Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 6048–6057
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Positive solutions of an integral boundary value problem for singular differential equations of mixed type with p-Laplacian

Changlong Yu*, Jufang Wang, Yanping Guo

College of Sciences, Hebei University of Science and Technology, Shijiazhuang, 050018, Hebei, P. R. China.

Communicated by M. Jleli

Abstract

In this paper, by Leggett-William fixed point theorem, we establish the existence of triple positive solutions of a new kind of integral boundary value problem for the nonlinear singular differential equations with p-Laplacian operator, in which $q(t)$ can be singular at $t = 0, 1$. We also show that the results obtained can be applied to study certain higher order mixed boundary value problems. At last, we give an example to demonstrate the use of the main result of this paper. The conclusions in this paper essentially extend and improve the known results.

Keywords: Positive solutions, integral boundary value problem, Leggett-William fixed point theorem, p-Laplacian operator, cone.
2010 MSC: 34B15, 34B16, 34B18, 34G20.

1. Introduction

In this paper, by using Leggett-William fixed point theorem, we will study the existence of triple positive solutions for the following a new kind of second order singular integral boundary value problem (IBVP):

$$\begin{cases} (\phi_p(u'))'(t) + q(t)f(u(t), u'(t), (Tu)(t), (Su)(t)) = 0, 0 \le t \le 1, \\ \phi_p(u'(0)) = \alpha \int_0^\xi \phi_p(u'(s))ds, \quad u(1) = \beta g(u'(\eta)), \end{cases} \tag{1.1}$$

*Corresponding author
Email addresses: changlongyu@126.com (Changlong Yu), wangjufang1981@126.com (Jufang Wang), guoyanping65@126.com (Yanping Guo)

Received 2016-04-23

where $\phi_p(s) = |s|^{p-2}s(p > 1)$ is an increasing function, $(\phi_p)^{-1}(s) = \phi_q(s), \frac{1}{p} + \frac{1}{q} = 1; 0 \le \xi \le 1, 0 \le \alpha\xi < 1, \beta > 0, 0 \le \eta \le 1;$ and T and S are two linear operators defined by

$$Tu(t) = \int_0^t k(t,s)u(s)ds, \quad Su(t) = \int_0^1 h(t,s)u(s)ds, \quad u \in C^1[0,1],$$

in which $k \in C[D, R^+],\ h \in C[D_0, R^+],\ D = \{(t,s) \in R^2 : 0 \le s \le t \le 1\},\ D_0 = \{(t,s) \in R^2 : 0 \le s,t \le 1\},\ R^+ = [0,+\infty),\ R = (-\infty,+\infty),\ k_0 = \max\{k(t,s) : (t,s) \in D\},$ and $h_0 = \max\{h(t,s) : (t,s) \in D_0\}$.

The existence and multiplicity of positive solutions for differential equations BVPs with the p-Laplacian operator have been extensively investigated in recent years, see [1–10, 12–16] and the references therein. Especially, the following differential equations with one-dimensional p-Laplacian:

$$(\phi_p(u'))' + q(t)f(t,u) = 0, \quad 0 \le t \le 1$$

has been studied subject to different kinds of boundary conditions, see [5, 9, 13, 16] and the references therein. The methods mainly depend on Kransnosel'skii fixed point theorem, upper and lower solution technique, Leggett-Williams fixed point theorem and some new fixed point theorems in cones and so on.

Recently, in [8], Kong et al. have studied the existence of triple positive solutions for the following BVP:

$$\begin{cases} (\phi_p(u'))'(t) + q(t)f(u(t), u'(t), (Tu)(t), (Su)(t)) = 0, \ 0 \le t \le 1, \\ u'(0) = 0, \quad u(1) = g(u'(1)). \end{cases} \tag{1.2}$$

In [7], Hu and Ma have studied the existence of triple positive solutions for the following BVP:

$$\begin{cases} (\phi_p(u'))'(t) + q(t)f(u(t), u'(t), (Tu)(t), (Su)(t)) = 0, \ 0 \le t \le 1, \\ u'(0) = \beta u'(\eta), \quad u(1) = g(u'(1)). \end{cases} \tag{1.3}$$

More recently, in [14], Wang and Yu have pointed out some mistakes in [8] and [7], and studied the existence of triple positive solution for the following BVP:

$$\begin{cases} (\phi_p(u'))'(t) + q(t)f(u(t), u'(t), (Tu)(t), (Su)(t)) = 0, \ 0 \le t \le 1, \\ \phi_p(u'(0)) = \sum_{i=1}^{m-2} \beta_i \phi_p(u'(\eta_i)), \qquad u(1) = g(u'(1)), \end{cases}$$

Motivated by the work above, in this paper, we will discuss the existence of solutions for the IBVP (1.1). Obviously, when $\alpha = 0, \beta = 1, \eta = 1$, IBVP (1.1) reduces to BVP (1.2). In addition, when $\beta = 1, \eta = 1$ and the integral boundary condition can be translated into $u'(0) = \phi_q(\alpha\xi)u'(\zeta)$ ($\zeta \in [0,\xi]$) by mean-value theorem of integrals, IBVP(1.1) reduces to BVP (1.3). Hence, BVPs (1.2) and (1.3) are special cases of IBVP (1.1).

