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Multivariate fuzzy perturbed neural network operators approximation

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Abstract

This article studies the determination of the rate of convergence to the unit of each of three newly introduced here multivariate fuzzy perturbed normalized neural network operators of one hidden layer. These are given through the multivariate fuzzy modulus of continuity of the involved multivariate fuzzy number valued function or its high order fuzzy partial derivatives and that appears in the right-hand side of the associated fuzzy multivariate Jackson type inequalities. The multivariate activation function is very general, especially it can derive from any sigmoid or bell-shaped function. The right hand sides of our multivariate fuzzy convergence inequalities do not depend on the activation function. The sample multivariate fuzzy functionals are of Stancu, Kantorovich and Quadrature types. We give applications for the first fuzzy partial derivatives of the involved function.

Keywords: Multivariate neural network fuzzy approximation, fuzzy partial derivative, multivariate fuzzy modulus of continuity, multivariate fuzzy operator.
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1. Introduction

The Cardaliaguet-Euvrard real neural network operators were studied extensively in [15], where the authors among many other things proved that these operators converge uniformly on compacta, to the unit over continuous and bounded functions. Our fuzzy "multivariate perturbed normalized neural network

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operators" are motivated and inspired by the "multivariate bell" and "multivariate squashing functions" of [15]. The work in [15] is qualitative where the used multivariate bell-shaped function is general. However, our work, though greatly motivated by [15], is quantitative and the used multivariate activation functions are of compact support. Here we extend to the fuzzy environment our initial real work, see [14]. We derive a series of multivariate fuzzy Jackson type inequalities giving close upper bounds to the errors in approximating the unit operator by the above multivariate fuzzy perturbed neural network induced operators. All involved constants there are well determined. These are pointwise and uniform estimates involving the multivariate first fuzzy modulus of continuity of the engaged multivariate fuzzy function or the fuzzy partial derivatives of the function under approximation. We give all necessary background of the multivariate fuzzy calculus needed.

Initial work of the subject was done in [11] and [12]. These works motivate the current work along with [16].

2. Fuzzy Multivariate Real Analysis Background

We need the following basic background

Definition 1. (see [22]) Let $\mu : \mathbb{R} \to [0,1]$ with the following properties:

(i) is normal, i.e., $\exists\, x_0 \in \mathbb{R} : \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\, x, y \in \mathbb{R}$, $\forall\, \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e., $\forall\, x_0 \in \mathbb{R}$ and $\forall\, \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\, x \in V(x_0)$.

(iv) the set $\overline{\operatorname{supp}(\mu)}$ is compact in $\mathbb{R}$ (where $\operatorname{supp}(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define

$$[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\} \tag{1}$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$ ([17]).

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \quad \forall\, r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [22]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \quad \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall\, r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ by

$$D(u,v) := \sup_{r \in [0,1]} \max\left\{\left|u_-^{(r)} - v_-^{(r)}\right|, \left|u_+^{(r)} - v_+^{(r)}\right|\right\}, \tag{2}$$

where

$$[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]; \quad u, v \in \mathbb{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [23], [22].

Let $f, g : \mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$. We define the distance

$$D^*(f, g) := \sup_{x \in \mathbb{R}^m} D(f(x), g(x)).$$

Here $\sum^*$ stands for fuzzy summation and $\widetilde{0} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \ \ \forall \ u \in \mathbb{R}_{\mathcal{F}}.$$

We need

Remark 2. *([4]). Here* $r \in [0,1]$, $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$, $i = 1, ..., m \in \mathbb{N}$. *Suppose that*

$$\sup_{r \in [0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right) \in \mathbb{R}, \text{ for } i = 1, ..., m.$$

Then one sees easily that

$$\sup_{r \in [0,1]} \max\left(\sum_{i=1}^m x_i^{(r)}, \sum_{i=1}^m y_i^{(r)}\right) \leq \sum_{i=1}^m \sup_{r \in [0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right). \tag{3}$$

Definition 3. Let $f \in C(\mathbb{R}^m)$, $m \in \mathbb{N}$, which is bounded or uniformly continuous, we define $(h > 0)$

$$\omega_1(f, h) := \sup_{\text{all } x_i, x_{i'} \in \mathbb{R}, \ \left|x_i - x_i'\right| \leq h, \text{ for } i=1,...,m} \left|f(x_1, ..., x_m) - f\left(x_1', ..., x_m'\right)\right|. \tag{4}$$

Definition 4. Let $f : \mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$, we define the fuzzy modulus of continuity of f by

$$\omega_1^{(\mathcal{F})}(f, \delta) = \sup_{x, y \in \mathbb{R}^m, \ |x_i - y_i| \leq \delta, \text{ for } i=1,...,m} D(f(x), f(y)), \quad \delta > 0, \tag{5}$$

where $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_m)$.

For $f : \mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$, we use

$$[f]^r = \left[f_-^{(r)}, f_+^{(r)}\right], \tag{6}$$

where $f_\pm^{(r)} : \mathbb{R}^m \to \mathbb{R}$, $\forall \ r \in [0,1]$.

We need

Proposition 5. *Let* $f : \mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$. *Assume that* $\omega_1^{\mathcal{F}}(f, \delta)$, $\omega_1\left(f_-^{(r)}, \delta\right)$, $\omega_1\left(f_+^{(r)}, \delta\right)$ *are finite for any* $\delta > 0$, $r \in [0,1]$.

Then

$$\omega_1^{(\mathcal{F})}(f, \delta) = \sup_{r \in [0,1]} \max\left\{\omega_1\left(f_-^{(r)}, \delta\right), \omega_1\left(f_+^{(r)}, \delta\right)\right\}. \tag{7}$$

Proof. By Proposition 1 of [8]. □

We define by $C_{\mathcal{F}}^U(\mathbb{R}^m)$ the space of fuzzy uniformly continuous functions from $\mathbb{R}^m \to \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(\mathbb{R}^m)$ is the space of fuzzy continuous functions on $\mathbb{R}^m$, and $C_B(\mathbb{R}^m, \mathbb{R}_{\mathcal{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 6. *([6]) Let $f \in C_{\mathcal{F}}^{U}(\mathbb{R}^m)$. Then $\omega_1^{(\mathcal{F})}(f,\delta) < \infty$, for any $\delta > 0$.*

Proposition 7. *([6]) It holds*

$$\lim_{\delta \to 0} \omega_1^{(\mathcal{F})}(f,\delta) = \omega_1^{(\mathcal{F})}(f,0) = 0, \tag{8}$$

iff $f \in C_{\mathcal{F}}^{U}(\mathbb{R}^m)$.

Proposition 8. *([6]) Let $f \in C_{\mathcal{F}}(\mathbb{R}^m)$. Then $f_{\pm}^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over $\mathbb{R}^m$, respectively in $\pm$.*

Note 9. *It is clear by Propositions 5, 7, that if $f \in C_{\mathcal{F}}^{U}(\mathbb{R}^m)$, then $f_{\pm}^{(r)} \in C_U(\mathbb{R}^m)$ (uniformly continuous on $\mathbb{R}^m$). Clearly also if $f \in C_B(\mathbb{R}^m, \mathbb{R}_{\mathcal{F}})$, then $f_{\pm}^{(r)} \in C_B(\mathbb{R}^m)$ (continuous and bounded functions on $\mathbb{R}^m$).*

We need

Definition 10. Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$, then we call z the H-difference on x and y, denoted $x - y$.

Definition 11. ([22]) Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f : T \to \mathbb{R}_{\mathcal{F}}$ is H-difference at $x \in T$ if there exists an $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that the limits (with respect to D)

$$\lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h} \tag{9}$$

exist and are equal to $f'(x)$.

We call f' the H-derivative or fuzzy derivative of f at x.

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exist in $\mathbb{R}_{\mathcal{F}}$ in a neighborhood of x.

