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On lightlike hypersurfaces and lightlike focal sets of null Cartan curves in Lorentz-Minkowski spacetime

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Abstract

In this paper, as a type of event horizons in astrophysics, a class of lightlike hypersurfaces that is generated by null curves will be investigated and discussed. Based on discussions of the properties of the local differential geometry of null curves and singularity theory, we provide classifications of the singularities of lightlike hypersurfaces and lightlike focal sets. In addition, we reveal the facts that the types of these singularities and the order of contact between a null Cartan curve and a pseudosphere are related closely to null Cartan curvatures. Finally, examples of lightlike hypersurfaces and lightlike focal set are used to demonstrate our theoretical results.

Keywords: null Cartan curve, lightlike hypersurface, singularity
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1. Introduction

As one of three types of submanifolds(i.e., spacelike submanifold, timelike submanifold and lightlike submanifold) in Lorentz-Minkowski spacetime, lightlike submanifolds have been widely studied by many physicists and practitioners of differential geometry. Lightlike submanifolds appear in many physics papers. For example, lightlike submanifolds are of interest because they provide models of different horizon types, such as event horizons of Kerr black holes, isolated horizons, Cauchy horizons, Kruskal horizons and Killing horizons[1, 4, 10, 11, 13, 16–18, 22]. Lightlike submanifolds are also studied in the theory of electromagnetism (see, for example, [5, 21].) It is well known that a null curve is a 1-dimensional lightlike submanifold and that a lightlike hypersurface is a lightlike submanifold of codimension one in terms of ambient space. Nersessian

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and Ramos clearly demonstrated the importance of the research of null curves in physics theories, and they demonstrated that there exists a geometric particle-model based entirely on the geometry of the null curves in Minkowskian 4-dimensional spacetime that yields wave equations under quantization corresponding to massive spinning particles of arbitrary spin[14]. Nersessian et al. have also studied the simplest geometric particle-model, which is associated with null curves in Minkowski 3-space[15]. In addition, Duggal et al. laid the foundation for the differential geometry theory of lightlike submanifolds[6–8], which, of course, includes the theory of the null curve. Thus, we have used these fundamental results as our basic tools in researching the geometry of null curves[9].

Many studies on singularities have established links between physics and geometry since singularity theory was founded in 1965 by R. Thom. R. Thom first proposed the idea of applying singularity theory to the study of differential geometry. The natural connection between geometry and singularities relies on the basic fact that the contacts of a submanifold with models (invariant under the action of a suitable transformation group) of the ambient space can be described by means of the analysis of the singularities of appropriate families of contact functions. Because many difficulties arise in generalizing the use of a singularity theory approach from nonlightlike submanifolds to lightlike submanifolds, the study of the singularity of submanifolds in Minkowski space remained at the nonlightlike submanifolds over a long period of time until we extended it to the lightlike submanifolds in 2010 and provided meaningful results[23–25]. Pei et al. also described the properties of the local differential geometry of the null curve and investigated the singularity of the null surface of the null curve on the 3-null cone[19, 20]. However, to the best of the authors knowledge, we are not aware of any literature report addressing the singularities of lightlike hypersurfaces involving null curves in $\mathbb{R}^4_1$. The current paper is an attempt to address this gap in knowledge. Because a singularity is a point (or a function) at which a function (or hypersurface) blows up, these singularities affect a hypersurface not only at a certain point but also around it; therefore, we focused our attention on germs in a local neighborhood around a fixed point. In this paper, to allow for a useful study of these singularities, we consider *lightlike distance squared functions* (denoted by locally around the point (s_0, v_0)). These functions are the unfoldings of these singularities in the local neighborhood of (s_0, v_0) and depend only on the germ that they are unfolding. In this paper, we obtain these functions by varying a fixed point in the lightlike distance squared function and obtain a family of functions. This function measures the contact between the null curve $\boldsymbol{\gamma}(s)$ and the pseudosphere with a vertex at the singularity. Furthermore, note the facts these singularities are versally unfolded by the family of lightlike distance squared functions and if the singularity of h_λ is A_k-type $(k = 1, 2, 3, 4)$ and the corresponding 4-parameter unfolding is versal, then the discriminant set of order ℓ of the 4-parameter unfolding is locally diffeomorphic to $C(2, 3)\times\mathbb{R}^2$, $C(2, 3, 4)\times\mathbb{R}^2$, $(2, 3, 4, 5)-$cusp, $SW\times\mathbb{R}$, butterfly, or $c-$butterfly; in this manner, we completed the classification of the singularities of the lightlike hypersurface and the lightlike focal set (note that the discriminant set of unfolding is precisely the lightlike hypersurface of the null curve, and the discriminant set of order 2 is the lightlike focal set).

This paper is organized as follows. We begin in Section 2 with the necessary background regarding the null curve in in $\mathbb{R}^4_1$. In Section 3, we introduce the lightlike distance squared function on the null curve $\boldsymbol{\gamma}$, which is useful for studying the singularities of the lightlike hypersurface. In Section 4, we provide classifications of the singularities of the lightlike hypersurface and the lightlike focal set along $\boldsymbol{\gamma}$. The main results in this paper are stated in Theorem 4.6. In Section 5, we provide the examples of the lightlike hypersurface and the lightlike focal set along a null curve and present the theoretical results by using graph-plotting software.

2. Preliminary

Let us review some ideas and properties of the null Cartan curve 4-dimensional Minkowski space (see [6]). The set of all 4-tuples $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ of real numbers is denoted by $\mathbb{R}^4$. *Minkowski 4-space*, which is denoted $\mathbb{R}^4_1$, is equipped with the *pseudo-inner product*

$$\langle \boldsymbol{x}, \boldsymbol{y}\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$$

for any vectors $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ and $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$ in $\mathbb{R}^4$. We also define the *pseudo-vector product* of $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ as follows:

$$\boldsymbol{x} \wedge \boldsymbol{y} \wedge \boldsymbol{z} = \begin{vmatrix} -\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 & \boldsymbol{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}.$$

where $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$, $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$, and $\boldsymbol{z} = (z_1, z_2, z_3, z_4)$ are in $\mathbb{R}^4_1$ and $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4)$ is the canonical basis of $\mathbb{R}^4_1$. We state that a vector $\boldsymbol{x} \in \mathbb{R}^4_1 \backslash \{\boldsymbol{0}\}$ is *spacelike, null,* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle$ is positive, zero, or negative, respectively. The *norm* of a non-null vector $\boldsymbol{x} \in \mathbb{R}^4_1$ is defined by $\parallel \boldsymbol{x} \parallel = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$, and we call $\boldsymbol{x}$ a *unit vector* if $\parallel \boldsymbol{x} \parallel = 1$. We define the signature of a vector as follows:

$$\operatorname{sign}(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \text{ is spacelike} \\ 0 & \boldsymbol{x} \text{ is null} \\ -1 & \boldsymbol{x} \text{ is timelike.} \end{cases}$$

For a vector $\boldsymbol{v} \in \mathbb{R}^4_1$ and a real number c, we define the hyperplane with pseudo-normal vector $\boldsymbol{v}$ by

$$HP(\boldsymbol{v}, c) = \Big\{ \boldsymbol{x} \in \mathbb{R}^4_1 : \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c \Big\}.$$

We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane,* a *timelike hyperplane*, or a *lightlike hyperplane* if $\boldsymbol{v}$ is *timelike, spacelike*, or *lightlike*, respectively.