Throughout this paper, we always suppose the following conditions hold:

(C_1) $f \in C(R^+ \times R \times R^+ \times R^+, (0,+\infty))$;

(C_2) $q(t) \in C([0,1], R^+)$ may be singular at $t = 0,1$ and $0 < \int_0^1 q(t)dt < +\infty$, so it is easy to see that there exists a constant $M > 0$ such that $0 < \int_0^1 q(t)dt < \phi_p(M)$;

(C_3) $g : R \to R^+$ is nonincreasing and continuous, and $0 \le g(v) \le |v|$ for $v \in R$.

We deal with the existence and uniqueness of solutions for BVP (1.1) by using the Schauder's fixed point theorem and Banach's contraction mapping principle and obtain multiplicity results which extend and improve the known results.

2. Preliminary results

In this section, we firstly present some definitions, theorems and lemmas, which will be needed in the proof of the main result.

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:

(i) $x \in P, \lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P, -x \in P$ implies $x = 0$.

Definition 2.2. Given a cone P in a real Banach space E. A continuous map α is called a concave (resp. convex) functional on P if and only if for all $x, y \in P$ and $0 \leq t \leq 1$, it holds

1. $\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$,
2. (resp. $\alpha(tx + (1-t)y) \leq t\alpha(x) + (1-t)\alpha(y)$).

We consider the Banach space $E = C^1[0,1]$ equipped with norm $\|u\| = \max_{0\leq t\leq 1}\{\|u(t)\|_0, \|u'(t)\|_0\}$, where $\|u\|_0 = \max_{0\leq t\leq 1}|u(t)|$.

We denote, for any fixed constants a, b, r,

$$\begin{aligned} C^+[0,1] &= \{u \in C[0,1] : u(t) \geq 0, t \in [0,1]\}, \\ P &= \{u \in E | u(t) \text{ is concave and nonincreasing on } [0,1]\}, \\ P_r &= \{u \in P : \|u\| < r\}, \\ P(\alpha, a, b) &= \{u \in P : a \leq \alpha(u), \|u\| < b\}. \end{aligned}$$

It is easy to see that P is a cone in E.

Theorem 2.3 (Leggett-William)**.** *Let $A : \overline{P}_c \to \overline{P}_c$ be a completely continuous map and let α be a nonnegative continuous concave functional on P with $\alpha(u) \leq \|u\|$ for any $u \in \overline{P}_c$. Suppose there exist constants a, b and d with $0 < a < b < d \leq c$ such that*

(i) *$\{u \in P(\alpha, b, d) : \alpha(u) > b\} \neq \phi$ and $\alpha(Au) > b$ for all $u \in P(\alpha, b, d)$;*
(ii) *$\|Au\| < a$ for all $u \in \overline{P}_a$;*
(iii) *$\alpha(Au) > b$ for all $u \in P(\alpha, b, c)$ with $\|Au\| > d$.*

Then A has at least three fixed points u_1, u_2, and u_3 satisfying

$$\|u_1\| < a, \quad b < \alpha(u_2), \quad \|u_3\| > a, \quad \text{and} \quad \alpha(u_3) < b.$$

Lemma 2.4. *Suppose $y \in C^1[0,1]$ with $(\phi_p(y'))' \in L^1[0,1]$ satisfies*

$$\begin{cases} (\phi_p(y'))'(t) \leq 0, \quad 0 \leq t \leq 1, \\ \phi_p(y'(0)) = \alpha \displaystyle\int_0^\xi \phi_p(y'(s))ds, \quad y(1) = \beta g(y'(\eta)). \end{cases}$$

Then, $y(t) \geq 0$ is concave and nonincreasing on $[0,1]$, that is, $y \in P$.

Proof. Since $(\phi_p(y'))'(t) \leq 0$, we know that $\phi_p(y')$ is nonincreasing, that is, $y'(t)$ is nonincreasing, which means $y(t)$ is concave. At the same time, we have $y'(t) \leq y'(0)$, so $y'(0) = \phi_q(\alpha \int_0^\xi \phi_p(y'(s))ds) \leq \phi_q(\alpha \int_0^\xi \phi_p(y'(0))ds) = \phi_q(\alpha\xi)y'(0)$, namely $y'(0) \leq 0$. Then $y'(t) \leq 0$, that is to say $y(t)$ is nonincreasing. So $y(t) \geq y(1) = \beta g(y'(\eta)) \geq 0$. Above all, $y \in P$. This completes the proof. $\square$

