{C^{kl}_{ij}} L^g \overline{\delta_{kl}} \rho e_\lambda(\sigma(s),s-\delta_{kl}(s))\big|\Big\}\Big|\Delta s \\
&\leq e_{-a_{ij}}(t_1,t_0)\|\psi\|_{\mathbb{B}} + cpMe_{\ominus\lambda}(t_1,t_0)\|\psi\|_{\mathbb{B}}\Big|\int_{t_0}^{t_1} e_{-a_{ij}\oplus\lambda}(t_1,\sigma(s)) \\
&\quad \times\Big\{\sum_{C_{kl}\in N_r(i,j)} \overline{B^{kl}_{ij}} M^f \rho \exp\big\{\lambda\sup_{s\in\mathbb{T}}\mu(s)\big\} \\
&\quad + \sum_{C_{kl}\in N_p(i,j)} \overline{C^{kl}_{ij}} M^g \overline{\delta_{kl}} \rho \exp\big\{\lambda(\overline{\delta_{kl}} + \sup_{s\in\mathbb{T}}\mu(s))\big\} \\
&\quad + \sum_{C_{kl}\in N_r(i,j)} \overline{B^{kl}_{ij}} L^f \rho \exp\big\{\lambda(\overline{\tau_{kl}} + \sup_{s\in\mathbb{T}}\mu(s))\big\} \\
&\quad + \sum_{C_{kl}\in N_p(i,j)} \overline{C^{kl}_{ij}} L^g \overline{\delta_{kl}} \rho \exp\big\{\lambda(\overline{\delta_{kl}} + \sup_{s\in\mathbb{T}}\mu(s))\big\}\Big\}\Big|\Delta s \\
&= cpMe_{\ominus\lambda}(t_1,t_0)\|\psi\|_{\mathbb{B}}\Big\{\frac{1}{cpM}e_{-a_{ij}\oplus\lambda}(t_1,t_0) + \exp\big\{\lambda\sup_{s\in\mathbb{T}}\mu(s)\big\}\Big[\sum_{C_{kl}\in N_r(i,j)} \overline{B^{kl}_{ij}} M^f \rho \\
&\quad + \sum_{C_{kl}\in N_p(i,j)} \overline{C^{kl}_{ij}} (M^g + L^g)\overline{\delta_{kl}} \rho \exp\big\{\lambda\overline{\delta_{kl}}\big\} + \sum_{C_{kl}\in N_r(i,j)} \overline{B^{kl}_{ij}} L^f \rho \exp\big\{\lambda\overline{\tau_{kl}}\big\}\Big] \\
&\quad \times\int_{t_0}^{t_1} e_{-a_{ij}\oplus\lambda}(t_1,\sigma(s))\Delta s \\
&< cpMe_{\ominus\lambda}(t_1,t_0)\|\psi\|_{\mathbb{B}}\Big\{\frac{1}{M}e_{-(\underline{a_{ij}}-\lambda)}(t_1,t_0) + \exp\big\{\lambda\sup_{s\in\mathbb{T}}\mu(s)\big\}\Big[\sum_{C_{kl}\in N_r(i,j)} \overline{B^{kl}_{ij}} M^f \rho \\
&\quad + \sum_{C_{kl}\in N_p(i,j)} \overline{C^{kl}_{ij}} (M^g + L^g)\overline{\delta_{kl}} \rho \exp\big\{\lambda\overline{\delta_{kl}}\big\} + \sum_{C_{kl}\in N_r(i,j)} \overline{B^{kl}_{ij}} L^f \rho \exp\big\{\lambda\overline{\tau_{kl}}\big\}\Big]
\end{aligned}$$