We define a pseudosphere at vertex $\boldsymbol{p}$ as follows:

$$S^3_{1,\boldsymbol{p}} = \Big\{ \boldsymbol{x} \in \mathbb{R}^4_1 : \ \langle \boldsymbol{x} - \boldsymbol{p}, \boldsymbol{x} - \boldsymbol{p} \rangle = 1 \Big\}.$$

For any $s \in I$, the curve, locally parameterized by $\boldsymbol{\gamma} : I \to \mathbb{R}^4_1$ is called a *spacelike curve*, a *null(lightlike) curve*, or a *timelike curve* if for each t, the velocity of the curve is $\langle \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}'(s) \rangle > 0, \langle \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}'(s) \rangle = 0$,or $\langle \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}'(s) \rangle < 0$, respectively. We call $\boldsymbol{\gamma}$ a *non-null curve* if $\boldsymbol{\gamma}$ is a timelike curve or a spacelike curve.

We will assume in the following that the null curve that we consider has no points at which the acceleration vector is null. Hence, $\langle \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}''(s) \rangle$ is never zero. We know that the acceleration vector of the null curve is always spacelike. Accordingly, we set

$$\boldsymbol{t}(s) = \frac{\boldsymbol{\gamma}'(s)}{\sqrt{\langle \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}''(s) \rangle}} = \frac{1}{\varphi(s)} \boldsymbol{\gamma}'(s), \tag{2.1}$$

where $\varphi(s) = \sqrt{\langle \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}''(s) \rangle} > 0$. Letting $\boldsymbol{n}_1(s) = \boldsymbol{t}'(s)$, we calculate

$$\boldsymbol{n}_1(s) = \boldsymbol{t}'(s) = \Big(\frac{1}{\varphi(s)} \Big)' \boldsymbol{\gamma}'(s) + \frac{1}{\varphi(s)} \boldsymbol{\gamma}''(s), \tag{2.2}$$

it is obvious that $\boldsymbol{n}_1(s)$ is a unit spacelike vector. Moreover, we have

$$\langle \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}''(s) \rangle + \langle \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}'''(s) \rangle = 0,$$

hence, $\boldsymbol{\gamma}'(s)$ cannot be perpendicular $\boldsymbol{\gamma}'''(s)$, i.e.,$\langle \boldsymbol{t}(s), \boldsymbol{\gamma}'''(s) \rangle = -\varphi(s) \neq 0$. Therefore, there always exists a null vector field, called the null transversal vector field (Duggal and Bejancu, [6]), uniquely determined by

$$\begin{aligned} \boldsymbol{\eta} &= \frac{1}{\langle \boldsymbol{t}, \boldsymbol{\gamma}''' \rangle} \Big\{ \boldsymbol{\gamma}''' - \frac{\langle \boldsymbol{\gamma}''', \boldsymbol{\gamma}''' \rangle}{2 \langle \boldsymbol{t}, \boldsymbol{\gamma}''' \rangle} \boldsymbol{t} \Big\} \\ &= -\frac{1}{\varphi} \boldsymbol{\gamma}''' - \frac{\langle \boldsymbol{\gamma}''', \boldsymbol{\gamma}''' \rangle}{2 \varphi^3} \boldsymbol{\gamma}'. \end{aligned} \tag{2.3}$$

Furthermore, we can choose a unit spacelike vector field $\boldsymbol{n}_2$ orthogonally spanned by $\{\boldsymbol{\gamma}', \boldsymbol{\gamma}'', \boldsymbol{\gamma}'''\}$ such that $\{\boldsymbol{t}, \boldsymbol{\eta}, \boldsymbol{n}_1, \boldsymbol{n}_2\}$ is positively oriented. $\boldsymbol{n}_2$ can be determined by

$$\begin{aligned} \boldsymbol{n}_2 = \boldsymbol{t} \wedge \boldsymbol{\eta} \wedge \boldsymbol{n}_1 &= \frac{1}{\varphi}\boldsymbol{\gamma}' \wedge \Big(- \frac{1}{\varphi}\boldsymbol{\gamma}'''\Big) \wedge \frac{1}{\varphi}\boldsymbol{\gamma}'' \\ &= \frac{1}{\varphi^3}\Big(\boldsymbol{\gamma}' \wedge \boldsymbol{\gamma}'' \wedge \boldsymbol{\gamma}'''\Big). \end{aligned} \tag{2.4}$$

For any given null curve $\boldsymbol{\gamma}$ satisfying $\varphi(s) = \sqrt{\langle \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}''(s)\rangle} \neq 0$, there exists one frame $\{\boldsymbol{t}(s), \boldsymbol{\eta}(s), \boldsymbol{n}_1(s), \boldsymbol{n}_2(s)\}$ of $\boldsymbol{\gamma}$ determined by Eqs.(2.1)-(2.4). It can be shown that

$$\langle \boldsymbol{t}(s), \boldsymbol{t}(s)\rangle = \langle \boldsymbol{\eta}(s), \boldsymbol{\eta}(s)\rangle = 0,$$

$$\langle \boldsymbol{n}_1(s), \boldsymbol{t}(s)\rangle = \langle \boldsymbol{n}_2(s), \boldsymbol{t}(s)\rangle = \langle \boldsymbol{n}_1(s), \boldsymbol{\eta}(s)\rangle = \langle \boldsymbol{n}_2(s), \boldsymbol{\eta}(s)\rangle = \langle \boldsymbol{n}_1(s), \boldsymbol{n}_2(s)\rangle = 0,$$

$$\langle \boldsymbol{n}_1(s), \boldsymbol{n}_1(s)\rangle = \langle \boldsymbol{n}_2(s), \boldsymbol{n}_2(s)\rangle = \langle \boldsymbol{t}(s), \boldsymbol{\eta}(s)\rangle = 1.$$