Definition 12. We denote by $C_{\mathcal{F}}^{N}(\mathbb{R}^m)$, $N \in \mathbb{N}$, the space of all N-times fuzzy continuously differentiable functions from $\mathbb{R}^m$ into $\mathbb{R}_{\mathcal{F}}$.

Here fuzzy partial derivatives are defined via Definition 11 in the obvious way as in the ordinary real case.

We mention

Theorem 13. *([18]) Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t \in [a,b]$, $0 \leq r \leq 1$. Clearly*

$$[f(t)]^r = \left[f(t)_{-}^{(r)}, f(t)_{+}^{(r)} \right] \subseteq \mathbb{R}.$$

Then $(f(t))_{\pm}^{(r)}$ are differentiable and

$$[f'(t)]^r = \left[\left(f(t)_{-}^{(r)} \right)', \left(f(t)_{+}^{(r)} \right)' \right].$$

I.e.

$$(f')_{\pm}^{(r)} = \left(f_{\pm}^{(r)} \right)', \quad \forall\, r \in [0,1]. \tag{10}$$

Remark 14. *(see also [5]) Let $f \in C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \geq 1$. Then by Theorem 13 we obtain $f_{\pm}^{(r)} \in C^N(\mathbb{R})$ and*

$$\left[f^{(i)}(t) \right]^r = \left[\left(f(t)_{-}^{(r)} \right)^{(i)}, \left(f(t)_{+}^{(r)} \right)^{(i)} \right],$$

for $i = 0, 1, 2, ..., N$, and in particular we have that

$$\left(f^{(i)} \right)_{\pm}^{(r)} = \left(f_{\pm}^{(r)} \right)^{(i)}, \tag{11}$$

for any $r \in [0,1]$.

Let $f \in C_{\mathcal{F}}^{N}(\mathbb{R}^m)$, denote $f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}$, where $\widetilde{\alpha} := (\widetilde{\alpha_1}, ..., \widetilde{\alpha_m})$, $\widetilde{\alpha}_i \in \mathbb{Z}^+$, $i = 1, ..., m$ and

$$0 < |\widetilde{\alpha}| := \sum_{i=1}^{m} \widetilde{\alpha}_i \leq N, \quad N > 1.$$

Then by Theorem 13 we get that

$$\left(f_{\pm}^{(r)}\right)_{\widetilde{\alpha}} = \left(f_{\widetilde{\alpha}}\right)_{\pm}^{(r)}, \ \forall\, r \in [0,1], \tag{12}$$

and any $\widetilde{\alpha} : |\widetilde{\alpha}| \leq N$. Here $f_{\pm}^{(r)} \in C^N(\mathbb{R}^m)$.

For the definition of general fuzzy integral we follow [19] next.

Definition 15. Let (Ω, Σ, μ) be a complete σ-finite measure space. We call $F : \Omega \to \mathbb{R}_{\mathcal{F}}$ measurable iff $\forall$ closed $B \subseteq \mathbb{R}$ the function $F^{-1}(B) : \Omega \to [0,1]$ defined by

$$F^{-1}(B)(w) := \sup_{x \in B} F(w)(x), \ \text{ all } w \in \Omega$$

is measurable, see [19].

Theorem 16. *([19]) For $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,*

$$F(w) = \{(F_{-}^{(r)}(w), F_{+}^{(r)}(w)) | 0 \leq r \leq 1\},$$

the following are equivalent

(1) F is measurable,

(2) $\forall\, r \in [0,1]$, $F_{-}^{(r)}$, $F_{+}^{(r)}$ are measurable.

Following [19], given that for each $r \in [0,1]$, $F_{-}^{(r)}$, $F_{+}^{(r)}$ are integrable we have that the parametrized representation

$$\left\{\left(\int_A F_{-}^{(r)} d\mu, \int_A F_{+}^{(r)} d\mu\right) | 0 \leq r \leq 1\right\}$$

is a fuzzy real number for each $A \in \Sigma$.

The last fact leads to

Definition 17. ([19]) A measurable function $F : \Omega \to \mathbb{R}_{\mathcal{F}}$,

$$F(w) = \{(F_{-}^{(r)}(w), F_{+}^{(r)}(w)) | 0 \leq r \leq 1\}$$

is called integrable if for each $r \in [0,1]$, $F_{\pm}^{(r)}$ are integrable, or equivalently, if $F_{\pm}^{(0)}$ are integrable.

In this case, the fuzzy integral of F over $A \in \Sigma$ is defined by

$$\int_A F d\mu := \left\{\left(\int_A F_{-}^{(r)} d\mu, \int_A F_{+}^{(r)} d\mu\right) | 0 \leq r \leq 1\right\}. \tag{13}$$

By [19] F is integrable iff $w \to \|F(w)\|_{\mathcal{F}}$ is real-valued integrable. Here

$$\|u\|_{\mathcal{F}} := D\left(u, \widetilde{0}\right), \ \forall\, u \in \mathbb{R}_{\mathcal{F}}.$$

We need also

Theorem 18. *([19]) Let $F, G : \Omega \to \mathbb{R}_{\mathcal{F}}$ be integrable. Then*

(1) Let $a, b \in \mathbb{R}$, then $aF + bG$ is integrable and for each $A \in \Sigma$,

$$\int_A (aF + bG)\, d\mu = a \int_A F d\mu + b \int_A G d\mu;$$

(2) $D(F, G)$ is a real-valued integrable function and for each $A \in \Sigma$,

$$D\left(\int_A F d\mu, \int_A G d\mu\right) \leq \int_A D(F, G)\, d\mu. \tag{14}$$

In particular,

$$\left\| \int_A F d\mu \right\|_{\mathcal{F}} \leq \int_A \|F\|_{\mathcal{F}}\, d\mu.$$

Above μ could be the Lebesgue measure, with all the basic properties valid here too.

Basically here we have

$$\left[\int_A F d\mu\right]^r := \left[\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu\right], \tag{15}$$

i.e.

$$\left(\int_A F d\mu\right)_{\pm}^{(r)} = \int_A F_{\pm}^{(r)} d\mu, \tag{16}$$

$\forall\, r \in [0, 1]$, respectively.

In this article we use the fuzzy integral with respect to Lebesgue measure on $\mathbb{R}^m$. See also Fubini's theorem from [19].

We also need

Notation 19. *We denote*

$$\left(\sum_{i=1}^{2} D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^2 f(x) := \tag{17}$$

$$D\left(\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2}, \widetilde{0}\right) + D\left(\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2}, \widetilde{0}\right) + 2D\left(\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2}, \widetilde{0}\right).$$

In general we denote ($j = 1, ..., N$)

$$\left(\sum_{i=1}^{m} D\left(\frac{\partial}{\partial x_i}, \widetilde{0}\right)\right)^j f(x) := \tag{18}$$

$$\sum_{(j_1,...,j_m) \in \mathbb{Z}_+^m : \sum_{i=1}^m j_i = j} \frac{j!}{j_1! j_2! ... j_m!} D\left(\frac{\partial^j f(x_1, ..., x_m)}{\partial x_1^{j_1} \partial x_2^{j_2} ... \partial x_m^{j_m}}, \widetilde{0}\right).$$

3. Real Neural Networks Multivariate Approximation Basics (see [14])

Here the activation function $b : \mathbb{R}^d \to \mathbb{R}_+$, $d \in \mathbb{N}$, is of compact support $\overline{B} := \prod_{j=1}^{d} [-T_j, T_j]$, $T_j > 0$, $j = 1, ..., d$. That is $b(x) > 0$ for any $x \in \overline{B}$, and clearly b may have jump discontinuities. Also the shape of the graph of b is immaterial.

Typically in neural networks approximation we take b to be a d-dimensional bell-shaped function (i.e. per coordinate is a centered bell-shaped function), or a product of univariate centered bell-shaped functions, or a product of sigmoid functions, in our case all of them are of compact support $\overline{B}$.