$$
\begin{aligned}
&\times\frac{1}{-(\underline{a_{ij}}-\lambda)}\int_{t_0}^{t_1}(-(\underline{a_{ij}}-\lambda))e_{-(\underline{a_{ij}}-\lambda)}(t_1,\sigma(s))\Delta s\\
=\;& cpMe_{\ominus\lambda}(t_1,t_0)\|\psi\|_{\mathbb{B}}\Big\{\frac{1}{M}-\frac{\exp\big\{\lambda\sup\limits_{s\in\mathbb{T}}\mu(s)\big\}}{\underline{a_{ij}}-\lambda}\Big(\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}M^f\rho\\
&+\sum_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}(M^g+L^g)\overline{\delta_{kl}}\rho\exp\big\{\lambda\overline{\delta_{kl}}\big\}+\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}L^f\rho\exp\big\{\lambda\overline{\tau_{kl}}\big\}\Big)\\
&\times e_{-(\underline{a_{ij}}-\lambda)}(t_1,t_0)+\frac{\exp\big\{\lambda\sup\limits_{s\in\mathbb{T}}\mu(s)\big\}}{\underline{a_{ij}}-\lambda}\Big(\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}M^f\rho\\
&+\sum_{C_{kl}\in N_p(i,j)}\overline{C_{ij}^{kl}}(M^g+L^g)\overline{\delta_{kl}}\rho\exp\big\{\lambda\overline{\delta_{kl}}\big\}+\sum_{C_{kl}\in N_r(i,j)}\overline{B_{ij}^{kl}}L^f\rho\exp\big\{\lambda\overline{\tau_{kl}}\big\}\Big)\Big\}\\
<\;& cpMe_{\ominus\lambda}(t_1,t_0)\|\psi\|_{\mathbb{B}},
\end{aligned}
$$

which is a contradiction. Therefore, (5.7) holds. Letting $p\to 1$, we obtain that (5.6) holds. Hence, we have that

$$\|y(t)\|_{\mathbb{B}}<Me_{\ominus\lambda}(t,t_0)\|\psi\|_{\mathbb{B}},\quad \forall t\in(t_0,+\infty)_{\mathbb{T}},$$

which means that the almost automorphic solution of system (5.1) is globally exponentially stable. The proof is complete. □

Remark 5.4. Theorems 5.2 and 5.3 are new even for the cases of differential equations ($\mathbb{T}=\mathbb{R}$) and difference equations ($\mathbb{T}=\mathbb{Z}$).

6. Examples

In this section, we will give examples to illustrate the feasibility and effectiveness of the results obtained in Section 5.

Consider the following SICNNs with time-varying delays:

$$
\begin{aligned}
x_{ij}^{\Delta}(t)=&-a_{ij}(t)x_{ij}(t)-\sum_{C_{kl}\in N_r(i,j)}B_{ij}^{kl}(t)f(x_{kl}(t-\tau_{kl}(t)))x_{ij}(t)\\
&-\sum_{C_{kl}\in N_p(i,j)}C_{ij}^{kl}(t)\int_{t-\delta_{kl}(t)}^{t}g(x_{kl}(u))\Delta u x_{ij}(t)+L_{ij}(t),
\end{aligned}\tag{6.1}
$$

where $i=1,2,3,\ j=1,2,3,\ r=p=1,\ f(x)=0.05|\cos x|,\ g(x)=\frac{|x|}{20},\ t\in\mathbb{T}$.

Example 6.1. If $\mathbb{T}=\mathbb{R}$, then $\mu(t)\equiv 0$. Take

$$\begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t)\\ a_{21}(t) & a_{22}(t) & a_{23}(t)\\ a_{31}(t) & a_{32}(t) & a_{33}(t)\end{pmatrix}=\begin{pmatrix} 3+|\sin t| & 4+|\cos\sqrt{2}t| & 2+|\sin t|\\ 2+|\cos(\frac{1}{3}t)| & 5+|\sin 2t| & 1+|\cos 2t|\\ 4+|\cos 2t| & 3+|\sin 2t| & 2+|\cos t|\end{pmatrix},$$

$$\begin{pmatrix} B_{11}(t) & B_{12}(t) & B_{13}(t)\\ B_{21}(t) & B_{22}(t) & B_{23}(t)\\ B_{31}(t) & B_{32}(t) & B_{33}(t)\end{pmatrix}=\begin{pmatrix} C_{11}(t) & C_{12}(t) & C_{13}(t)\\ C_{21}(t) & C_{22}(t) & C_{23}(t)\\ C_{31}(t) & C_{32}(t) & C_{33}(t)\end{pmatrix}=\begin{pmatrix} 0.1|\cos t| & 0.2|\cos 2t| & 0.1|\sin t|\\ 0.1|\sin 2t| & 0.1|\cos t| & 0.2|\cos 2t|\\ 0.1|\cos t| & 0.2|\cos 2t| & 0.1|\cos t|\end{pmatrix},$$