In the circumstances, Frenet equations associated with the *Frenet frame* $\{\boldsymbol{t}(s), \boldsymbol{\eta}(s), \boldsymbol{n}_1(s), \boldsymbol{n}_2(s)\}$ are given by(see also [3])

$$\begin{cases} \boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)\big/\sqrt{\langle \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}''(s)\rangle} \\ \boldsymbol{t}'(s) = \boldsymbol{n}_1(s) \\ \boldsymbol{\eta}'(s) = k_1(s)\boldsymbol{n}_1(s) + k_2\boldsymbol{n}_2(s) \\ \boldsymbol{n}_1'(s) = -k_1(s)\boldsymbol{t}(s) - \boldsymbol{\eta}(s) \\ \boldsymbol{n}_2'(s) = -k_2(s)\boldsymbol{t}(s), \end{cases} \tag{2.5}$$

where

$$\begin{aligned} k_1(s) &= \tfrac{1}{2\varphi^2(s)}\Big(\langle \boldsymbol{\gamma}'''(s), \boldsymbol{\gamma}'''(s)\rangle + 2\varphi(s)\varphi''(s) - 4(\varphi'(s))^2\Big), \\ k_2(s) &= -\tfrac{1}{\varphi^4(s)}\det\Big(\boldsymbol{\gamma}'(s), \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}'''(s), \boldsymbol{\gamma}^{(4)}(s)\Big). \end{aligned} \tag{2.6}$$

Here, curve $C = \boldsymbol{\gamma}(I)$, which satisfies the assumptions above, is called a *Cartan curve* with a *Cartan frame* $\{\boldsymbol{t}(s), \boldsymbol{\eta}(s), \boldsymbol{n}_1(s), \boldsymbol{n}_2(s)\}$ and *Cartan curvatures* $\{k_1(s), k_2(s)\}$. We can check that

$$\begin{aligned} &\boldsymbol{n}_1(s) \wedge \boldsymbol{t}(s) \wedge \boldsymbol{\eta}(s) = \boldsymbol{n}_2(s), \quad \boldsymbol{n}_1(s) \wedge \boldsymbol{n}_2(s) \wedge \boldsymbol{t}(s) = \boldsymbol{t}(s), \\ &\boldsymbol{n}_1(s) \wedge \boldsymbol{\eta}(s) \wedge \boldsymbol{n}_2(s) = \boldsymbol{\eta}(s), \quad \boldsymbol{t}(s) \wedge \boldsymbol{n}_2(s) \wedge \boldsymbol{\eta}(s) = \boldsymbol{n}_1(s). \end{aligned} \tag{2.7}$$

If we use a parameter u, known as the *pseudo-arc parameter*, defined as

$$u(s) = \int_{s_0}^{s} \Big\langle \boldsymbol{\gamma}''(t), \boldsymbol{\gamma}''(t)\Big\rangle^{1/4} dt,$$

then, by a simple calculation, we have $\varphi(u) = 1$; thus, Eqs.(2.1)-(2.6)can be describe by simpler formulas. However, we still adopt the general parameter s in the following study so that the results are obtained more generally.

Theorem 2.1 ([6]). *Because the Cartan frame is unique up to the orientation, the number of the Cartan curvatures is minimum and the Cartan curvatures are invariant under Lorentz transformations.*

Remark 2.2. Null straight lines are the only null curves that are not null Cartan curves.

Let $\boldsymbol{\gamma} : I \to \mathbb{R}^4_1$ be a null Cartan curve with Frenet frame $\{\boldsymbol{t}(s), \boldsymbol{\eta}(s), \boldsymbol{n}_1(s), \boldsymbol{n}_2(s)\}$ along with the following definitions.

Definition 2.3. We define the map

$$\mathbb{DG}_C : U \times \mathbb{R} \to \mathbb{S}^3_1$$

as

$$\mathbb{DG}_C(s, \mu, \theta) = \mu\boldsymbol{t}(s) + \cos\theta\boldsymbol{n}_1(s) + \sin\theta\boldsymbol{n}_2(s),$$

we call $\mathbb{DG}^{\pm}_C(u, \theta)$ the *de Sitter Gauss image* of C.

Definition 2.4. We define the hypersurfaces

$$\mathbb{LH}_C : U \times \mathbb{R} \to \mathbb{R}^4_1$$

as

$$\mathbb{LH}_C(s,\mu,\theta) = \boldsymbol{\gamma}(s) + \mu\boldsymbol{t}(s) + \cos\theta\boldsymbol{n}_1(s) + \sin\theta\boldsymbol{n}_2(s) = \boldsymbol{\gamma}(s) + \mathbb{DG}_C(s,\mu,\theta),$$

we call $\mathbb{LH}_C$ the *lightlike hypersurface along* C.

3. Lightlike distance squared function and null Cartan curves

In this section, we consider the lightlike hypersurface along $C = \boldsymbol{\gamma}(I)$ and calculate the *lightlike distance squared function* on C, which is useful for studying the singularities of the lightlike hypersurface. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^4_1$ be a null Cartan curve with the Frenet frame $\{\boldsymbol{t}(s), \boldsymbol{\eta}(s), \boldsymbol{n}_1(s), \boldsymbol{n}_2(s)\}$; then, the lightlike distance squared function is defined as

$$H : \mathbb{R}^4_1 \times \mathbb{R}^4_1 \to \mathbb{R}, \quad H(p,\boldsymbol{\lambda}) = \langle \boldsymbol{\lambda} - \boldsymbol{\gamma}(s), \boldsymbol{\lambda} - \boldsymbol{\gamma}(s)\rangle - 1,$$

where $p = \boldsymbol{\gamma}(s)$. For any fixed $\boldsymbol{\lambda}_0 \in \mathbb{R}^4_1$, we write $h(p) = h_{\lambda_0}(p) = H(p, \boldsymbol{\lambda}_0)$.

We calculate

$$h'(p) = -2\langle \varphi(s)\boldsymbol{t}(s), \boldsymbol{\lambda}_0 - \boldsymbol{\gamma}(s)\rangle,$$

thus we conclude that the discriminant set of the lightlike distance squared function H is given by

$$\mathscr{D}_H = \big\{\boldsymbol{\lambda} = \boldsymbol{\gamma}(s) + \mu\boldsymbol{t}(s) + \cos\theta\boldsymbol{n}_1(s) + \sin\theta\boldsymbol{n}_2(s) | \theta \in [0,2\pi), s \in I, \mu \in \mathbb{R}\big\},$$

which is the image of the lightlike hypersurface along C and

$$\begin{aligned} h''(p) &= -2\langle \varphi'(s)\boldsymbol{t}(s) + \varphi(s)\boldsymbol{t}'(s), \boldsymbol{\lambda}_0 - \boldsymbol{\gamma}(s)\rangle + 2\langle \varphi(s)\boldsymbol{t}(s), \varphi(s)\boldsymbol{t}(s)\rangle \\ &= -2\langle \varphi'(s)\boldsymbol{t}(s) + \varphi(s)\boldsymbol{n}_1(s), \cos\theta\boldsymbol{n}_1(s) + \sin\theta\boldsymbol{n}_2(s)\rangle \\ &= -2\varphi(s)\cos\theta, \end{aligned}$$

therefore $h(p) = h'(p) = h''(p) = 0$ if and only if $\cos\theta = 0$, that is, a singular point of the lightlike hypersurface is a point $\boldsymbol{\lambda}_0 = \boldsymbol{\gamma}(s_0) + \mu_0\boldsymbol{t}(s_0) \pm \boldsymbol{n}_2(s_0)$ for $\mu_0 \in \mathbb{R}$. Thus we define *lightlike focal set* as

$$\mathbb{LFS}^{\pm}_C = \left\{\boldsymbol{\lambda} = \gamma(s) + \mathbb{DG}_C\big(s,\mu,\pi \mp \frac{\pi}{2}\big) \middle| s \in I, \mu \in \mathbb{R}\right\}.$$

Moreover, by calculating the third, fourth, fifth derivatives of $h(s)$, we have the following proposition.