Example 20. *Take $b(x) = \beta(x_1)\beta(x_2)...\beta(x_d)$, where β is any of the following functions, $j = 1, ..., d$:*
(i) $\beta(x_j)$ is the characteristic function on $[-1,1]$,
(ii) $\beta(x_j)$ is the hat function over $[-1,1]$, that is,

$$\beta(x_j) = \begin{cases} 1 + x_j, & -1 \leq x_j \leq 0, \\ 1 - x_j, & 0 < x_j \leq 1, \\ 0, & \textit{elsewhere}, \end{cases}$$

(iii) the truncated sigmoids

$$\beta(x_j) = \begin{cases} \frac{1}{1+e^{-x_j}} \textit{ or } \tanh x_j \textit{ or } \quad \mathit{erf}(x_j), & \textit{for } x_j \in [-T_j, T_j], \textit{ with large } T_j > 0, \\ 0, \quad x_j \in \mathbb{R} - [-T_j, T_j], \end{cases}$$

(iv) the truncated Gompertz function

$$\beta(x_j) = \begin{cases} e^{-\alpha e^{-\beta x_j}}, & x_j \in [-T_j, T_j]; \ \alpha, \beta > 0; \ \textit{large } T_j > 0, \\ 0, & x_j \in \mathbb{R} - [-T_j, T_j], \end{cases}$$

The Gompertz functions are also sigmoid functions, with wide applications to many applied fields, e.g. demography and tumor growth modeling, etc.

Thus the general activation function b we will be using here includes all kinds of activation functions in neural network approximations.

Here we consider functions $f : \mathbb{R}^d \to \mathbb{R}$ that either continuous and bounded, or uniformly continuous.

Let here the parameters: $0 < \alpha < 1$, $x = (x_1, ..., x_d) \in \mathbb{R}^d$, $n \in \mathbb{N}$; $\overline{r} = (r_1, ..., r_d) \in \mathbb{N}^d$, $i = (i_1, ..., i_d) \in \mathbb{N}^d$, with $i_j = 1, 2, ..., r_j$, $j = 1, ..., d$; also let $w_i = w_{i_1,...,i_d} \geq 0$, such that $\sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} ... \sum_{i_d=1}^{r_d} w_{i_1,...,i_d} = 1$, in brief written as $\sum_{i=1}^{\overline{r}} w_i = 1$. We further consider the parameters $k = (k_1, ..., k_d) \in \mathbb{Z}^d$; $\mu_i = (\mu_{i_1}, ..., \mu_{i_d}) \in \mathbb{R}_+^d$, $\nu_i = (\nu_{i_1}, ..., \nu_{i_d}) \in \mathbb{R}_+^d$; and $\lambda_i = \lambda_{i_1,...,i_d}$, $\rho_i = \rho_{i_1,...,i_d} \geq 0$; $\mu, \nu \geq 0$. Call $\nu_i^{\min} = \min\{\nu_{i_1}, ..., \nu_{i_d}\}$.

We use here the first modulus of continuity, with $\delta > 0$,

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in \mathbb{R}^d \\ \|x - y\|_\infty \leq \delta}} |f(x) - f(y)|, \tag{19}$$

where $\|x\|_\infty = \max(|x_1|, ..., |x_d|)$. Notice (19) is equivalent to (4).

Given that f is uniformly continuous we get $\lim_{\delta \to 0} \omega_1(f, \delta) = 0$.

Here we mention from [14] about the pointwise convergence with rates over $\mathbb{R}^d$, to the unit operator, of the following one hidden layer multivariate normalized neural network perturbed operators,

(i) the Stancu type (see [20])

$$(H_n(f))(x) = (H_n(f))(x_1, ..., x_d) = \tag{20}$$

$$\frac{\sum_{k=-n^2}^{n^2} \left(\sum_{i=1}^{\overline{r}} w_i f\left(\frac{k+\mu_i}{n+\nu_i} \right) \right) b\left(n^{1-\alpha}\left(x - \frac{k}{n} \right) \right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n} \right) \right)} =$$

$$\frac{\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} \left(\sum_{i_1=1}^{r_1} ... \sum_{i_d=1}^{r_d} w_{i_1,...,i_d} f\left(\frac{k_1+\mu_{i_1}}{n+\nu_{i_1}}, ..., \frac{k_d+\mu_{i_d}}{n+\nu_{i_d}} \right) \right) \cdot}{\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n} \right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n} \right) \right)}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right),$$

(ii) the Kantorovich type

$$(K_n(f))(x)= \tag{21}$$

$$\frac{\sum_{k=-n^2}^{n^2}\left(\sum_{i=1}^{\overline{r}}w_i(n+\rho_i)^d\int_0^{\frac{1}{n+\rho_i}}f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right)dt\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}=$$

$$\frac{\sum_{k_1=-n^2}^{n^2}...\sum_{k_d=-n^2}^{n^2}\left(\sum_{i_1=1}^{r_1}...\sum_{i_d=1}^{r_d}w_{i_1,...,i_d}\left(n+\rho_{i_1,...,i_d}\right)^d\cdot\int...\int\int_0^{\frac{1}{n+\rho_{i_1,...,i_d}}}...\int f\left(t_1+\frac{k_1+\lambda_{i_1,...,i_d}}{n+\rho_{i_1,...,i_d}},...,t_d+\frac{k_d+\lambda_{i_1,...,i_d}}{n+\rho_{i_1,...,i_d}}\right)dt_1...dt_d\right)\cdot}{\sum_{k_1=-n^2}^{n^2}...\sum_{k_d=-n^2}^{n^2}b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)} \tag{22}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right),$$

and

(iii) the quadrature type

$$(M_n(f))(x)=\frac{\sum_{k=-n^2}^{n^2}\left(\sum_{i=1}^{\overline{r}}w_if\left(\frac{k}{n}+\frac{i}{n\overline{r}}\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}= \tag{23}$$

$$\frac{\sum_{k_1=-n^2}^{n^2}...\sum_{k_d=-n^2}^{n^2}\left(\sum_{i_1=1}^{r_1}...\sum_{i_d=1}^{r_d}w_{i_1,...,i_d}f\left(\frac{k_1}{n}+\frac{i_1}{nr_1},...,\frac{k_d}{n}+\frac{i_d}{nr_d}\right)\right)\cdot}{\sum_{k_1=-n^2}^{n^2}...\sum_{k_d=-n^2}^{n^2}b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right).$$

Similar operators defined for d-dimensional bell-shaped activation functions and sample coefficients $f\left(\frac{k}{n}\right)=f\left(\frac{k_1}{n},...,\frac{k_d}{n}\right)$ were studied initially in [15], [1], [2], [3], [9], [10], etc.

Here we care about the multivariate generalized perturbed cases of these operators (see [13], [14]).

Operator K_n in the corresponding Signal Processing context, represents the natural so called "time-jitter" error, where the sample information is calculated in a perturbed neighborhood of $\frac{k+\mu}{n+\nu}$ rather than exactly at the node $\frac{k}{n}$.

The perturbed sample coefficients $f\left(\frac{k+\mu}{n+\nu}\right)$ with $0\leq\mu\leq\nu$, were first used by D. Stancu [20], in a totally different context, generalizing Bernstein operators approximation on $C([0,1])$.