Lemma 2.5. *Let $y \in C[0,1]$ and $(\phi_p(u'))' \in L^1[0,1]$, then BVP*

$$\begin{cases} (\phi_p(u'))'(t) = -y(t), \quad 0 \le t \le 1, \\ \phi_p(u'(0)) = \alpha \displaystyle\int_0^{\xi} \phi_p(u'(s))ds, \quad u(1) = \beta g(u'(\eta)), \end{cases}$$

has a unique solution

$$\begin{aligned} u(t) = &\int_t^1 \phi_q\Big(\frac{\alpha}{1-\alpha\xi}\int_0^{\xi}(\xi - r)y(r)dr + \int_0^s y(r)dr\Big)ds \\ &+ \beta g\Big(-\phi_q\Big(\frac{\alpha}{1-\alpha\xi}\int_0^{\xi}(\xi - r)y(r)dr + \int_0^{\eta} y(r)dr\Big)\Big). \end{aligned}$$

Define the operator $A : P \to E$ by

$$\begin{aligned} (Au)(t) = &\int_t^1 \phi_q\Big(\frac{\alpha}{1-\alpha\xi}\int_0^{\xi}(\xi - r)q(r)f(u,u',Tu,Su)dr + \int_0^s q(r)f(u,u',Tu,Su)dr\Big)ds \\ &+ \beta g\Big(-\phi_q\Big(\frac{\alpha}{1-\alpha\xi}\int_0^{\xi}(\xi - r)q(r)f(u,u',Tu,Su)dr + \int_0^{\eta} q(r)f(u,u',Tu,Su)dr\Big)\Big), \end{aligned} \tag{2.1}$$

obviously, A is well-defined and $u \in E$ is a solution of BVP (1.1) if and only if u is a fixed point of A.

Lemma 2.6. *$A : P \to P$ is completely continuous.*

Proof. It is similar to the proof of Lemma 2.2 in [8]. □

Lemma 2.7. *For any $u \in P$, we have $\|Au\|_0 \le (1+\beta)\|(Au)'\|_0$, $\|Au\| \le (1+\beta)\|(Au)'\|_0$.*

Proof. From (2.1), we obtain

$$\begin{aligned} \|(Au)\|_0 = &\int_0^1 \phi_q\Big(\frac{\alpha}{1-\alpha\xi}\int_0^{\xi}(\xi - r)q(r)f(u,u',Tu,Su)dr + \int_0^s q(r)f(u,u',Tu,Su)dr\Big)ds \\ &+ \beta g\Big(-\phi_q\Big(\frac{\alpha}{1-\alpha\xi}\int_0^{\xi}(\xi - r)q(r)f(u,u',Tu,Su)dr + \int_0^{\eta} q(r)f(u,u',Tu,Su)dr\Big)\Big) \\ \le &\int_0^1 \phi_q\Big(\frac{\alpha}{1-\alpha\xi}\int_0^{\xi}(\xi - r)q(r)f(u,u',Tu,Su)dr + \int_0^s q(r)f(u,u',Tu,Su)dr\Big)ds \\ &+ \beta \phi_q\Big(\frac{\alpha}{1-\alpha\xi}\int_0^{\xi}(\xi - r)q(r)f(u,u',Tu,Su)dr + \int_0^{\eta} q(r)f(u,u',Tu,Su)dr\Big)\Big) \\ \le &(1+\beta)\phi_q\Big(\frac{\alpha}{1-\alpha\xi}\int_0^{\xi}(\xi - r)q(r)f(u,u',Tu,Su)dr + \int_0^1 q(r)f(u,u',Tu,Su)dr\Big) \\ = &(1+\beta)|(Au)'(1)| \\ = &(1+\beta)\|(Au)'\|_0. \end{aligned}$$

Since $\|Au\| = \max\{\|Au\|_0, \|(Au)'\|_0\}$, so we have $\|Au\| \le (1+\beta)\|(Au)'\|_0$, which completes the proof. □

3. Existence and uniqueness results

For any $\delta \in (0, \min\{\xi, 1-\xi, \eta, 1-\eta\})$, we define a nonnegative continuous concave function $\alpha : P \to R^+$ by $\alpha(u) = \min_{\delta \le t \le (1-\delta)} u(t)$. Obviously, the following two conclusions hold:

$$\alpha(u) = u(1-\delta) \le \|u\|_0 \quad \text{and} \quad \alpha(Au) = Au(1-\delta) \quad \forall u \in P.$$

The main result of this paper is the following.

Theorem 3.1. *Let* $m = \min_{0\le t\le 1} q(t)$ *and* $0 \le \alpha\xi < 1$. *Suppose* $(C_1), (C_2)$ *and* (C_3) *hold. Suppose further that there exist numbers* $\delta \in (0, \min\{\xi, 1-\xi, \eta, 1-\eta\}), a, b, c$ *and* d *such that* $0 < a < b \le \frac{m\delta d}{M} < d \le c$, *and*

(H_1) $f(u, v, w, l) \le (1-\alpha\xi)\phi_p(\frac{a}{(1+\beta)M})$ *for* $(u, v, w, l) \in [0, a] \times [-a, 0] \times [0, k_0 a] \times [0, h_0 a]$;