$$\begin{pmatrix} L_{11}(t) & L_{12}(t) & L_{13}(t)\\ L_{21}(t) & L_{22}(t) & L_{23}(t)\\ L_{31}(t) & L_{32}(t) & L_{33}(t)\end{pmatrix}=\begin{pmatrix} 0.1\sin t & \cos t & 0.2\sin t\\ 0.2\sin t & 0.4\sin t & 0.1\sin t\\ 0.1\sin t & 0.3\sin t & 0.1\cos t\end{pmatrix},$$

$$\begin{pmatrix} \delta_{11}(t) & \delta_{12}(t) & \delta_{13}(t) \\ \delta_{21}(t) & \delta_{22}(t) & \delta_{23}(t) \\ \delta_{31}(t) & \delta_{32}(t) & \delta_{33}(t) \end{pmatrix} = \begin{pmatrix} 0.5+0.5\sin t & \cos t & 0.7+0.2\sin t \\ 0.8+0.2\sin t & 0.2+0.4\sin t & 0.7+0.2\sin t \\ 0.9+0.1\sin t & 0.6+0.3\sin t & 0.4+0.4\cos t \end{pmatrix}.$$

Obviously, (H_1) holds. Clearly, we have

$$M^f = M^g = 0.05, \quad L^f = L^g = 0.05, \quad \sum_{C_{kl}\in N_1(1,1)} \overline{B_{11}^{kl}} = \sum_{C_{kl}\in N_1(1,1)} \overline{C_{11}^{kl}} = 0.5,$$

$$\sum_{C_{kl}\in N_1(1,2)} \overline{B_{12}^{kl}} = \sum_{C_{kl}\in N_1(1,2)} \overline{C_{12}^{kl}} = 1, \quad \sum_{C_{kl}\in N_1(1,3)} \overline{B_{13}^{kl}} = \sum_{C_{kl}\in N_1(1,3)} \overline{C_{13}^{kl}} = 0.6,$$

$$\sum_{C_{kl}\in N_1(2,1)} \overline{B_{21}^{kl}} = \sum_{C_{kl}\in N_1(2,1)} \overline{C_{21}^{kl}} = 0.8, \quad \sum_{C_{kl}\in N_1(2,2)} \overline{B_{22}^{kl}} = \sum_{C_{kl}\in N_1(2,2)} \overline{C_{22}^{kl}} = 1.2,$$

$$\sum_{C_{kl}\in N_1(2,3)} \overline{B_{23}^{kl}} = \sum_{C_{kl}\in N_1(2,3)} \overline{C_{23}^{kl}} = 1, \quad \sum_{C_{kl}\in N_1(3,1)} \overline{B_{31}^{kl}} = \sum_{C_{kl}\in N_1(3,1)} \overline{C_{31}^{kl}} = 0.5,$$

$$\sum_{C_{kl}\in N_1(3,2)} \overline{B_{32}^{kl}} = \sum_{C_{kl}\in N_1(3,2)} \overline{C_{32}^{kl}} = 0.8, \quad \sum_{C_{kl}\in N_1(3,3)} \overline{B_{33}^{kl}} = \sum_{C_{kl}\in N_1(3,3)} \overline{C_{33}^{kl}} = 0.6$$

and we can easily check that $\max\limits_{i,j} \frac{\overline{L_{ij}}}{\underline{a_{ij}}} = 0.25$. Take $\rho = 1$. Then

$$\max_{i,j}\left\{ \frac{\sum\limits_{C_{kl}\in N_r(i,j)} \overline{B_{ij}^{kl}} M^f \rho^2 + \sum\limits_{C_{kl}\in N_p(i,j)} \overline{C_{ij}^{kl}} M^g \rho^2 \overline{\delta_{kl}} + \overline{L_{ij}}}{\underline{a_{ij}}} \right\} \approx 0.2669 < \rho = 1$$

and

$$\max_{i,j}\left\{ \frac{\sum\limits_{C_{kl}\in N_r(i,j)} \overline{B_{ij}^{kl}} (M^f + L^f)\rho + \sum\limits_{C_{kl}\in N_p(i,j)} \overline{C_{ij}^{kl}} (M^g + L^g)\rho \overline{\delta_{kl}}}{\underline{a_{ij}}} \right\} \approx 0.5337 < 1.$$

The combination of the above two inequalities means that (H_3) is satisfied for $\rho = 1$.