The terms in the ratio of sums (20)-(23) can be nonzero, iff simultaneously

$$\left|n^{1-\alpha}\left(x_j-\frac{k_j}{n}\right)\right|\leq T_j,\ \ \text{all } j=1,...,d, \tag{24}$$

i.e. $\left|x_j-\frac{k_j}{n}\right|\leq\frac{T_j}{n^{1-\alpha}}$, all $j=1,...,d$, iff

$$nx_j-T_jn^{\alpha}\leq k_j\leq nx_j+T_jn^{\alpha},\ \ \text{all } j=1,...,d. \tag{25}$$

To have the order

$$-n^2 \leq nx_j - T_j n^\alpha \leq k_j \leq nx_j + T_j n^\alpha \leq n^2, \tag{26}$$

we need $n \geq T_j + |x_j|$, all $j = 1, ..., d$. So (26) is true when we take

$$n \geq \max_{j \in \{1,...,d\}} (T_j + |x_j|). \tag{27}$$

When $x \in \overline{B}$ in order to have (26) it is enough to assume that $n \geq 2T^*$, where $T^* := \max\{T_1, ..., T_d\} > 0$. Consider

$$\widetilde{I}_j := [nx_j - T_j n^\alpha, nx_j + T_j n^\alpha], \ \ j = 1, ..., d, \ n \in \mathbb{N}.$$

The length of $\widetilde{I}_j$ is $2T_j n^\alpha$. By Proposition 1 of [1], we get that the cardinality of $k_j \in \mathbb{Z}$ that belong to $\widetilde{I}_j := card(k_j) \geq \max(2T_j n^\alpha - 1, 0)$, any $j \in \{1, ..., d\}$. In order to have $card(k_j) \geq 1$, we need $2T_j n^\alpha - 1 \geq 1$ iff $n \geq T_j^{-\frac{1}{\alpha}}$, any $j \in \{1, ..., d\}$.

Therefore, a sufficient condition in order to obtain the order (26) along with the interval $\widetilde{I}_j$ to contain at least one integer for all $j = 1, ..., d$ is that

$$n \geq \max_{j \in \{1,...,d\}} \left\{ T_j + |x_j|, T_j^{-\frac{1}{\alpha}} \right\}. \tag{28}$$

Clearly as $n \to +\infty$ we get that $card(k_j) \to +\infty$, all $j = 1, ..., d$. Also notice that $card(k_j)$ equals to the cardinality of integers in $[\lceil nx_j - T_j n^\alpha \rceil, [nx_j + T_j n^\alpha]]$ for all $j = 1, ..., d$. Here $[\cdot]$ denotes the integral part of the number while $\lceil \cdot \rceil$ denotes its ceiling.

From now on, in this article we assume (28).

We denote by $T = (T_1, ..., T_d)$, $[nx + Tn^\alpha] = ([nx_1 + T_1 n^\alpha], ..., [nx_d + T_d n^\alpha])$, and $\lceil nx - Tn^\alpha \rceil = (\lceil nx_1 - T_1 n^\alpha \rceil, ..., \lceil nx_d - T_d n^\alpha \rceil)$. Furthermore it holds

(i)

$$(H_n(f))(x) = (H_n(f))(x_1, ..., x_d) = \tag{29}$$

$$\frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^{\overline{r}} w_i f\left(\frac{k+\mu_i}{n+\nu_i} \right) \right) b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)} =$$

$$\frac{\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1+T_1n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d+T_dn^\alpha]} \left(\sum_{i_1=1}^{r_1} \cdots \sum_{i_d=1}^{r_d} w_{i_1,...,i_d} f\left(\frac{k_1+\mu_{i_1}}{n+\nu_{i_1}}, ..., \frac{k_d+\mu_{i_d}}{n+\nu_{i_d}} \right) \right) \cdot}{\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1+T_1n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d+T_dn^\alpha]} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n} \right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n} \right) \right)}$$

$$b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n} \right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n} \right) \right),$$

(ii)

$$(K_n(f))(x) = \tag{30}$$

$$\frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^{\overline{r}} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} f\left(t + \frac{k+\lambda_i}{n+\rho_i} \right) dt \right) b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)} =$$

$$\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1+T_1n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d+T_dn^\alpha]} \left(\sum_{i_1=1}^{r_1} \cdots \sum_{i_d=1}^{r_d} w_{i_1,...,i_d} (n + \rho_{i_1,...,i_d})^d \cdot \right.$$

$$\frac{\int \dots \int_0^{\frac{1}{n+\rho_{i_1,\dots,i_d}}} \dots \int f\left(t_1+\frac{k_1+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}},\dots,t_d+\frac{k_d+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}\right)dt_1\dots dt_d\Bigg)\cdot}{\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]}\dots\sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\dots,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)} \tag{31}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\dots,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right),$$

and

(iii)

$$(M_n(f))(x)=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{\bar{r}} w_i f\left(\frac{k}{n}+\frac{i}{n\bar{r}}\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}= \tag{32}$$

$$\frac{\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]}\dots\sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]}\left(\sum_{i_1=1}^{r_1}\dots\sum_{i_d=1}^{r_d} w_{i_1,\dots,i_d} f\left(\frac{k_1}{n}+\frac{i_1}{nr_1},\dots,\frac{k_d}{n}+\frac{i_d}{nr_d}\right)\right)\cdot}{\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]}\dots\sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\dots,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\dots,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right).$$

So if $\left|n^{1-\alpha}\left(x_j-\frac{k_j}{n}\right)\right|\leq T_j$, all $j=1,\dots,d$, we get that

$$\left\|x-\frac{k}{n}\right\|_{\infty}\leq\frac{T^*}{n^{1-\alpha}}. \tag{33}$$

For convinience we call

$$V(x)=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)=$$

$$\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]}\dots\sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\dots,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right). \tag{34}$$

We make

Remark 21. *(see [14]) Here always k is as in (26).*

I) We have that

$$\left\|\frac{k+\mu_i}{n+\nu_i}-x\right\|_{\infty}\leq\left(\frac{\|\nu_i\|_{\infty}\|x\|_{\infty}+\|\mu_i\|_{\infty}}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_{\infty}}{(n+\nu_i^{\min})}\right)\frac{T^*}{n^{1-\alpha}}. \tag{35}$$

Hence we derive

$$\omega_1\left(f,\left\|\frac{k+\mu_i}{n+\nu_i}-x\right\|_{\infty}\right)\leq$$

$$\omega_1\left(f,\left(\frac{\|\nu_i\|_{\infty}\|x\|_{\infty}+\|\mu_i\|_{\infty}}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_{\infty}}{(n+\nu_i^{\min})}\right)\frac{T^*}{n^{1-\alpha}}\right), \tag{36}$$

with dominant speed of convergence $\frac{1}{n^{1-\alpha}}$.

II) We also have for

$$0\leq t_j\leq\frac{1}{n+\rho_{i_1,\dots,i_d}},\quad j=1,\dots,d, \tag{37}$$

that it holds

$$\left\| t + \frac{k + \lambda_{i_1,...,i_d}}{n + \rho_{i_1,...,i_d}} - x \right\|_\infty \leq$$

$$\left(\frac{\rho_{i_1,...,i_d} \|x\|_\infty + \lambda_{i_1,...,i_d} + 1}{n + \rho_{i_1,...,i_d}} \right) + \left(1 + \frac{\rho_{i_1,...,i_d}}{n + \rho_{i_1,...,i_d}} \right) \frac{T^*}{n^{1-\alpha}}, \tag{38}$$

and

$$\omega_1 \left(f, \left\| t + \frac{k + \lambda_{i_1,...,i_d}}{n + \rho_{i_1,...,i_d}} - x \right\|_\infty \right) \leq$$

$$\omega_1 \left(f, \left(\frac{\rho_{i_1,...,i_d} \|x\|_\infty + \lambda_{i_1,...,i_d} + 1}{n + \rho_{i_1,...,i_d}} \right) + \left(1 + \frac{\rho_{i_1,...,i_d}}{n + \rho_{i_1,...,i_d}} \right) \frac{T^*}{n^{1-\alpha}} \right), \tag{39}$$

with dominant speed $\frac{1}{n^{1-\alpha}}$.