(H_2) $f(u, v, w, l) \le (1-\alpha\xi)\phi_p(\frac{c}{(1+\beta)M})$ *for* $(u, v, w, l) \in [0, c] \times [-c, 0] \times [0, k_0 c] \times [0, h_0 c]$;

(H_3) $f(u, v, w, l) > \phi_p(\frac{b}{\delta L})$ *for* $(u, v, w, l) \in [b, d] \times [-d, 0] \times [0, k_0 d] \times [0, h_0 d]$, *where* $L = \phi_q(\int_0^{1-\delta} q(t)dt)$;

(H_4) $\min_{(u,v,w,l)\in J} f(u, v, w, l)\phi_p(\frac{M}{(1+\beta)m}) \int_0^{1-\delta} q(t)dt \ge \max_{(u,v,w,l)\in J} f(u, v, w, l) \int_0^1 q(t)dt$, *where*

$$J = [0, c] \times [-c, 0] \times [0, k_0 c] \times [0, h_0 c].$$

Then IBVP (1.1) *has at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$\|u_1\| < a, \quad b < \min_{\delta\le t<1-\delta} u_2(t), \quad \|u_3\| > a \quad \text{and} \quad \min_{\delta\le t<1-\delta} u_3(t) < b.$$

Proof. We divide the proof into three steps.

Step 1. We prove $A\overline{P}_c \subset \overline{P}_c, A\overline{P}_a \subset \overline{P}_a$, that is, (ii) of Theorem 2.3.

By Lemma 2.6, we have $A\overline{P}_c \subset P$, so for all $u \in \overline{P}_c$, we get $0 \le u(t) \le c, -c \le u'(t) \le 0, 0 \le (Tu)(t) \le k_0 c, 0 \le (Su)(t) \le h_0 c$. For $t \in [0, 1]$, and by (H_2)

$$\begin{aligned}
\|(Au)'\|_0 &= \phi_q\Big(\frac{\alpha}{1-\alpha\xi}\int_0^\xi (\xi - r)q(r)f(u, u', Tu, Su)dr + \int_0^1 q(r)f(u, u', Tu, Su)dr\Big)\\
&\le \phi_q\Big(\frac{\alpha\xi}{1-\alpha\xi}\int_0^1 q(r)f(u, u', Tu, Su)dr + \int_0^1 q(r)f(u, u', Tu, Su)dr\Big)\\
&\le \phi_q\Big(\frac{1}{1-\alpha\xi}\int_0^1 q(r)f(u, u', Tu, Su)dr\Big)\\
&\le \phi_q\Big(\phi_p\Big(\frac{c}{(1+\beta)M}\Big)\Big)\phi_q\Big(\int_0^1 q(r)dr\Big)\\
&\le \frac{c}{1+\beta} \le c,
\end{aligned}$$

and by Lemma 2.7, it is obvious that

$$\|(Au)\|_0 \le (1+\beta)\|(Au)'\|_0 \le (1+\beta)\frac{c}{1+\beta} = c.$$

Hence, $\|Au\| < c$ and $A\overline{P}_c \subset \overline{P}_c$. Similarly, we obtain $A\overline{P}_a \subset \overline{P}_a$.

Step 2. We show

$$\{u \in P(\alpha, b, d) : \alpha(u) > b\} \ne \phi, \tag{3.1}$$

and

$$\alpha(Au) > b \;\text{ for all }\; u \in P(\alpha, b, d), \tag{3.2}$$

that is, (i) of the Theorem 2.3.

Let $u = \frac{b+d}{2}$, then $u \in P(\alpha, b, d), \alpha(u) = \frac{b+d}{2} > b$. Hence (3.1) holds. For any $u \in P(\alpha, b, d)$, we have $b \le u(t) \le d, -d \le u'(t) \le 0, 0 \le (Tu)(t) \le k_0 d, 0 \le (Su)(t) \le h_0 d, t \in [0, 1-\delta]$, so by (H_3), we have

$$\begin{aligned}
\alpha(Au) &= \min_{t\in[\delta,1-\delta]}(Au)(t) = (Au)(1-\delta)\\
&= \int_{1-\delta}^{1}\phi_q\bigg(\frac{\alpha}{1-\alpha\xi}\int_0^\xi(\xi-r)q(r)f(u,u',Tu,Su)dr + \int_0^s q(r)f(u,u',Tu,Su)dr\bigg)ds\\
&\quad + \beta g\bigg(-\phi_q\bigg(\frac{\alpha}{1-\alpha\xi}\int_0^\xi(\xi-r)q(r)f(u,u',Tu,Su)dr + \int_0^\eta q(r)f(u,u',Tu,Su)dr\bigg)\bigg)\\
&\geq \delta\phi_q\bigg(\frac{\alpha}{1-\alpha\xi}\int_0^\xi(\xi-r)q(r)f(u,u',Tu,Su)dr + \int_0^{1-\delta} q(r)f(u,u',Tu,Su)dr\bigg)\\
&\quad + \beta g\bigg(-\phi_q\bigg(\frac{\alpha}{1-\alpha\xi}\int_0^\xi(\xi-r)q(r)f(u,u',Tu,Su)dr + \int_0^\eta q(r)f(u,u',Tu,Su)dr\bigg)\bigg)\\
&\geq \delta\phi_q\bigg(\int_0^{1-\delta} q(r)f(u,u',Tu,Su)dr\bigg)\\
&\geq \delta\phi_q\bigg(\phi_p\bigg(\frac{b}{\delta L}\bigg)\phi_q\bigg(\int_0^{1-\delta}q(r)dr\bigg)\\
&= b.
\end{aligned}$$

Hence (3.2) holds.