Therefore, we have shown that assumptions (H_1)-(H_3) are satisfied. By Theorem 5.2, system (6.1) has exactly one almost automorphic solution in $\mathbb{E} = \{\varphi \in \mathbb{B} : \|\varphi\|_{\mathbb{B}} \le \rho\}$. Moreover, by Theorem 5.3 this solution is globally exponentially stable.

Example 6.2. If $\mathbb{T} = \mathbb{Z}$, then $\mu(t) \equiv 1$. Take

$$\begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{pmatrix} = \begin{pmatrix} 0.3+0.1|\sin t| & 0.4+0.1|\cos\sqrt{2}t| & 0.2+0.1|\sin t| \\ 0.2+0.2|\cos(\frac{1}{3}t)| & 0.5+0.1|\sin 2t| & 0.1+0.1|\cos 2t| \\ 0.4+0.1|\cos 2t| & 0.3+0.2|\sin 2t| & 0.2+0.1|\cos t| \end{pmatrix},$$

$$\begin{pmatrix} B_{11}(t) & B_{12}(t) & B_{13}(t) \\ B_{21}(t) & B_{22}(t) & B_{23}(t) \\ B_{31}(t) & B_{32}(t) & B_{33}(t) \end{pmatrix} = \begin{pmatrix} C_{11}(t) & C_{12}(t) & C_{13}(t) \\ C_{21}(t) & C_{22}(t) & C_{23}(t) \\ C_{31}(t) & C_{32}(t) & C_{33}(t) \end{pmatrix} = \begin{pmatrix} 0.02|\cos t| & 0.02|\cos 2t| & 0.03|\sin t| \\ 0.01|\sin 2t| & 0.01|\cos t| & 0.02|\cos 2t| \\ 0.03|\cos t| & 0.02|\cos 2t| & 0.01|\cos t| \end{pmatrix},$$

$$\begin{pmatrix} L_{11}(t) & L_{12}(t) & L_{13}(t) \\ L_{21}(t) & L_{22}(t) & L_{23}(t) \\ L_{31}(t) & L_{32}(t) & L_{33}(t) \end{pmatrix} = \begin{pmatrix} 0.01\sin t & 0.01\cos t & 0.02\sin t \\ 0.02\sin t & 0.04\sin t & 0.03\sin t \\ 0.01\sin t & 0.03\sin t & 0.02\cos t \end{pmatrix},$$

$$\begin{pmatrix} \delta_{11}(t) & \delta_{12}(t) & \delta_{13}(t) \\ \delta_{21}(t) & \delta_{22}(t) & \delta_{23}(t) \\ \delta_{31}(t) & \delta_{32}(t) & \delta_{33}(t) \end{pmatrix} = \begin{pmatrix} \sin t & 0.9\cos t & 0.2+0.5\sin t \\ 0.2+0.8\sin t & 0.4+0.6\sin t & 0.1+0.9\sin t \\ \sin t & \sin t & 0.7+0.2\cos t \end{pmatrix}.$$

Obviously, (H_1) holds. Clearly, we have

$$M^f = M^g = 0.05,\ \ L^f = L^g = 0.05,\quad \sum_{C_{kl}\in N_1(1,1)} \overline{B_{11}^{kl}} = \sum_{C_{kl}\in N_1(1,1)} \overline{C_{11}^{kl}} = 0.06,$$

$$\sum_{C_{kl}\in N_1(1,2)} \overline{B_{12}^{kl}} = \sum_{C_{kl}\in N_1(1,2)} \overline{C_{12}^{kl}} = 0.11,\quad \sum_{C_{kl}\in N_1(1,3)} \overline{B_{13}^{kl}} = \sum_{C_{kl}\in N_1(1,3)} \overline{C_{13}^{kl}} = 0.08,$$

$$\sum_{C_{kl}\in N_1(2,1)} \overline{B_{21}^{kl}} = \sum_{C_{kl}\in N_1(2,1)} \overline{C_{21}^{kl}} = 0.11,\quad \sum_{C_{kl}\in N_1(2,2)} \overline{B_{22}^{kl}} = \sum_{C_{kl}\in N_1(2,2)} \overline{C_{22}^{kl}} = 0.17,$$

$$\sum_{C_{kl}\in N_1(2,3)} \overline{B_{23}^{kl}} = \sum_{C_{kl}\in N_1(2,3)} \overline{C_{23}^{kl}} = 0.11,\quad \sum_{C_{kl}\in N_1(3,1)} \overline{B_{31}^{kl}} = \sum_{C_{kl}\in N_1(3,1)} \overline{C_{31}^{kl}} = 0.07,$$