III) We also have

$$\left\| \frac{k}{n} + \frac{i}{n\overline{r}} - x \right\|_\infty \leq \frac{T^*}{n^{1-\alpha}} + \frac{1}{n}, \tag{40}$$

and

$$\omega_1 \left(f, \left\| \frac{k}{n} + \frac{i}{n\overline{r}} - x \right\|_\infty \right) \leq \omega_1 \left(f, \frac{T^*}{n^{1-\alpha}} + \frac{1}{n} \right), \tag{41}$$

with dominant speed $\frac{1}{n^{1-\alpha}}$.

Inequalities (35)-(41) were essential in the proofs of the next Theorems 22-27, proved and presented in [14].

So we mention from [14] to use in this article the following results.

Theorem 22. *Let* $x \in \mathbb{R}^d$ *and* $n \in \mathbb{N}$ *such that* $n \geq \max_{j \in \{1,...,d\}} \left(T_j + |x_j|, T_j^{-\frac{1}{\alpha}} \right)$, $T_j > 0$, $0 < \alpha < 1$; $f \in C_B(\mathbb{R}^d)$ *or* $f \in C_U(\mathbb{R}^d)$. *Then*

$$|(H_n(f))(x) - f(x)| \leq$$

$$\sum_{i=1}^{\overline{r}} w_i \omega_1 \left(f, \left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right) = \tag{42}$$

$$\sum_{i_1=1}^{r_1} ... \sum_{i_d=1}^{r_d} w_{i_1,...,i_d} \omega_1 \left(f, \left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right),$$

where $i = (i_1, ..., i_d)$.

Theorem 23. *All assumptions as in Theorem 22. Then*

$$|(K_n(f))(x) - f(x)| \leq$$

$$\sum_{i=1}^{\overline{r}} w_i \omega_1 \left(f, \left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right) = \tag{43}$$

$$\sum_{i_1=1}^{r_1} ... \sum_{i_d=1}^{r_d} w_{i_1,...,i_d} \omega_1 \left(f, \left(\frac{\rho_{i_1,...,i_d} \|x\|_\infty + \lambda_{i_1,...,i_d} + 1}{n + \rho_{i_1,...,i_d}} \right) + \left(1 + \frac{\rho_{i_1,...,i_d}}{n + \rho_{i_1,...,i_d}} \right) \frac{T^*}{n^{1-\alpha}} \right).$$

Theorem 24. *All here as in Theorem 22. Then*

$$|M_n(f)(x) - f(x)| \leq \omega_1 \left(f, \frac{T^*}{n^{1-\alpha}} + \frac{1}{n} \right). \tag{44}$$

All convergences in (42)-(44) are at the rate of $\frac{1}{n^{1-\alpha}}$, when $f \in C_U\left(\mathbb{R}^d\right)$.

Theorem 25. *Let $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$ such that $n \geq \max_{j \in \{1,\dots,d\}} \left(T_j + |x_j|, T_j^{-\frac{1}{\alpha}}\right)$, $T_j > 0$, $0 < \alpha < 1$. Let also $f \in C^N\left(\mathbb{R}^d\right)$, $N \in \mathbb{N}$, such that all of its partial derivatives $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha} : |\widetilde{\alpha}| = \sum_{j=1}^{d} \alpha_j = N$, are uniformly continuous or continuous and bounded. Then*

$$|(H_n(f))(x) - f(x)| \leq \sum_{l=1}^{N} \frac{1}{l!} \left(\left(\sum_{j=1}^{d} \left| \frac{\partial}{\partial x_j} \right| \right)^l f(x) \right) \cdot$$
$$\left[\sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}}\right) \frac{T^*}{n^{1-\alpha}} \right]^l \right] +$$
$$\frac{d^N}{N!} \sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}}\right) \frac{T^*}{n^{1-\alpha}} \right]^N \cdot$$
$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}}\right) \frac{T^*}{n^{1-\alpha}} \right] \right). \tag{45}$$

Inequality (45) implies the pointwise convergence with rates on $(H_n(f))(x)$to $f(x)$, as $n \to \infty$, at speed $\frac{1}{n^{1-\alpha}}$.

Theorem 26. *All here as in Theorem 25. Then*

$$|(K_n(f))(x) - f(x)| \leq$$
$$\sum_{l=1}^{N} \frac{1}{l!} \left(\sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i}\right) \frac{T^*}{n^{1-\alpha}} \right]^l \right) \cdot$$
$$\left(\left(\sum_{j=1}^{d} \left| \frac{\partial}{\partial x_j} \right| \right)^l f(x) \right) + \tag{46}$$
$$\frac{d^N}{N!} \sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i}\right) \frac{T^*}{n^{1-\alpha}} \right]^N \cdot$$
$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i}\right) \frac{T^*}{n^{1-\alpha}} \right] \right).$$

Inequality (46) implies the pointwise convergence with rates of $(K_n(f))(x)$ to $f(x)$, as $n \to \infty$, at speed $\frac{1}{n^{1-\alpha}}$.

Theorem 27. *All here as in Theorem 25. Then*

$$|(M_n(f))(x) - f(x)| \leq \sum_{l=1}^{N} \frac{1}{l!} \left(\left(\sum_{j=1}^{d} \left| \frac{\partial}{\partial x_j} \right| \right)^l f(x) \right) \cdot$$
$$\left(\frac{T^*}{n^{1-\alpha}} + \frac{1}{n} \right)^l + \frac{d^N}{N!} \left(\frac{T^*}{n^{1-\alpha}} + \frac{1}{n} \right)^N \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} + \frac{1}{n} \right). \tag{47}$$

Inequality (47) implies the pointwise convergence with rates on $(M_n(f))(x)$to $f(x)$, as $n \to \infty$, at speed $\frac{1}{n^{1-\alpha}}$.

In this article we extend Theorems 22-27 to the fuzzy setting and environment. We give also important special cases applications of these fuzzy main results.

4. Fuzzy Multivariate Neural Network Approximations

Here we consider $f \in C_{\mathcal{F}}^{U}\left(\mathbb{R}^{d}\right)$ or $f \in C_{B}\left(\mathbb{R}^{d}, \mathbb{R}_{\mathcal{F}}\right)$, b is as in Section 3, $0<\alpha<1$, also the rest of the parameters are as in section 3. For $x \in \mathbb{R}^{d}$, we take always that $n \geq \max_{j \in\{1, \ldots, d\}}\left(T_{j}+\left|x_{j}\right|, T_{j}^{-\frac{1}{\alpha}}\right)$, see (28). The fuzzy analogs of the operators H_{n}, K_{n}, M_{n}, see (29), (30) together with (31), and (32), respectivelly, follow, $n \in \mathbb{N}$.

We define the corresponding fuzzy multivariate neural network operators next:

(i)

$$\left(H_{n}^{\mathcal{F}}(f)\right)(x)=\left(H_{n}^{\mathcal{F}}(f)\right)\left(x_{1}, \ldots, x_{d}\right)=$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]^{*}}\left(\sum_{i=1}^{\bar{r}*} w_{i} \odot f\left(\frac{k+\mu_{i}}{n+\nu_{i}}\right)\right) \odot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}= \tag{48}$$

$$\frac{\sum_{k_{1}=\lceil nx_{1}-T_{1}n^{\alpha}\rceil}^{[nx_{1}+T_{1}n^{\alpha}]} \ldots \sum_{k_{d}=\lceil nx_{d}-T_{d}n^{\alpha}\rceil}^{[nx_{d}+T_{d}n^{\alpha}]^{*}}\left(\sum_{i_{1}=1}^{r_{1}} \ldots \sum_{i_{d}=1}^{r_{d}*} w_{i_{1}, \ldots, i_{d}} \odot f\left(\frac{k_{1}+\mu_{i_{1}}}{n+\nu_{i_{1}}}, \ldots, \frac{k_{d}+\mu_{i_{d}}}{n+\nu_{i_{d}}}\right)\right) \odot}{\sum_{k_{1}=\lceil nx_{1}-T_{1}n^{\alpha}\rceil}^{[nx_{1}+T_{1}n^{\alpha}]} \ldots \sum_{k_{d}=\lceil nx_{d}-T_{d}n^{\alpha}\rceil}^{[nx_{d}+T_{d}n^{\alpha}]} b\left(n^{1-\alpha}\left(x_{1}-\frac{k_{1}}{n}\right), \ldots, n^{1-\alpha}\left(x_{d}-\frac{k_{d}}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_{1}-\frac{k_{1}}{n}\right), \ldots, n^{1-\alpha}\left(x_{d}-\frac{k_{d}}{n}\right)\right),$$