Proposition 3.1. *Suppose that $\boldsymbol{\gamma} : I \to \mathbb{R}^4_1$ is a null Cartan curve with the null Cartan Frenet frame $\{\boldsymbol{t}(s), \boldsymbol{\eta}(s), \boldsymbol{n}_1(s), \boldsymbol{n}_2(s)\}$. Then*
(1) $h(p_0) = h'(p_0) = 0$ *if and only if there exists $\theta_0 \in [0, 2\pi)$ and $\mu_0 \in \mathbb{R}$ such that*

$$\boldsymbol{\lambda}_0 - p_0 = \mu\boldsymbol{t}(s_0) + \cos\theta_0\boldsymbol{n}_1(s_0) + \sin\theta_0\boldsymbol{n}_2(s_0).$$

(2) $h(p_0) = h'(p_0) = h''(p_0) = 0$ *if and only if there exists $\theta_0 = \pi \mp \frac{\pi}{2}$ and $\mu_0 \in \mathbb{R}$ such that*

$$\boldsymbol{\lambda}_0 - p_0 = \mathbb{DG}_C(s_0, \mu_0, \pi \mp \frac{\pi}{2}) = \mu_0\boldsymbol{t}(s_0) \pm \boldsymbol{n}_2(s_0).$$

(3) $h(p_0) = h'(p_0) = h''(p_0) = h'''(p_0) = 0$ *if and only if there exists $\theta_0 = \pi \mp \frac{\pi}{2}$ and $\mu_0 = 0$ such that*

$$\boldsymbol{\lambda}_0 - p_0 = \mathbb{DG}_C(s_0, 0, \pi \mp \frac{\pi}{2}) = \pm\boldsymbol{n}_2(s_0).$$

(4)$h(p_0) = h'(p_0) = h''(p_0) = h'''(p_0) = h^{(4)}(p_0) = 0$ *if and only if there exists* $\theta_0 = \pi \mp \frac{\pi}{2}$ *and* $\mu_0 = 0$ *such that*

$$\boldsymbol{\lambda}_0 - p_0 = \mathbb{DG}_C(s_0, 0, \pi \mp \frac{\pi}{2}) = \pm \boldsymbol{n}_2(s_0)$$

and $k_2(s_0) = \pm\varphi(s_0)$.
(5)$h(p_0) = h'(p_0) = h''(p_0) = h'''(p_0) = h^{(4)}(p_0) = h^{(5)}(p_0) = 0$ *if and only if*

$$\boldsymbol{\lambda}_0 - p_0 = \mathbb{DG}_C(s_0, 0, \pi \mp \frac{\pi}{2}) = \pm \boldsymbol{n}_2(s_0)$$

and $k_2(s_0) = \pm\varphi(s_0), k_2'(s_0) = \pm\varphi'(s_0)$.

Definition 3.2. Let $F : \mathbb{R}^4_1 \to \mathbb{R}^4_1$ be a submersion and let $\boldsymbol{\alpha} : I \to \mathbb{R}^4_1$ be a null Cartan curve. We say that $\boldsymbol{\alpha}$ and $F^{-1}(0)$ have *k-point contact* for $s = s_0$ if the function $g(s) = F \circ \boldsymbol{\alpha}(s)$ satisfies

$$g(s_0) = g'(s_0) = \cdots = g^{k-1}(s_0) = 0,\ g^k(s_0) \neq 0.$$

In addition, we say $\boldsymbol{\alpha}$ and $F^{-1}(0)$ have *at least k-point contact* for $s = s_0$ if the function $g(s) = F \circ \boldsymbol{\alpha}(s)$ satisfies $g(s_0) = g'(s_0) = \cdots = g^{k-1}(s_0) = 0$.

Definition 3.3. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^4_1$ be a Cartan curve in $\mathbb{R}^4_1$. Then, the pseudo-sphere having five-point contact with $\boldsymbol{\gamma}$ is called the osculating pseudo-sphere of $\boldsymbol{\gamma}$(Ref. [3]).

Remark 3.4. If we define the lightlike distance squared function as

$$H(p, \boldsymbol{\lambda}) = \langle \boldsymbol{\lambda} - \boldsymbol{\gamma}(s), \boldsymbol{\lambda} - \boldsymbol{\gamma}(s)\rangle - r^2,$$

where $r \in \mathbb{R}^+$, then the assertion (4) of Proposition 3.1 becomes $h(p_0) = h'(p_0) = h''(p_0) = h'''(p_0) = h^{(4)}(p_0) = 0$ if and only if there exists $\theta_0 = \pi \pm \frac{\pi}{2}$ and $\mu_0 = 0$ such that

$$\boldsymbol{\lambda}_0 - p_0 = \mathbb{DG}_C(s_0, 0, \pi \mp \frac{\pi}{2}) = \pm r\boldsymbol{n}_2(s_0)$$

and $rk_2(s_0) = \pm\varphi(s_0)$.
We can see more clearly the relations between $k_2(s_0)$ and the radius r of the pseudo-sphere with center at $\boldsymbol{\lambda}_0$. Therefore, we can show that the center point of the osculating pseudo-sphere at a point $\boldsymbol{\gamma}(s_0)$ is

$$\boldsymbol{\lambda}_0 = \boldsymbol{\gamma}(s_0) + \mathbb{DG}_C(s_0, 0, \pi \mp \frac{\pi}{2}) = \boldsymbol{\gamma}(s_0) \pm \frac{\varphi(s_0)}{k_2(s_0)}\boldsymbol{n}_2(s_0).$$

4. Classifications of singularities and null Cartan curvatures

We classify the singularities of the lightlike hypersurface and lightlike focal set along $\boldsymbol{\gamma}$ as an application of the unfolding theory of functions. For a function $f(s)$, we say that $f(s)$ has A_k-singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \le p \le k$ and $f^{(k+1)}(s_0) \neq 0$. Let F be an unfolding of f, and $f(s)$ has A_k $(k \ge 1)$ at s_0. We denote the $(k-1)$-jet of the partial derivative $\partial F/\partial x_i$ at s_0 by

$$j^{(k-1)}\Big(\frac{\partial F}{\partial x_i}(s, \boldsymbol{x}_0)\Big)(s_0) = \sum_{j=1}^{k-1} \alpha_{ji} s^j, (i = 1 \cdots r).$$

If the $k \times r$ matrix of coefficients $(\alpha_{0i}, \alpha_{ji})$ has rank k $(k \le r)$, then F is called a *versal unfolding* , where $\alpha_{0i} = \frac{\partial F}{\partial x_i}(s_0, x_0)$.