$$\sum_{C_{kl}\in N_1(3,2)} \overline{B_{32}^{kl}} = \sum_{C_{kl}\in N_1(3,2)} \overline{C_{32}^{kl}} = 0.1,\quad \sum_{C_{kl}\in N_1(3,3)} \overline{B_{33}^{kl}} = \sum_{C_{kl}\in N_1(3,3)} \overline{C_{33}^{kl}} = 0.06$$

and we can easily check that $\max\limits_{i,j} \frac{\overline{L_{ij}}}{\underline{a_{ij}}} = 0.15$. Take $\rho = 1$, then

$$\max_{i,j}\left\{\frac{\sum\limits_{C_{kl}\in N_r(i,j)} \overline{B_{ij}^{kl}} M^f \rho^2 + \sum\limits_{C_{kl}\in N_p(i,j)} \overline{C_{ij}^{kl}} M^g \rho^2 \overline{\delta_{kl}} + \overline{L_{ij}}}{\underline{a_{ij}}}\right\} \approx 0.1904 < \rho = 1$$

and

$$\max_{i,j}\left\{\frac{\sum\limits_{C_{kl}\in N_r(i,j)} \overline{B_{ij}^{kl}} (M^f + L^f)\rho + \sum\limits_{C_{kl}\in N_p(i,j)} \overline{C_{ij}^{kl}} (M^g + L^g)\rho \overline{\delta_{kl}}}{\underline{a_{ij}}}\right\} \approx 0.3657 < 1.$$

The combination of the above two inequalities means that (H_3) is satisfied for $\rho = 1$.

Therefore, we have shown that assumptions (H_1)-(H_3) are satisfied. By Theorem 5.2, system (6.1) has exactly one almost automorphic solution in $\mathbb{E} = \{\varphi \in \mathbb{B} : \|\varphi\|_{\mathbb{B}} \le \rho\}$. Moreover, by Theorem 5.3 this solution is globally exponentially stable.

7. Conclusion

In this paper, we propose a new concept of almost automorphic functions on almost periodic time scales and study some basic properties. We state two open problems concerning the relationship between the algebraic operation property of elements of a time scale and the analytical property of the time scale. Then, based on these concepts and results, we establish the existence of an almost automorphic solution for both the linear nonhomogeneous dynamic equation on time scales and its associated homogeneous equation. Finally, as an application of the results, we study the existence and global exponential stability of almost automorphic solutions to a class of shunting inhibitory cellular neural networks with time-varying delays on time scales.

Acknowledgements:

This work is supported by the National Natural Sciences Foundation of People's Republic of China under Grant 11361072.

References

[1] M. Adıvar, H. C. Koyuncuoğlu, Y. N. Raffoul, *Existence of periodic solutions in shifts $\delta_{\pm}$ for neutral nonlinear dynamic systems*, Appl. Math. Comput., **242** (2014), 328–339. 1

[2] A. Ardjouni, A. Djoudi, *Existence of periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale*, Commun. Nonlinear Sci. Numer. Simul., **17** (2012), 3061–3069. 1

[3] L. Bi, M. Bohner, M. Fan, *Periodic solutions of functional dynamic equations with infinite delay*, Nonlinear Anal., **68** (2008), 1226–1245. 1

[4] Y. T. Bian, Y. K. Chang, J. J. Nieto, *Weighted asymptotic behavior of solutions to semilinear integro-differential equations in Banach spaces*, Electron. J. Differential Equations, **2014** (2014), 16 pages. 1

[5] S. Bochner, *Curvature and Betti numbers in real and complex vector bundles*, Univ. e Politec. Torino. Rend. Sem. Mat., **15** (1955–56), 225–253. 1

[6] S. Bochner, *A new approach to almost periodicity*, Proc. Nat. Acad. Sci. U.S.A., **48** (1962), 2039–2043. 1

[7] S. Bochner, *Continuous mappings of almost automorphic and almost periodic functions*, Proc. Nat. Acad. Sci. U.S.A., **52** (1964), 907–910. 1

[8] M. Bohner, M. Fan, J. Zhang, *Existence of periodic solutions in predator-prey and competition dynamic systems*, Nonlinear Anal. Real World Appl., **7** (2006), 1193–1204. 1