(ii)

$$\left(K_{n}^{\mathcal{F}}(f)\right)(x)=$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]^{*}}\left(\sum_{i=1}^{\bar{r}*} w_{i}\left(n+\rho_{i}\right)^{d} \odot \int_{0}^{\frac{1}{n+\rho_{i}}} f\left(t+\frac{k+\lambda_{i}}{n+\rho_{i}}\right) dt\right) \odot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}=$$

$$\sum_{k_{1}=\lceil nx_{1}-T_{1}n^{\alpha}\rceil}^{[nx_{1}+T_{1}n^{\alpha}]} \ldots \sum_{k_{d}=\lceil nx_{d}-T_{d}n^{\alpha}\rceil}^{[nx_{d}+T_{d}n^{\alpha}]*}\left(\sum_{i_{1}=1}^{r_{1}} \ldots \sum_{i_{d}=1}^{r_{d}*} w_{i_{1}, \ldots, i_{d}}\left(n+\rho_{i_{1}, \ldots, i_{d}}\right)^{d} \odot\right. \tag{49}$$

$$\frac{\left.\int \ldots \int_{0}^{\frac{1}{n+\rho_{i_{1}, \ldots, i_{d}}}} \ldots \int f\left(t_{1}+\frac{k_{1}+\lambda_{i_{1}, \ldots, i_{d}}}{n+\rho_{i_{1}, \ldots, i_{d}}}, \ldots, t_{d}+\frac{k_{d}+\lambda_{i_{1}, \ldots, i_{d}}}{n+\rho_{i_{1}, \ldots, i_{d}}}\right) dt_{1} \ldots dt_{d}\right) \odot}{\sum_{k_{1}=\lceil nx_{1}-T_{1}n^{\alpha}\rceil}^{[nx_{1}+T_{1}n^{\alpha}]} \ldots \sum_{k_{d}=\lceil nx_{d}-T_{d}n^{\alpha}\rceil}^{[nx_{d}+T_{d}n^{\alpha}]} b\left(n^{1-\alpha}\left(x_{1}-\frac{k_{1}}{n}\right), \ldots, n^{1-\alpha}\left(x_{d}-\frac{k_{d}}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_{1}-\frac{k_{1}}{n}\right), \ldots, n^{1-\alpha}\left(x_{d}-\frac{k_{d}}{n}\right)\right),$$

and

(iii)

$$\left(M_{n}^{\mathcal{F}}(f)\right)(x)=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]^{*}}\left(\sum_{i=1}^{\bar{r}*} w_{i} \odot f\left(\frac{k}{n}+\frac{i}{n\bar{r}}\right)\right) \odot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}= \tag{50}$$

$$\frac{\sum_{k_{1}=\lceil nx_{1}-T_{1}n^{\alpha}\rceil}^{[nx_{1}+T_{1}n^{\alpha}]} \ldots \sum_{k_{d}=\lceil nx_{d}-T_{d}n^{\alpha}\rceil}^{[nx_{d}+T_{d}n^{\alpha}]^{*}}\left(\sum_{i_{1}=1}^{r_{1}} \ldots \sum_{i_{d}=1}^{r_{d}*} w_{i_{1}, \ldots, i_{d}} \odot f\left(\frac{k_{1}}{n}+\frac{i_{1}}{nr_{1}}, \ldots, \frac{k_{d}}{n}+\frac{i_{d}}{nr_{d}}\right)\right) \odot}{\sum_{k_{1}=\lceil nx_{1}-T_{1}n^{\alpha}\rceil}^{[nx_{1}+T_{1}n^{\alpha}]} \ldots \sum_{k_{d}=\lceil nx_{d}-T_{d}n^{\alpha}\rceil}^{[nx_{d}+T_{d}n^{\alpha}]} b\left(n^{1-\alpha}\left(x_{1}-\frac{k_{1}}{n}\right), \ldots, n^{1-\alpha}\left(x_{d}-\frac{k_{d}}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_{1}-\frac{k_{1}}{n}\right), \ldots, n^{1-\alpha}\left(x_{d}-\frac{k_{d}}{n}\right)\right).$$

We notice that ($r \in [0,1]$)

$$\left[\left(H_n^{\mathcal{F}}(f)\right)(x)\right]^r = \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{\overline{r}} w_i \left[f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right]^r\right) \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}$$

$$= \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{\overline{r}} w_i \left[f_-^{(r)}\left(\frac{k+\mu_i}{n+\nu_i}\right), f_+^{(r)}\left(\frac{k+\mu_i}{n+\nu_i}\right)\right]\right) \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}$$

$$= \left[\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{\overline{r}} w_i f_-^{(r)}\left(\frac{k+\mu_i}{n+\nu_i}\right)\right) \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)},\right. \tag{51}$$

$$\left.\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{\overline{r}} w_i f_+^{(r)}\left(\frac{k+\mu_i}{n+\nu_i}\right)\right) \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\right]$$

$$= \left[\left(H_n\left(f_-^{(r)}\right)\right)(x), \left(H_n\left(f_+^{(r)}\right)\right)(x)\right].$$

We have proved that

$$\left(H_n^{\mathcal{F}}(f)\right)_{\pm}^{(r)} = H_n\left(f_{\pm}^{(r)}\right), \quad \forall\, r \in [0,1], \tag{52}$$

respectively.

For convinience also we call

$$A(x) = \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}. \tag{53}$$

We observe that

$$\left[\left(K_n^{\mathcal{F}}(f)\right)(x)\right]^r =$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{\overline{r}} w_i (n+\rho_i)^d \left[\int_0^{\frac{1}{n+\rho_i}} f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right) dt\right]^r\right) A(x) \overset{(15)}{=}$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{\overline{r}} w_i (n+\rho_i)^d \left[\int_0^{\frac{1}{n+\rho_i}} f_-^{(r)}\left(t+\frac{k+\lambda_i}{n+\rho_i}\right) dt,\right.\right.$$

$$\left.\left.\int_0^{\frac{1}{n+\rho_i}} f_+^{(r)}\left(t+\frac{k+\lambda_i}{n+\rho_i}\right) dt\right]\right) A(x) = \tag{54}$$

$$\left[\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{\overline{r}} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} f_-^{(r)}\left(t+\frac{k+\lambda_i}{n+\rho_i}\right) dt\right) A(x),\right.$$

$$\left.\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{\overline{r}} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} f_+^{(r)}\left(t+\frac{k+\lambda_i}{n+\rho_i}\right) dt\right) A(x)\right]$$

$$= \left[\left(K_n\left(f_-^{(r)}\right)\right)(x), \left(K_n\left(f_+^{(r)}\right)\right)(x)\right].$$

We have proved that

$$\left(K_n^{\mathcal{F}}(f)\right)_{\pm}^{(r)} = K_n\left(f_{\pm}^{(r)}\right), \quad \forall\, r \in [0,1], \tag{55}$$

respectively.