We now introduce the following important sets concerning the unfolding:

$$D^\ell_F = \{\boldsymbol{x} \in \mathbb{R}^r, F(s, \boldsymbol{x}) = \frac{\partial F}{\partial s} = \cdots = \frac{\partial^\ell F}{\partial s^\ell} = 0\},$$

which is called a *discriminant set of order* ℓ. Of course, $D^1_F = D_F$, and D^2_F is the set of singular points of D_F . Therefore, we have the following proposition.

Proposition 4.1. *$D_H = D^1_H = \mathbb{LH}_C(I \times \mathbb{R} \times \mathbb{S}^1)$, $D^2_H = \mathbb{LFS}^{\pm}_C$, and D^3_H is the critical value set of $\mathbb{LFS}^{\pm}_C$.*

To understand the geometric properties of the discriminant set of order ℓ, we introduce an equivalence relationship among the unfoldings of functions. Let F and G be the r-parameter unfoldings of $f(s)$ and $g(s)$, respectively. We state that F and G are P-$\mathscr{R}$-equivalent if there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \to (\mathbb{R} \times \mathbb{R}^r, (s'_0, \boldsymbol{x}'_0))$ of the form $\Phi(s, \boldsymbol{x}) = (\Phi_1(s, \boldsymbol{x}), \phi(\boldsymbol{x}))$ such that $G \circ \Phi = F$. Straightforward calculations yield the following proposition.

Proposition 4.2. *Let F and G be the r-parameter unfoldings of $f(s)$ and $g(s)$, respectively. If F and G are P-$\mathscr{R}$-equivalent by a diffeomorphism germ $\Phi : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \to (\mathbb{R} \times \mathbb{R}^r, (s'_0, \boldsymbol{x}'_0))$ of the form $\Phi(s, \boldsymbol{x}) = (\Phi_1(s, \boldsymbol{x}), \phi(\boldsymbol{x}))$, then $\phi(D^\ell_F) = D^\ell_G$ are set germs.*

We have the following classification theorem of versal unfoldings (see page 149, [2]).

Theorem 4.3. *. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \boldsymbol{x}_0)) \to \mathbb{R}$ be an r-parameter unfolding of f with A_k-singularity at s_0. Suppose F is a versal unfolding of f. Then, F is P-$\mathscr{R}$-equivalent to one of the following unfoldings:*
(a)$k = 1 : \pm s^2 + x_1$.
(b)$k = 2 : \pm s^3 + x_1 + sx_2$.
(c)$k = 3 : \pm s^4 + x_1 + sx_2 + s^2x_3$.
(d)$k = 4 : \pm s^5 + x_1 + sx_2 + s^2x_3 + s^3x_4$.

Izumiya et al. give the following classification result as a corollary of the above theorem[12].

Corollary 4.4. *Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be an r-parameter of $f(s)$ which has the A_k singularity at s_0. Suppose that F is a versal unfolding. Then, we have the following assertions:*
(a) *If $k = 1$, then D_F is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$ and $D^2_F = \varnothing$.*
(b) *If $k = 2$, then D_F is locally diffeomorphic to $C(2, 3) \times \mathbb{R}^{r-2}$,$D^2_F$ is locally diffeomorphic to $\{\mathbf{0}\} \times \mathbb{R}^{r-2}$, $D^3_F = \varnothing$*
(c) *If $k = 3$, then D_F is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$, D^2_F is locally diffeomorphic to $C(2, 3, 4) \times \mathbb{R}^{r-2}$, and D^3_F is locally diffeomorphic to $\{\mathbf{0}\} \times \mathbb{R}^{r-3}$, and $D^4_F = \varnothing$.*
(d)*If $k = 4$, then D_F is locally diffeomorphic to $BF \times \mathbb{R}^{r-4}$, D^2_F is locally diffeomorphic to $C(BF) \times \mathbb{R}^{r-4}$, D^3_F is locally diffeomorphic to $C(2, 3, 4, 5) \times \mathbb{R}^{r-4}$, D^4_F is locally diffeomorphic to $\{\mathbf{0}\} \times \mathbb{R}^{r-4}$, and $D^5_F = \varnothing$.*

Note that all the diffeomorphisms in the above assertions are diffeomorphism germs. We call
$C(2, 3) = \{(x_1, x_2)|x_1 = u^2, x_2 = u^3\}$ *a (2,3)-cusp,*
$C(2, 3, 4) = \{(x_1, x_2, x_3)|x_1 = u^2, x_2 = u^3, x_3 = u^4\}$ *a (2,3,4)-cusp,*
$C(2, 3, 4, 5) = \{(x_1, x_2, x_3, x_4)|x_1 = u^2, x_2 = u^3, x_3 = u^4, x_4 = u^5\}$ *a (2,3,4,5)-cusp,*
$SW = \{(x_1, x_2, x_3)|x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$ *a swallowtail,*
$BF = \{(x_1, x_2, x_3, x_4)|x_1 = 5u^4 + 3vu^2 + 2wu, x_2 = 4u^5 + 2vu^3 + wu^2, x_3 = u, x_4 = v\}$ *a butterfly,*
and $C(BF) = \{(x_1, x_2, x_3, x_4)|x_1 = 6u^5 + u^3v, x_2 = 25u^4 + 9u^2v, x_3 = 10u^3 + 3uv, x_4 = v\}$ *a c-butterfly (i.e., the critical value set of the butterfly). We have the following key proposition for H.*

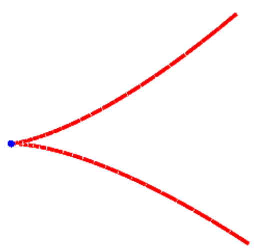

Fig.1 (2,3)-cusp.

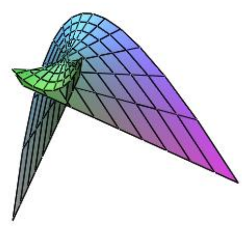

Fig.2 Swallowtail.

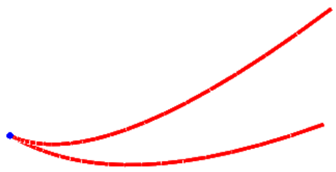

Fig.3 (2,3,4)-cusp.

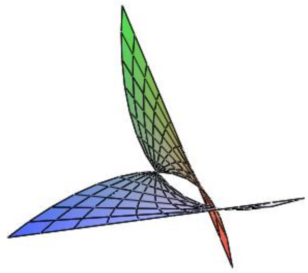

Fig.4 The projection of butterfly on 3-space when the parameter $v = 0$ of BF.