Step 3. We show that $\alpha(Au) > b$ for all $u \in P(\alpha, b, c)$ with $\|Au\| > d$, that is, (iii) of the Theorem 2.3.

If $u \in P(\alpha, b, c)$ with $\|Au\| > d$, we obtain $0 \leq u(t) \leq c, -c \leq u'(t) \leq 0, 0 \leq (Tu)(t) \leq k_0c, 0 \leq (Su)(t) \leq h_0c$, for any $t \in [0, 1]$, and so by (H_4), we have

$$\phi_p\bigg(\frac{M}{(1+\beta)m}\bigg)\int_0^{1-\delta}q(r)f(u,u',Tu,Su)dr \geq \int_0^1 q(r)f(u,u',Tu,Su)dr.$$

Furthermore, we have

$$\begin{aligned}
&\phi_p\bigg(\frac{M}{(1+\beta)m}\bigg)\int_0^{1-\delta}q(r)f(u,u',Tu,Su)dr + \phi_p\bigg(\frac{M}{(1+\beta)m}\bigg)\frac{\alpha}{1-\alpha\xi}\int_0^\xi(\xi-r)q(r)f(u,u',Tu,Su)dr\\
&\quad\geq \int_0^1 q(r)f(u,u',Tu,Su)dr + \frac{\alpha}{1-\alpha\xi}\int_0^\xi(\xi-r)q(r)f(u,u',Tu,Su)dr.
\end{aligned}$$

Therefore, by Lemma 2.7, we have

$$\begin{aligned}
\alpha(Au) &= (Au)(1-\delta)\\
&= \int_{1-\delta}^{1}\phi_q\bigg(\frac{\alpha}{1-\alpha\xi}\int_0^\xi(\xi-r)q(r)f(u,u',Tu,Su)dr + \int_0^s q(r)f(u,u',Tu,Su)dr\bigg)ds\\
&\quad + \beta g\bigg(-\phi_q\bigg(\frac{\alpha}{1-\alpha\xi}\int_0^\xi(\xi-r)q(r)f(u,u',Tu,Su)dr + \int_0^\eta q(r)f(u,u',Tu,Su)dr\bigg)\bigg)\\
&\geq \delta\phi_q\bigg(\frac{\alpha}{1-\alpha\xi}\int_0^\xi(\xi-r)q(r)f(u,u',Tu,Su)dr + \int_0^{1-\delta} q(r)f(u,u',Tu,Su)dr\bigg)\\
&\geq \delta\phi_q\bigg(\frac{\frac{\alpha}{1-\alpha\xi}\int_0^\xi(\xi-r)q(r)f(u,u',Tu,Su)dr + \int_0^1 q(r)f(u,u',Tu,Su)dr}{\phi_p(\frac{M}{(1+\beta)m})}\bigg)\\
&= \frac{(1+\beta)m\delta}{M}\phi_q\bigg(\frac{\alpha}{1-\alpha\xi}\int_0^\xi(\xi-r)q(r)f(u,u',Tu,Su)dr + \int_0^1 q(r)f(u,u',Tu,Su)dr\bigg)\\
&= \frac{(1+\beta)m\delta}{M}|(Au)'(1)| = \frac{(1+\beta)m\delta}{M}\|(Au)'\|_0\\
&\geq \frac{m\delta}{M}d \geq b.
\end{aligned}$$

Hence, by Theorem 2.3, the results of Theorem 3.1 hold. This completes the proof of Theorem 3.1. $\square$

Corollary 3.2. *Suppose there exist constants $a, b, c,$ and d with $0 < a < b \leq \frac{m\delta}{M}d < d \leq c$. Suppose also that f satisfies* (H$_1$)-(H$_3$) *of Theorem* 3.1 *and the following conditions:*

(H$_5$) *$k(t,s) \equiv c_1, h(t,s) \equiv c_2$, c_1 and c_2 are constants, and $0 < c_1, c_2 \leq 1$, $f(u,v,w,l)$ is increasing with respect to (u,v) and decreasing with respect to (w,l) for $0 \leq u \leq c, -c \leq v \leq 0, 0 \leq w \leq k_0c, 0 \leq l \leq h_0c$;*

(H$_6$) $\phi_q\left(\frac{M}{(1+\beta)m}\right) \geq \frac{\int_0^1 q(t)dt}{\int_0^{1-\delta} q(t)dt}$.