[9] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales, An Introuduction with Applications*, Birkhäuser, Boston, (2001). 2

[10] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, (2003). 2

[11] Y. K. Chang, Z. H. Zhao, J. J. Nieto, *Pseudo almost automorphic and weighted pseudo almost automorphic mild solutions to semi-linear differential equations in Hilbert spaces*, Rev. Mat. Complut., **24** (2011), 421–438. 1

[12] H. S. Ding, G. M. N'Guérékata, J. J. Nieto, *Weighted pseudo almost periodic solutions for a class of discrete hematopoiesis model*, Rev. Mat. Complut., **26** (2013), 427–443. 1

[13] Q. Fan, J. Shao, *Positive almost periodic solutions for shunting inhibitory cellular neural networks with time-varying and continuously distributed delays*, Commun. Nonlinear Sci. Numer. Simulat., **15** (2010), 1655–1663. 5

[14] M. Fazly, M. Hesaaraki, *Periodic solutions for predator-prey systems with Beddington-DeAngelis functional response on time scales*, Nonlinear Anal. Real World Appl., **9** (2008), 1224–1235. 1

[15] J. Gao, Q. R. Wang, L. W. Zhang, *Existence and stability of almost-periodic solutions for cellular neural networks with time-varying delays in leakage terms on time scales*, Appl. Math. Comput., **237** (2014), 639–649. 1

[16] W. Guan, D. G. Li, S. H. Ma, *Nonlinear first-order periodic boundary-value problems of impulsive dynamic equations on time scales*, Electron. J. Differential Equations **2012** (2012), 8 pages. 1

[17] S. Hilger, *Analysis on measure chains-a unified approach to continuous and discrete calculus*, Result. Math., **18** (1990), 18–56. 1

[18] E. R. Kaufmann, Y. N. Raffoul, *Periodic solutions for a neutral nonlinear dynamical equation on a time scale*, J. Math. Anal. Appl., **319** (2006), 315–325. 1, 1

[19] L. Li, Z. Fang, Y. Yang, *A shunting inhibitory cellular neural network with continuously distributed delays of neutral type*, Nonlinear Anal. Real World Appl., **13** (2012), 1186–1196. 5

[20] Y. K. Li, B. Li, *Almost periodic time scales and almost periodic functions on time scales*, J. Appl. Math., **2015** (2015), 8 pages. 1, 2.2, 2.3, 2, 2.5, 3

[21] Y. K. Li, C. C. Liu, L. F. Zhu, *Global exponential stability of periodic solution for shunting inhibitory CNNs with delays*, Phys. Lett. A, **337** (2005) 46–54. 5

[22] Y. K. Li, L. J. Sun, *Infinite many positive solutions for nonlinear first-order BVPs with integral boundary conditions on time scales*, Topol. Methods Nonlinear Anal., **41** (2013), 305–321. 1

[23] Y. K. Li, C. Wang, *Almost periodic functions on time scales and applications*, Discrete Dyn. Nat. Soc., **2011** (2011), 20 pages. 2.1, 4, 5

[24] Y. K. Li, C. Wang, *Uniformly almost periodic functions and almost periodic solutions to dynamic equations on time scales*, Abstr. Appl. Anal., **2011** (2011), 22 pages. 1, 1.2, 1

[25] Y. Li, C. Wang, *Pseudo almost periodic functions and pseudo almost periodic solutions to dynamic equations on time scales*, Adv. Difference Equ., **2012** (2012), 24 pages. 1

[26] Y. K. Li, C. Wang, *Almost periodic solutions of shunting inhibitory cellular neural networks on time scales*, Commun. Nonlinear Sci. Numer. Simul., **17** (2012), 3258–3266. 5

[27] Y. K. Li, L. Yang, W. Q. Wu, *Anti-periodic solution for impulsive BAM neural networks with time-varying leakage delays on time scales*, Neurocomputing, **149** (2015), 536–545. 1

[28] Y. K. Li, L. L. Zhao, L. Yang, *C^1-Almost periodic solutions of BAM neural networks with time-varying delays on time scales*, The Scientific World J., **2015** (2015), 15 pages. 1, 3

[29] T. Liang, Y. Yang, Y. Liu, L. Li, *Existence and global exponential stability of almost periodic solutions to Cohen-Grossberg neural networks with distributed delays on time scales*, Neurocomputing, **123** (2014), 207–215. 1