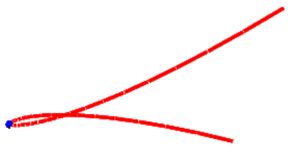

Fig.5 The projection of (2,3,4,5)-cusp on 3-space.

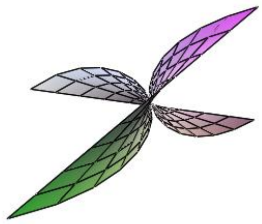

Fig.6 The projection of c-butterfly on 3-space.

Proposition 4.5. *If $h_{\lambda_0}(s)$ has A_k-singularity at s_0 $(k = 1, 2, 3, 4)$, then H is a versal unfolding of $h_{\lambda_0}(s)$.*

Proof. We denote

$$\boldsymbol{\gamma}(s) = \big(x_0(s), x_1(s), x_2(s), x_3(s)\big) \text{ and } \boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \text{ in } \mathbb{R}^4_1,$$

and by definition, we obtain that

$$\begin{aligned}
H(s, \boldsymbol{\lambda}) &= -(\lambda_0 - x_0(s))^2 + (\lambda_1 - x_1(s))^2 + (\lambda_2 - x_2(s))^2 + (\lambda_3 - x_3(s))^2,\\
\frac{\partial H}{\partial \lambda_0}(s, \boldsymbol{\lambda}) &= -2(\lambda_0 - x_0(s)),\ \frac{\partial H}{\partial \lambda_i}(s, \boldsymbol{\lambda}) = 2(\lambda_i - x_i(s)),\ (i = 1, 2, 3),\\
\frac{\partial^2 H}{\partial s \partial \lambda_0}(s, \boldsymbol{\lambda}) &= 2x_0'(s),\ \frac{\partial^2 H}{\partial s \partial \lambda_i}(s, \boldsymbol{\lambda}) = -2x_i'(s),\ (i = 1, 2, 3),\\
\frac{\partial^3 H}{\partial s^2 \partial \lambda_0}(s, \boldsymbol{\lambda}) &= 2x_0''(s),\ \frac{\partial^3 H}{\partial s^2 \partial \lambda_i}(s, \boldsymbol{\lambda}) = -2x_i''(s),\ (i = 1, 2, 3),\\
\frac{\partial^4 H}{\partial s^3 \partial \lambda_0}(s, \boldsymbol{\lambda}) &= 2x_0'''(s),\ \frac{\partial^4 H}{\partial s^3 \partial \lambda_i}(s, \boldsymbol{\lambda}) = -2x_i'''(s),\ (i = 1, 2, 3).
\end{aligned}$$

For a fixed $\boldsymbol{\lambda}_0 = (\lambda_{00}, \lambda_{01}, \lambda_{02}, \lambda_{03})$, let $j^3 \frac{\partial H}{\partial \lambda_i}(s, \boldsymbol{\lambda}_0)(s_0)$ be the 3-jet of $\frac{\partial H}{\partial \lambda_i}(s, \boldsymbol{\lambda})$ at $s_0 (i = 0, 1, 2, 3)$, so

$$\begin{aligned}
&\tfrac{\partial H}{\partial \lambda_i}(s_0, \boldsymbol{\lambda}_0) + j^2(\tfrac{\partial H}{\partial \lambda_i}(s, \boldsymbol{\lambda}_0))(s_0)\\
&\quad = \tfrac{\partial H}{\partial \lambda_i}(s_0, \boldsymbol{\lambda}_0) + \tfrac{\partial^2 H}{\partial s \partial \lambda_i}(s_0, \boldsymbol{\lambda}_0)(s - s_0) + \tfrac{1}{2} \tfrac{\partial^3 H}{\partial s^2 \partial \lambda_i}(s_0, \boldsymbol{\lambda}_0)(s - s_0)^2 + \tfrac{1}{6} \tfrac{\partial^4 H}{\partial s^3 \partial \lambda_i}(s_0, \boldsymbol{\lambda}_0)(s - s_0)^3\\
&\quad = \alpha_{0,i} + \alpha_{1,i}(s - s_0) + \tfrac{1}{2}\alpha_{2,i}(s - s_0)^2 + \tfrac{1}{6}\alpha_{3,i}(s - s_0)^3.
\end{aligned}$$

When $h(s)$ has A_1-singularity at s_0, we have

$$\begin{aligned} B &= \begin{pmatrix} \alpha_{0,1} & \alpha_{0,2} & \alpha_{0,3} & \alpha_{0,4} \end{pmatrix} \\ &= \begin{pmatrix} -2(\lambda_0 - x_1(s_0)) & 2(\lambda_1 - x_0(s_0)) & 2(\lambda_2 - x_2(s_0)) & 2(\lambda_3 - x_3(s_0)) \end{pmatrix}. \end{aligned}$$

It is easy to see from $\boldsymbol{\lambda} - \boldsymbol{\gamma}(s) \in \mathbb{S}^3_{1,\boldsymbol{\lambda}}$ that the rank of B is 1.

Assume that $h(s)$ has A_k-singularity at s_0 $(k = 2, 3, 4)$, under these conditions; $\boldsymbol{\lambda} - \boldsymbol{\gamma}(s) = \mu\boldsymbol{t}(s) \pm \boldsymbol{n}_2(s)$, and we prove the rank of the matrix

$$\begin{aligned} C &= \begin{pmatrix} \alpha_{0,1} & \alpha_{0,2} & \alpha_{0,3} & \alpha_{0,4} \\ \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{3,4} \end{pmatrix} \\ &= \begin{pmatrix} -2(\lambda_0 - x_0(s_0)) & 2(\lambda_1 - x_1(s_0)) & 2(\lambda_2 - x_2(s_0)) & 2(\lambda_3 - x_3(s_0)) \\ 2x_0'(s_0) & -2x_1'(s_0) & -2x_2'(s_0) & -2x_3'(s_0) \\ 2x_0''(s_0) & -2x_1''(s_0) & -2x_2''(s_0) & -2x_3''(s_0) \\ 2x_0'''(s_0) & -2x_1'''(s_0) & -2x_2'''(s_0) & -2x_3'''(s_0) \end{pmatrix} \end{aligned}$$

is four. In fact, we can use an elementary transformation such that matrix C becomes matrix C_1, where

$$C_1 = \begin{pmatrix} (\lambda_0 - x_1(s_0)) & (\lambda_1 - x_1(s_0)) & (\lambda_2 - x_2(s_0)) & (\lambda_3 - x_3(s_0)) \\ x_0'(s_0) & x_1'(s_0) & x_2'(s_0) & x_3'(s_0) \\ x_0''(s_0) & x_1''(s_0) & x_2''(s_0) & x_3''(s_0) \\ x_0'''(s_0) & x_1'''(s_0) & x_2'''(s_0) & x_3'''(s_0) \end{pmatrix}.$$