Then IBVP (1.1) *has at least three positive solutions $u_1, u_2,$ and u_3 such that*

$$\|u_1\| < a, \quad b < \min_{\delta \leq t < (1-\delta)} u_2(t), \quad \|u_3\| > a \quad \text{and} \quad \min_{\delta \leq t < (1-\delta)} u_3(t) < b.$$

Proof. By the proof of Theorem 3.1, we only need to show $\alpha(Au) > b$ for all $u \in P(\alpha, b, c)$ with $\|Au\| > d$. In fact, for any $u \in P(\alpha, b, c)$, we have $0 \leq u(t) \leq c, -c \leq u'(t) \leq 0, 0 \leq (Tu)(t) \leq k_0c, 0 \leq (Su)(t) \leq h_0c$. From the fact that the functions $u(t)$ and $u'(t)$ are decreasing, and the functions $(Tu)(t)$ and $(Su)(t)$ are increasing, we get

$$\int_{1-\delta}^1 q(r)f(u(r), u'(r), (Tu)(r), (Su)(r))dr \leq f(u(1-\delta), u'(1-\delta), (Tu)(1-\delta), (Su)(1-\delta)) \int_{1-\delta}^1 q(r)dr,$$
$$\int_0^{1-\delta} q(r)f(u(r), u'(r), (Tu)(r), (Su)(r))dr \geq f(u(1-\delta), u'(1-\delta), (Tu)(1-\delta), (Su)(1-\delta)) \int_0^{1-\delta} q(r)dr,$$

and so

$$\frac{\int_{1-\delta}^1 q(r)f(u(r), u'(r), (Tu)(r), (Su)(r))dr}{\int_0^{1-\delta} q(r)f(u(r), u'(r), (Tu)(r), (Su)(r))dr} \leq \frac{\int_{1-\delta}^1 q(r)dr}{\int_0^{1-\delta} q(r)dr},$$

then

$$\frac{\int_0^1 q(r)f(u(r), u'(r), (Tu)(r), (Su)(r))dr}{\int_0^{1-\delta} q(r)f(u(r), u'(r), (Tu)(r), (Su)(r))dr} \leq \frac{\int_0^1 q(r)dr}{\int_0^{1-\delta} q(r)dr},$$

which yields

$$\begin{aligned}&\frac{\frac{\alpha}{1-\alpha\xi}\int_0^\xi (\xi - r)q(r)f(u, u', Tu, Su)dr + \int_0^1 q(r)f(u, u', Tu, Su)dr}{\frac{\alpha}{1-\alpha\xi}\int_0^\xi (\xi - r)q(r)f(u, u', Tu, Su)dr + \int_0^{1-\delta} q(r)f(u, u', Tu, Su)dr} \\ &\quad \leq \frac{\int_0^1 q(r)f(u(r), u'(r), (Tu)(r), (Su)(r))dr}{\int_0^{1-\delta} q(r)f(u(r), u'(r), (Tu)(r), (Su)(r))dr} \leq \frac{\int_0^1 q(r)dr}{\int_0^{1-\delta} q(r)dr} \leq \phi_p\left(\frac{M}{(1+\beta)m}\right).\end{aligned}$$

Then proceeding as in Step 3 in the proof of Theorem 3.1, we complete the proof of Corollary 3.2. $\square$

Remark 3.3. Similarly, we can study the existence of three positive solutions for the following two boundary value problems:

$$\begin{cases} (\phi_p(u'))'(t) + q(t)f(u(t), u'(t), (Tu)(t), (Su)(t)) = 0, 0 \leq t \leq 1, \\ \phi_p(u'(1)) = \alpha \int_0^\xi \phi_p(u'(s))ds, \quad u(0) = \beta g(u'(\eta)), \end{cases}$$

and

$$\begin{cases} (\phi_p(u'))'(t) + q(t)f(u(t), u'(t), (Tu)(t), (Su)(t)) = 0, 0 \leq t \leq 1, \\ \phi_p(u'(0)) = \alpha \int_0^\xi \phi_p(u'(s))ds, \quad u(1) = \beta \int_0^1 g(u'(s))ds. \end{cases}$$

4. Applications

Our results can also be applied to study the solution of some higher order mixed boundary value problems (MBVPs; see[11, 14]). Here we consider the following IBVP:

$$\begin{cases} (\phi_p(x'''))'(t) + q(t)f(x'', x''', x', x) = 0,\ 0 \le t \le 1, \\ \alpha_1 x(0) + \alpha_2 x'(0) = 0, \\ \beta_1 x(1) + \beta_2 x'(1) = 0, \\ x'''(0) = \alpha \int_0^\xi \phi_p(u'''(s))ds, \qquad x''(1) = \beta g(x'''(\eta)). \end{cases} \tag{4.1}$$