Next we observe that

$$\left[\left(M_n^{\mathcal{F}}(f)\right)(x)\right]^r = \sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^{\overline{r}} w_i \left[f\left(\frac{k}{n}+\frac{i}{n\overline{r}}\right)\right]^r\right) A(x) =$$

$$\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^{\overline{r}} w_i \left[f_-^{(r)}\left(\frac{k}{n}+\frac{i}{n\overline{r}}\right), f_+^{(r)}\left(\frac{k}{n}+\frac{i}{n\overline{r}}\right)\right]\right) A(x) = \tag{56}$$

$$\left[\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^{\overline{r}} w_i f_-^{(r)}\left(\frac{k}{n}+\frac{i}{n\overline{r}}\right)\right) A(x),\right.$$

$$\left.\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^{\overline{r}} w_i f_+^{(r)}\left(\frac{k}{n}+\frac{i}{n\overline{r}}\right)\right) A(x)\right]$$

$$= \left[\left(M_n\left(f_-^{(r)}\right)\right)(x), \left(M_n\left(f_+^{(r)}\right)\right)(x)\right].$$

That is proving

$$\left(M_n^{\mathcal{F}}(f)\right)_\pm^{(r)} = M_n\left(f_\pm^{(r)}\right), \quad \forall\, r\in[0,1], \tag{57}$$

respectively.

It follows are main results.

We present

Theorem 28. *All as at the begining of Section 4. It holds*

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \le$$

$$\sum_{i=1}^{\overline{r}} w_i \omega_1^{(\mathcal{F})}\left(f, \left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right) = \tag{58}$$

$$\sum_{i_1=1}^{r_1} \cdots \sum_{i_d=1}^{r_d} w_{i_1,\ldots,i_d} \omega_1^{(\mathcal{F})}\left(f, \left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right).$$

Proof. We observe that

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) =$$

$$\sup_{r\in[0,1]} \max\left\{\left|\left(H_n^{\mathcal{F}}(f)\right)_-^{(r)}(x) - f_-^{(r)}(x)\right|, \left|\left(H_n^{\mathcal{F}}(f)\right)_+^{(r)}(x) - f_+^{(r)}(x)\right|\right\} \overset{(52)}{=} \tag{59}$$

$$\sup_{r\in[0,1]} \max\left\{\left|\left(H_n\left(f_-^{(r)}\right)\right)(x) - f_-^{(r)}(x)\right|, \left|\left(H_n\left(f_+^{(r)}\right)\right)(x) - f_+^{(r)}(x)\right|\right\} \overset{(42)}{\le}$$

$$\sup_{r\in[0,1]} \max\left\{\sum_{i=1}^{\overline{r}} w_i \omega_1\left(f_-^{(r)}, \left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right),\right.$$

$$\left.\sum_{i=1}^{\overline{r}} w_i \omega_1\left(f_+^{(r)}, \left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right)\right\} \overset{(3)}{\le}$$

$$\sum_{i=1}^{\overline{r}} w_i \sup_{r\in[0,1]} \max\left\{\omega_1\left(f_-^{(r)}, \left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right),\right. \tag{60}$$

$$\omega_1\left(f_+^{(r)},\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right)\Bigg\}\overset{(7)}{=}$$

$$\sum_{i=1}^{\overline{r}}w_i\omega_1^{(\mathcal{F})}\left(f,\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right), \tag{61}$$

proving the claim. □

Theorem 29. *All as at the beginning of Section 4. It holds*

$$D\left(\left(K_n^{\mathcal{F}}(f)\right)(x),f(x)\right)\leq$$

$$\sum_{i=1}^{\overline{r}}w_i\omega_1^{(\mathcal{F})}\left(f,\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right)= \tag{62}$$

$$\sum_{i_1=1}^{r_1}\ldots\sum_{i_d=1}^{r_d}w_{i_1,\ldots,i_d}\omega_1^{(\mathcal{F})}\left(f,\left(\frac{\rho_{i_1,\ldots,i_d}\|x\|_\infty+\lambda_{i_1,\ldots,i_d}+1}{n+\rho_{i_1,\ldots,i_d}}\right)+\left(1+\frac{\rho_{i_1,\ldots,i_d}}{n+\rho_{i_1,\ldots,i_d}}\right)\frac{T^*}{n^{1-\alpha}}\right).$$

Proof. Using (43), (55), similar to Theorem 28. □

Theorem 30. *All as in Theorem 28. It holds*

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)(x),f(x)\right)\leq\omega_1^{(\mathcal{F})}\left(f,\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right). \tag{63}$$

Proof. Using (44), (57), similar to Theorem 28. □

Remark 31. *When $f\in C_{\mathcal{F}}^U\left(\mathbb{R}^d\right)$, as $n\to\infty$, from Theorems 28-30 we obtain the pointwise convergence with rates of $\left(H_n^{\mathcal{F}}(f)\right)(x)\overset{D}{\to}f(x)$, $\left(K_n^{\mathcal{F}}(f)\right)(x)\overset{D}{\to}f(x)$ and $\left(M_n^{\mathcal{F}}(f)\right)(x)\overset{D}{\to}f(x)$, at the speed of $\frac{1}{n^{1-\alpha}}$.*

In the next three corollaries we take $x\in\prod_{j=1}^{d}\left[-\gamma_j,\gamma_j\right]\subset\mathbb{R}^d$, $\gamma_j>0,\gamma^*=\max\left\{\gamma_1,\ldots,\gamma_d\right\}$ and $n\in\mathbb{N}$ such that $n\geq\max_{j\in\{1,\ldots,d\}}\left\{T_j+\gamma_j,T_j^{-\frac{1}{\alpha}}\right\}$.

We derive the following

Corollary 32. *(to Theorem 28) It holds*

$$D^*\left(H_n^{\mathcal{F}}(f),f\right)_{\prod_{j=1}^{d}[-\gamma_j,\gamma_j]}\leq$$

$$\sum_{i=1}^{\overline{r}}w_i\omega_1^{(\mathcal{F})}\left(f,\left(\frac{\|\nu_i\|_\infty\gamma^*+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right)= \tag{64}$$

$$\sum_{i_1=1}^{r_1}\ldots\sum_{i_d=1}^{r_d}w_{i_1,\ldots,i_d}\omega_1^{(\mathcal{F})}\left(f,\left(\frac{\|\nu_i\|_\infty\gamma^*+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right).$$

Proof. By (58). □

Corollary 33. *(to Theorem 29) It holds*

$$D^*\left(K_n^{\mathcal{F}}(f),f\right)_{\prod_{j=1}^{d}[-\gamma_j,\gamma_j]}\leq$$

$$\sum_{i=1}^{\overline{r}}w_i\omega_1^{(\mathcal{F})}\left(f,\left(\frac{\rho_i\gamma^*+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right)= \tag{65}$$

$$\sum_{i_1=1}^{r_1}\ldots\sum_{i_d=1}^{r_d}w_{i_1,\ldots,i_d}\omega_1^{(\mathcal{F})}\left(f,\left(\frac{\rho_{i_1,\ldots,i_d}\gamma^*+\lambda_{i_1,\ldots,i_d}+1}{n+\rho_{i_1,\ldots,i_d}}\right)+\left(1+\frac{\rho_{i_1,\ldots,i_d}}{n+\rho_{i_1,\ldots,i_d}}\right)\frac{T^*}{n^{1-\alpha}}\right).$$

Proof. By (62). □

Corollary 34. *(to Theorem 30) It holds*

$$D^*\left(M_n^{\mathcal{F}}(f), f\right)_{\prod_{j=1}^{d}[-\gamma_j,\gamma_j]} \leq \omega_1^{(\mathcal{F})}\left(f, \frac{T^*}{n^{1-\alpha}} + \frac{1}{n}\right). \tag{66}$$

Proof. By (63). □

Remark 35. *When $f \in C_{\mathcal{F}}^U(\mathbb{R}^d)$, as $n \to \infty$, from Corollaries 32-34 we obtain the uniform convergence with rates of $H_n^{\mathcal{F}}(f) \xrightarrow{D^*} f$, $K_n^{\mathcal{F}}(f) \xrightarrow{D^*} f$ and $M_n^{\mathcal{F}}(f) \xrightarrow{D^*} f$, at the speed of $\frac{1}{n^{1-\alpha}}$.*

Next we present higher order of fuzzy approximation results based on the high order fuzzy differentiability of the approximated function.