The rank of B is equal to the rank of C_1; we can calculate the determinant of C_1

$$\begin{aligned} \det C_1 &= \Big\langle \Big(\boldsymbol{\lambda} - \boldsymbol{\gamma}(s)\Big) \wedge \boldsymbol{\gamma}'(s) \wedge \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}'''(s) \Big\rangle \\ &= \Big\langle \Big(\mu\boldsymbol{t}(s) \pm \boldsymbol{n}_2(s)\Big) \wedge \Big(\varphi(s)\boldsymbol{t}(s)\Big) \wedge \Big(\varphi'(s)\boldsymbol{t}(s) + \varphi(s)\boldsymbol{n}_1(s)\Big), \\ &\qquad \Big(\varphi''(s) - k_1(s)\varphi(s)\Big)\boldsymbol{t}(s) + 2\varphi'(s)\boldsymbol{n}_1(s) - \varphi(s)\boldsymbol{\eta}(s) \Big\rangle \\ &= \Big\langle \pm\varphi^2(s)\boldsymbol{t}(s), \Big(\varphi''(s) - k_1(s)\varphi(s)\Big)\boldsymbol{t}(s) + 2\varphi'(s)\boldsymbol{n}_1(s) - \varphi(s)\boldsymbol{\eta}(s) \Big\rangle \\ &= \pm\varphi^3(s). \\ &\neq 0, \end{aligned}$$

where $\sqrt{\langle \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}''(s) \rangle} = \varphi(s) > 0$. This completes the proof. □

Finally, in combination with Proposition 3.1, Definition 3.2, Proposition 4.5, we can apply Corollary 4.4 to our situation. Then, we have the following theorem.

Theorem 4.6. *Let $\boldsymbol{\gamma} : I \to \mathbb{R}^4_1$ be a regular null Cartan curve with the Cartan Frenet frame $\{\boldsymbol{t}(s), \boldsymbol{\eta}(s), \boldsymbol{n}_1(s), \boldsymbol{n}_2(s)\}$. For the lightlike hypersurfaces $\mathbb{LH}_C(s, \mu, \theta) = \boldsymbol{\gamma}(s) + \mathbb{DG}_C(s, \mu, \theta)$ of $C = \boldsymbol{\gamma}(I)$, we have the following assertions:*
(1) The null Cartan curve $\boldsymbol{\gamma}(s)$ and pseudosphere $\mathbb{S}^3_{1,\lambda_0}$ have at least a two-point contact.
(2) The null Cartan curve $\boldsymbol{\gamma}(s)$ and pseudosphere $\mathbb{S}^3_{1,\lambda_0}$ have a three-point contact if and only if there exist $\theta_0 = \pi \pm \frac{\pi}{2}$ and $\mu_0 \in \mathbb{R}\backslash\{0\}$ such that

$$\boldsymbol{\lambda}_0 - p_0 = \mathbb{DG}_C(s_0, \mu_0, \pi \pm \frac{\pi}{2}).$$

Under this condition, the lightlike hypersurfaces is locally diffeomorphic to $C(2,3) \times \mathbb{R}^2$ at $\boldsymbol{\lambda}_0$, and the lightlike focal set $\mathbb{LFS}^{\pm}_C$ is non-singular.

(3) *The null Cartan curve* $\boldsymbol{\gamma}(s)$ *and pseudosphere* $\mathbb{S}^3_{1,\lambda_0}$ *have a four-point contact if and only if there exist* $\theta_0 = \pi \pm \frac{\pi}{2}$ *and* $\mu_0 = 0$ *such that*

$$\boldsymbol{\lambda}_0 - p_0 = \mathbb{DG}_C(s_0, \mu_0, \pi \pm \frac{\pi}{2})$$

and $k_2(s_0) \neq \pm\varphi(s_0)$.
Under this condition, the lightlike hypersurfaces is locally diffeomorphic to $SW \times \mathbb{R}$ *at* $\boldsymbol{\lambda}_0$*, the lightlike focal set* $\mathbb{LFS}^{\pm}_C$ *is locally diffeomorphic to* $C(2,3,4) \times \mathbb{R}^2$*, and the critical value set of* $\mathbb{LFS}^{\pm}_C$ *is a regular curve.*
(4) *The null Cartan curve* $\boldsymbol{\gamma}(s)$ *and pseudosphere* $\mathbb{S}^3_{1,\lambda_0}$ *have a five-point contact if and only if there exist* $\theta_0 = \pi \pm \frac{\pi}{2}$ *and* $\mu_0 \in \mathbb{R}\backslash\{0\}$ *such that*

$$\boldsymbol{\lambda}_0 - p_0 = \mathbb{DG}_C(s_0, 0, \pi \pm \frac{\pi}{2})$$

and $k_2(s_0) = \pm\varphi(s_0), k_2'(s_0) \neq \pm\varphi'(s_0)$.
Under this condition, the lightlike hypersurfaces is locally diffeomorphic to BF *at* $\boldsymbol{\lambda}_0$*; in other words, the locus of the center of the osculating pseudo sphere of* $\boldsymbol{\gamma}(s)$ *is* BF*. The lightlike focal set* $\mathbb{LFS}^{\pm}_C$ *is locally diffeomorphic to* $C(BF)$*, and the critical value set of* $\mathbb{LFS}^{\pm}_C$ *is locally diffeomorphic to the* $C(2,3,4,5)$*-cusp.*

5. Example

To better illustrate our results, we provide examples of lightlike hypersurface and lightlike focal set along a null curve in $\mathbb{R}^4_1$. Furthermore, we depict the types of the singularities of the lightlike hypersurface by using appropriate graph-plotting software. Consider the null curve $\boldsymbol{\gamma} : (0, +\infty) \to \mathbb{R}^4_1$ in $\mathbb{R}^4_1$ given by

$$\boldsymbol{\gamma}(s) = \Bigg(\frac{1}{6}s^3 + \frac{1}{2}s, \frac{1}{6}s^3 - \frac{1}{2}s, \frac{-\frac{1}{5}s^2 + \frac{4}{5}s^2\tan(\frac{1}{2}\ln(s)) + \frac{1}{5}s^2\tan^2(\frac{1}{2}\ln(s))}{1+\tan^2(\frac{1}{2}\ln(s))}, \frac{\frac{2}{5}s^2 + \frac{2}{5}s^2\tan(\frac{1}{2}\ln(s)) - \frac{2}{5}s^2\tan^2(\frac{1}{2}\ln(s))}{1+\tan^2(\frac{1}{2}\ln(s))}\Bigg).$$