Assume $\alpha_1^2 + \alpha_2^2 > 0, \beta_1^2 + \beta_2^2 > 0, \Delta = \alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_2\beta_1 \neq 0$ and $f \in C[R \times R, (0, +\infty)]$. Let $x'' = u$; then

$$x'(t) = \int_0^t u(s)ds \equiv Tu(t), \quad k(t,s) \equiv 1, \quad 0 < s \le t < 1,$$
$$x(t) = \int_0^1 h(t,s)u(s)ds \equiv Su(t), \quad t \in (0,1),$$

where $h(t,s)$ is the Green function of the following BVP:

$$\begin{cases} x''(t) = 0, \\ \alpha_1 x(0) + \alpha_2 x'(0) = 0, \\ \beta_1 x(1) + \beta_2 x'(1) = 0. \end{cases}$$

Thus MBVP(4.1) can be transformed to the following IBVP:

$$\begin{cases} (\phi_p(u'))'(t) + q(t)f(u(t), u'(t), (Tu)(t), (Su)(t)) = 0, 0 \le t \le 1, \\ \phi_p(u'(0)) = \alpha \int_0^\xi \phi_p(u'(s))ds, \qquad u(1) = \beta g(u'(\eta)), \end{cases}$$

and by Theorem 3.1, MBVP(4.1) has at least three solutions under proper conditions.

Example 4.1. Consider the following IBVP:

$$\begin{cases} (|u'|u')'(t) + q(t)f(u(t), u'(t), (Tu)(t), (Su)(t)) = 0, 0 \le t \le 1, \\ |u'(0)|u'(0) = \frac{1}{2} \int_0^{\frac{1}{2}} \phi_p(u'(s))ds, \qquad u(1) = \frac{1}{2} g(u'(\frac{7}{16})), \end{cases} \tag{4.2}$$

where

$$q(t) = \begin{cases} t^{-\frac{1}{2}}, & 0 \le t \le \frac{9}{16}, \\ -\dfrac{76}{25}t + \dfrac{913}{300}, & \frac{9}{16} \le t \le 1, \end{cases}$$

$$f(u,v,w,l) = \begin{cases} 10u^{10} + \dfrac{2+e^v}{500} + \dfrac{\sqrt[3]{w}}{500} + \dfrac{\sqrt{l}}{500}, & 0 \le u \le 1, v \le 0, w, l \ge 0, \\ 10\sqrt[10]{u} + \dfrac{2+e^v}{500} + \dfrac{\sqrt[3]{w}}{500} + \dfrac{\sqrt{l}}{500}, & 1 \le u, v \le 0, w, l \ge 0, \end{cases}$$

$$g(v) = \begin{cases} v^{\frac{1}{2}}, & |v| \ge 1, \\ v^2, & |v| < 1. \end{cases}$$

Since $p = 3, M = \sqrt{2}$ and $m = \frac{1}{300}, \alpha = \frac{1}{2}, \xi = \frac{1}{2}, \beta = \frac{1}{2}, \eta = \frac{7}{16}, \delta = \frac{7}{16}, a = \frac{1}{4}, b = 1, d = 1600, c = 3600,$ $k(t,s) = 1$ and $h(t,s) = 1$, then we can obtain $0 < a < b \le \frac{m\delta d}{M} < d \le c$, and

$$L = \sqrt{\int_0^{1-\frac{7}{16}} q(t)dt} = \sqrt{\int_0^{\frac{9}{16}} t^{-\frac{1}{2}} dt} = \sqrt{\frac{3}{2}}, \qquad \phi_p(\frac{a}{(1+\beta)M}) = \phi_p(\frac{\frac{1}{4}}{\frac{3}{2}\sqrt{2}}) = \frac{1}{72},$$

$$\phi_p(\frac{c}{(1+\beta)M}) = \phi_p(\frac{3600}{\frac{3}{2}\sqrt{2}}) = \frac{2400^2}{2}, \qquad \phi_p(\frac{b}{\delta L}) = \phi_p(\frac{1}{\frac{7}{16}\sqrt{\frac{3}{2}}}) = \frac{16^2}{7^2}\frac{2}{3} = \frac{512}{147}.$$

Next, we show that (H_1)-(H_4) are satisfied.

If $0 \le u \le \frac{1}{4}, -\frac{1}{4} \le v \le 0, 0 \le w, l \le \frac{1}{2}$, then

$$f(u,v,w,l) < 10 \cdot (\frac{1}{4})^{10} + \frac{3}{500} + \frac{2}{500} < (1 - \alpha\xi)\phi_p(\frac{a}{(1+\beta)M}) = \frac{1}{96}.$$

So (H_1) is satisfied.

If $0 \le u \le 3600, -3600 \le v \le 0, 0 \le w, l \le 3600$, then

$$f(u,v,w,l) < 10 \sqrt[10]{3600} + \frac{3}{500} + \frac{60 \times 2}{500} < (1 - \alpha\xi)\phi_p(\frac{c}{(1+\beta)M}) = 720000.$$

So (H_2) is satisfied.

If $1 \le u \le 1600, -1600 \le v \le 0, 0 \le w, l \le 1600$, then

$$f(u,v,w,l) > 10 \sqrt[10]{1} + \frac{2}{500} > \phi_p(\frac{b}{\delta L}) = \frac{512}{147}.$$

So (H_3) is satisfied.