[30] Y. Liao, L. Xu, *Almost periodic solution for a delayed Lotka-Volterra system on time scales*, Adv. Difference Equ., **2014** (2014), 19 pages. 1

[31] C. Lizama, J. G. Mesquita, *Almost automorphic solutions of dynamic equations on time scales*, J. Funct. Anal., **265** (2013), 2267–2311. 1

[32] C. Ou, *Almost periodic solutions for shunting inhibitory cellular neural networks*, Nonlinear Anal. Real World Appl., **10** (2009), 2652–2658. 5

[33] Y. H. Su, Z. S. Feng, *Homoclinic orbits and periodic solutions for a class of Hamiltonian systems on time scales*, J. Math. Anal. Appl., **411** (2014), 37–62. 1

[34] C. Wang, *Almost periodic solutions of impulsive BAM neural networks with variable delays on time scales*, Commun. Nonlinear Sci. Numer. Simul., **19** (2014), 2828–2842. 1

[35] S. P. Wang, *Existence of periodic solutions for higher order dynamic equations on time scales*, Taiwanese J. Math., **16** (2012), 2259–2273. 1

[36] C. Wang, R. P. Agarwal, *Weighted piecewise pseudo almost automorphic functions with applications to abstract impulsive* ∇*-dynamic equations on time scales*, Adv. Difference Equ., **2014**, (2014), 29 pages. 1

[37] C. Wang, R. P. Agarwal, *A further study of almost periodic time scales with some notes and applications*, Abstr. Appl. Anal., **2014** (2014), 11 pages. 1

[38] Y. Wang, L. Li, *Almost periodic solutions for second order dynamic equations on time scales*, Discrete Dyn. Nat. Soc., **2013** (2013), 11 pages. 1

[39] C. Wang, Y. Li, *Weighted pseudo almost automorphic functions with applications to abstract dynamic equations on time scales*, Ann. Polon. Math., **108** (2013), 225–240. 1, 1.3

[40] L. Wang, M. Yu, *Favard's theorem of piecewise continuous almost periodic functions and its application*, J. Math. Anal. Appl., **413**, (2014), 35–46. 1

[41] W. Wu, *Existence and uniqueness of globally attractive positive almost periodic solution in a predator-prey dynamic system with Beddington-DeAngelis functional response*, Abstr. Appl. Anal., **2014** (2014), 9 pages. 1

[42] Y. H. Xia, J. D. Cao, Z. K. Huang, *Existence and exponential stability of almost periodic solution for shunting inhibitory cellular neural networks with impulses*, Chaos Solitons Fractals, **34** (2007), 1599–1607. 5

[43] Z. J. Yao, *Uniqueness and global exponential stability of almost periodic solution for Hematopoiesis model on time scales*, J. Nonlinear Sci. Appl., **8** (2015), 142–152. 1

[44] X. S. Yang, *Existence and global exponential stability of periodic solution for Cohen-Grossberg shunting inhibitory cellular neural networks with delays and impulses*, Neurocomputing, **72** (2009), 2219–2226. 5

[45] A. Zafer, *The stability of linear periodic Hamiltonian systems on time scales*, Appl. Math. Lett., **26** (2013), 330–336. 1

[46] J. Zhang, M. Fan, M. Bohner, *Periodic solutions of nonlinear dynamic systems with feedback control*, Int. J. Difference Equ., **6** (2011), 59–79. 1

[47] H. Zhang, Y. Li, *Existence of positive periodic solutions for functional differential equations with impulse effects on time scales*, Commun. Nonlinear Sci. Numer. Simul., **14** (2009), 19–26. 1

[48] Q. Zhang, L. Yang, J. Liu, *Existence and stability of anti-periodic solutions for impulsive fuzzy Cohen-Grossberge neural networks on time scales*, Math. Slovaca, **64** (2014), 119–138. 1

[49] W. R. Zhao, H. S. Zhang, *On almost periodic solution of shunting inhibitory cellular neural networks with variable coefficients and time-varying delays*, Nonlinear Anal. Real World Appl., **9** (2008), 2326–2336. 5

[50] H. Zhou, Z. F. Zhou, W. Jiang, *Almost periodic solutions for neutral type BAM neural networks with distributed leakage delays on time scales*, Neurocomputing, **157** (2015), 223–230. 1