We calculate

$$\boldsymbol{\gamma}'(s) = \Big\{\frac{1}{2}s^2 + \frac{1}{2}, \frac{1}{2}s^2 - \frac{1}{2}, s\sin\big(\ln(s)\big), s\cos\big(\ln(s)\big)\Big\},$$

$$\boldsymbol{\gamma}''(s) = \Big\{s, s, \sin\big(\ln(s)\big) + \cos\big(\ln(s)\big), \cos\big(\ln(s)\big) - \sin\big(\ln(s)\big)\Big\},$$

$$\boldsymbol{\gamma}'''(s) = \Big\{1, 1, -\frac{1}{s}\Big(\sin\big(\ln(s)\big) - \cos\big(\ln(s)\big)\Big), -\frac{1}{s}\Big(\cos\big(\ln(s)\big) + \sin\big(\ln(s)\big)\Big)\Big\},$$

$$\boldsymbol{\gamma}^{(4)}(s) = \Big\{0, 0, -\frac{2}{s^2}\cos\big(\ln(s)\big), \frac{2}{s^2}\sin\big(\ln(s)\big)\Big\},$$

then

$$\varphi(s) = \sqrt{2}, \quad k_2(s) = \frac{1}{s^2},$$

thus

$$\boldsymbol{t}(s) = \frac{\sqrt{2}}{2}\Bigg(\frac{1}{2}s^2 + \frac{1}{2}, \frac{1}{2}s^2 - \frac{1}{2}, s\sin\big(\ln(s)\big), s\cos\big(\ln(s)\big)\Bigg),$$

$$\boldsymbol{\eta}(s) = -\frac{\sqrt{2}}{2}\Bigg(\frac{5}{4} + \frac{1}{4s^2}, \frac{5}{4} - \frac{1}{4s^2}, \frac{1}{2s}\Big(2\cos\big(\ln(s)\big) - \sin\big(\ln(s)\big)\Big), -\frac{1}{2s}\Big(\cos\big(\ln(s)\big) + 2\sin\big(\ln(s)\big)\Big)\Bigg),$$

$$\boldsymbol{n}_1(s) = \frac{\sqrt{2}}{2}\Bigg(s, s, \cos\big(\ln(s)\big) + \sin\big(\ln(s)\big), \cos\big(\ln(s)\big) - \sin\big(\ln(s)\big)\Bigg),$$

$$\boldsymbol{n}_2(s) = \frac{\sqrt{2}}{4}\left(s - \frac{1}{s}, s + \frac{1}{s}, -2\cos\big(\ln(s)\big), 2\sin\big(\ln(s)\big)\right).$$

Although we cannot draw the figure of the lightlike hypersurface $\mathbb{LH}_C(s,\mu,\theta) = \boldsymbol{\gamma}(s)+\mu\boldsymbol{t}(s)+\cos\theta\boldsymbol{n}_1(s)+\sin\theta\boldsymbol{n}_1(s)$, when $\mu = 0$, the lightlike hypersurface $\mathbb{LH}_C(s,\mu,\theta)$ is a surface in $\mathbb{R}^4_1$ and the lightlike hypersurface $\mathbb{LH}_C(s,\mu,\theta)$ is a ruled hypersurface generated by the surface $\mathbb{LH}_C(s,0,\theta)$ (see figs.7 and 8) along the tangent curve $\boldsymbol{t}(s)$, so we can draw the projection of surface $\mathbb{LH}_C(s,0,\theta)$ on 3-space and we can obtain the information on the image of the ruled lightlike hypersurface $\mathbb{LH}_C(s,\mu,\theta)$ by the projection.

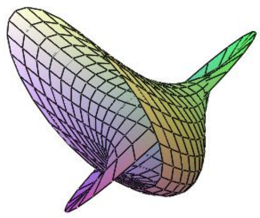

Fig.7 Projection of surface $\mathbb{LH}_C(s,0,\theta)$ on 3-space.

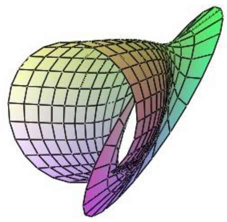

Fig.8 Projection of surface $\mathbb{LH}_C(s,0,\theta)$ on 3-space from another viewpoint.

We can draw the projection of the lightlike focal set $\mathbb{LFS}^{\pm}_C$ on 3-space (set of the critical values of $\mathbb{LH}_C(s,\mu,\theta)$, see figs.9 and 10.

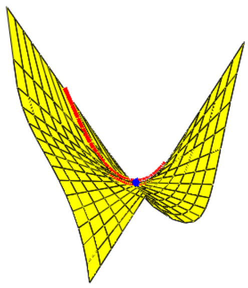

Fig.9 Projection of $\mathbb{LFS}^{+}_C$ on 3-space.

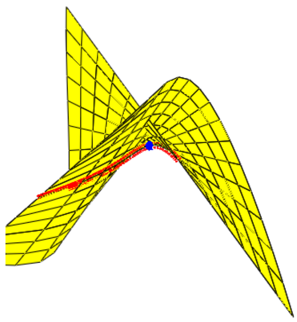

Fig.10 Projection of $\mathbb{LFS}^{+}_C$ on 3-space from another viewpoint.

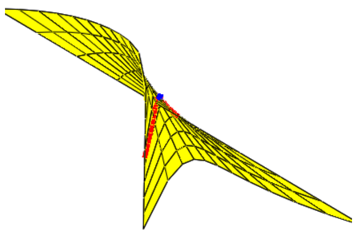

Fig.11 Projection of $\mathbb{LFS}^{-}_C$ on 3-space.

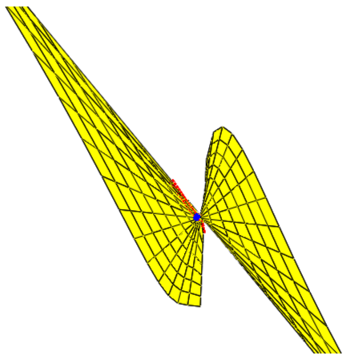

Fig.12 Projection of $\mathbb{LFS}^{-}_C$ on 3-space from another viewpoint.

Remark 5.1. When $s \in (0, +\infty)$, solving $k_2(s) - \varphi(s) = 0$ gives one real root $s_0 = 2^{1/4}$ and $k_1'(s_0) - \varphi'(s_0) \neq 0$ at $s_0 = 2^{1/4}$. Hence, the set of critical values of $\mathbb{LH}_C(s, \mu, \theta)$ are classified into three parts: the yellow parts correspond to the points $(s_0, \mu_0, \theta_0), \mu_0 \neq 0, \theta_0 = \pi \pm \frac{\pi}{2}$; the red parts correspond to the points $(s_0, \mu_0, \theta_0), s_0 \neq 2^{1/4}, \mu_0 = 0$, and $\theta_0 = \pi \pm \frac{\pi}{2}$; and the blue one corresponds to the point $(s_0, \mu_0, \theta_0), s_0 = 2^{1/4}, \mu_0 = 0$, and $\theta_0 = \pi \pm \frac{\pi}{2}$.

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