For any $(u,v,w,l) \in [0,3600] \times [-3600,0] \times [0,3600] \times [0,3600]$, we have

$$\min f(u,v,w,l) \ge \frac{2}{500}, \qquad \max f(u,v,w,l) \le 10 \sqrt[10]{3600} + \frac{3}{500} + \frac{60 \times 2}{500},$$

$$\phi_p(\frac{M}{(1+\beta)m}) = 2 \cdot 200^2, \qquad \int_0^{1-\delta} q(t)dt = \frac{3}{2}, \qquad \int_0^1 q(t)dt = \frac{17207}{9600}.$$

Hence, it is easy to know that (H_4) is satisfied.

So by Theorem 3.1, we conclude that the IBVP (4.2) has three positive solutions u_1, u_2, and u_3 satisfying

$$\|u_1\| < \frac{1}{4}, \quad 1 < \min_{\delta \le t < 1-\delta} u_2(t), \quad \|u_3\| > \frac{1}{4}, \quad \text{and} \quad \min_{\delta \le t < 1-\delta} u_3(t) < 1 \quad \text{for} \quad \delta = \frac{7}{16}.$$

Acknowledgment

This paper is supported by the Natural Science Foundation of China (11201112), the Natural Science Foundation of Hebei Province (A2014208152) and (A2015208114), the Foundation of Hebei Education Department (QN2015175) and (QN2016165), and the Foundation of Hebei University of Science and Technology (2016ZDYY24).

References

[1] N. Al Arifi, I. Altun, M. Jleli, A. Lashin, B. Samet, *Lyapunov-type inequalities for a fractional p-Laplacian equation*, J. Inequal. Appl., **2016** (2016), 11 pages. 1

[2] H.-Y. Feng, W.-G. Ge, *Existence of three positive solutions for m-point boundary-value problems with one-dimensional p-Laplacian*, Nonlinear Anal., **68** (2008), 2017–2026.

[3] Y.-P. Guo, C.-L. Yu, J.-F. Wang, *Existence of three positive solutions for m-point boundary value problems on infinite intervals*, Nonlinear Anal., **71** (2009), 717–722.

[4] X.-M. He, W.-G. Ge, *Triple solutions for second-order three-point boundary value problems*, J. Math. Anal. Appl., **268** (2002), 256–265.

[5] X.-M. He, W.-G. Ge, *Twin positive solutions for the one-dimensional p-Laplacian boundary value problems*, Nonlinear Anal., **56** (2004), 975–984. 1

[6] X.-M. He, W.-G. Ge, M.-S. Peng, *Multiple positive solutions for one-dimensional p-Laplacian boundary value problems*, Appl. Math. Lett., **15** (2002), 937–943.

[7] J.-X. Hu, D.-X. Ma, *Triple positive solutions of a boundary value problem for second order three-point differential equations with p-Laplacian operator*, J. Appl. Math. Comput., **36** (2011), 251–261. 1, 1

[8] D. Kong, L.-S. Liu, Y.-H. Wu, *Triple positive solutions of a boundary value problem for nonlinear singular second-order differential equations of mixed type with p-Laplacian*, Comput. Math. Appl., **58** (2009), 1425–1432. 1, 1, 2

[9] L.-B. Kong, J.-Y. Wang, *Multiple positive solutions for the one-dimensional p-Laplacian*, Nonlinear Anal., **42** (2000), 1327–1333. 1

[10] A. Lakmeche, A. Hammoudi, *Multiple positive solutions of the one-dimensional p-Laplacian*, J. Math. Anal. Appl., **317** (2006), 43–49. 1

[11] L.-S. Liu, C.-X. Wu, F. Guo, *Existence theorems of global solutions of initial value problems for nonlinear integrodifferential equations of mixed type in Banach spaces and applications*, Comput. Math. Appl., **47** (2004), 13–22. 4

[12] D.-X. Ma, *Existence and iteration of positive solution for a three-point boundary value problem with a p-Laplacian operator*, J. Appl. Math. Comput., **25** (2007), 329–337. 1

[13] J.-Y. Wang, *The existence of positive solutions for the one-dimensional p-Laplacian*, Proc. Amer. Math. Soc., **125** (1997), 2275–2283. 1

[14] J.-F. Wang, C.-L. Yu, Y.-P. Guo, *Triple positive solutions of a nonlocal boundary value problem for singular differential equations with p-Laplacian*, Abstr. Appl. Anal., **2013** (2013), 7 pages. 1, 4

[15] C.-L. Yu, J.-F. Wang, C.-Y. Zuo, *Existence of positive solutions for second-order three-point boundary value problems with p-Laplacian operator*, J. Hebei Univ. Sci. Technol., **32** (2014), 127–133.

[16] D.-X. Zhao, H.-Z. Wang, W.-G. Ge, *Existence of triple positive solutions to a class of p-Laplacian boundary value problems*, J. Math. Anal. Appl., **328** (2007), 972–983. 1