Theorem 36. *Let $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$ such that $n \geq \max_{j\in\{1,\dots,d\}}\left(T_j + |x_j|, T_j^{-\frac{1}{\alpha}}\right)$, $T_j > 0$, $0 < \alpha < 1$. Let also $f \in C_{\mathcal{F}}^N(\mathbb{R}^d)$, $N \in \mathbb{N}$, such that all of its fuzzy partial derivatives $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha} : |\widetilde{\alpha}| = \sum_{j=1}^{d}\alpha_j = N$, are fuzzy uniformly continuous or fuzzy continuous and bounded. Then*

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$
$$\sum_{i=1}^{N}\frac{1}{l!}\left[\sum_{i=1}^{\bar{r}} w_i\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1 + \frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^l\right]$$
$$\cdot\left[\left(\sum_{j=1}^{d} D\left(\frac{\partial}{\partial x_j}, \widetilde{o}\right)\right)^l f(x)\right] +$$
$$\frac{d^N}{N!}\sum_{i=1}^{\bar{r}} w_i\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1 + \frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^N \cdot \tag{67}$$
$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1 + \frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]\right).$$

Proof. We observe that

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) =$$
$$\sup_{r\in[0,1]}\max\left\{\left|\left(H_n^{\mathcal{F}}(f)\right)_-^{(r)}(x) - f_-^{(r)}(x)\right|, \left|\left(H_n^{\mathcal{F}}(f)\right)_+^{(r)}(x) - f_+^{(r)}(x)\right|\right\} \overset{(52)}{=}$$
$$\sup_{r\in[0,1]}\max\left\{\left|\left(H_n\left(f_-^{(r)}\right)\right)(x) - f_-^{(r)}(x)\right|, \left|\left(H_n\left(f_+^{(r)}\right)\right)(x) - f_+^{(r)}(x)\right|\right\} \overset{(45)}{\leq}$$
$$\sup_{r\in[0,1]}\max\left\{\sum_{l=1}^{N}\frac{1}{l!}\left(\left(\sum_{j=1}^{d}\left|\frac{\partial}{\partial x_j}\right|\right)^l f_-^{(r)}(x)\right)\cdot\right. \tag{68}$$
$$\left[\sum_{i=1}^{\bar{r}} w_i\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1 + \frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^l\right] +$$
$$\frac{d^N}{N!}\sum_{i=1}^{\bar{r}} w_i\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1 + \frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^N \cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(\left(f_-^{(r)}\right)_{\widetilde{\alpha}},\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]\right),$$

$$\sum_{l=1}^N\frac{1}{l!}\left(\left(\sum_{j=1}^d\left|\frac{\partial}{\partial x_j}\right|\right)^l f_+^{(r)}(x)\right)\cdot$$

$$\left[\sum_{i=1}^{\overline{r}}w_i\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^l\right]+$$

$$\frac{d^N}{N!}\sum_{i=1}^{\overline{r}}w_i\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^N\cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(\left(f_+^{(r)}\right)_{\widetilde{\alpha}},\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]\right)\Bigg\}$$

$$\overset{(12)}{=}\sup_{r\in[0,1]}\max\left\{\sum_{l=1}^N\frac{1}{l!}\left(\left(\sum_{j=1}^d\left|\frac{\partial}{\partial x_j}\right|\right)^l f(x)\right)_-^{(r)}\cdot\right.$$

$$\left[\sum_{i=1}^{\overline{r}}w_i\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^l\right]+$$

$$\frac{d^N}{N!}\sum_{i=1}^{\overline{r}}w_i\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^N\cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(\left(f_{\widetilde{\alpha}}\right)_-^{(r)},\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]\right),$$

$$\sum_{l=1}^N\frac{1}{l!}\left(\left(\sum_{j=1}^d\left|\frac{\partial}{\partial x_j}\right|\right)^l f(x)\right)_+^{(r)}\cdot \tag{69}$$

$$\left[\sum_{i=1}^{\overline{r}}w_i\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^l\right]+$$

$$\frac{d^N}{N!}\sum_{i=1}^{\overline{r}}w_i\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^N\cdot$$

$$\left.\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(\left(f_{\widetilde{\alpha}}\right)_+^{(r)},\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]\right)\right\}$$

$$\overset{(3)}{\leq}\sum_{l=1}^N\frac{1}{l!}\left[\sum_{i=1}^{\overline{r}}w_i\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^l\right]$$

$$\cdot\sup_{r\in[0,1]}\max\left\{\left(\left(\sum_{j=1}^d\left|\frac{\partial}{\partial x_j}\right|\right)^l f(x)\right)_-^{(r)},\left(\left(\sum_{j=1}^d\left|\frac{\partial}{\partial x_j}\right|\right)^l f(x)\right)_+^{(r)}\right\}+$$

$$\frac{d^N}{N!}\sum_{i=1}^{\overline{r}}w_i\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^N\cdot \tag{70}$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \sup_{r\in[0,1]} \max\left\{\omega_1\left(\left(f_{\widetilde{\alpha}}\right)_-^{(r)}, \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]\right),\right.$$

$$\left.\omega_1\left(\left(f_{\widetilde{\alpha}}\right)_+^{(r)}, \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]\right)\right\}$$

$$\overset{\text{(by (3), (7), (18))}}{\leq} \sum_{l=1}^{N}\frac{1}{l!}\left[\sum_{i=1}^{\bar{r}} w_i \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^l\right]$$

$$\cdot\left[\left(\sum_{j=1}^{d} D\left(\frac{\partial}{\partial x_j},\widetilde{o}\right)\right)^l f(x)\right] +$$

$$\frac{d^N}{N!}\sum_{i=1}^{\bar{r}} w_i \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^N \cdot \tag{71}$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]\right).$$

The theorem is proved. □

Similarly we find

Theorem 37. *All as in Theorem 36. Then*

$$D\left(\left(K_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\sum_{i=1}^{N}\frac{1}{l!}\left[\sum_{i=1}^{\bar{r}} w_i\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^l\right]$$

$$\cdot\left[\left(\sum_{j=1}^{d} D\left(\frac{\partial}{\partial x_j},\widetilde{o}\right)\right)^l f(x)\right] +$$

$$\frac{d^N}{N!}\sum_{i=1}^{\bar{r}} w_i\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^N \cdot \tag{72}$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]\right).$$

Proof. Similar to Theorem 36, using (46), (55), etc. □

We continue with

Theorem 38. *All as in Theorem 36. Then*

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\sum_{i=1}^{N}\frac{1}{l!}\left(\left(\sum_{j=1}^{d} D\left(\frac{\partial}{\partial x_j},\widetilde{o}\right)\right)^l f(x)\right)\left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right)^l +$$

$$\frac{d^N}{N!}\left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right)^N \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right). \tag{73}$$

Proof. Similar to Theorem 36, using (47), (57), etc. □

Note 39. *Inequalities (67), (72) and (73) imply the pointwise convergence with rates of* $\left(H_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, $\left(K_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$ *and* $\left(M_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, *as* $n \to \infty$, *at speed* $\frac{1}{n^{1-\alpha}}$.

Note 40. *In the next three corolaries additionally we assume that*

$$f_{\widetilde{\alpha}}(x) = \widetilde{o}, \text{ for all } \widetilde{\alpha} : |\widetilde{\alpha}| = 1, ..., N, \tag{74}$$

for $x \in \mathbb{R}^d$ *fixed.*

The last implies $D\left(f_{\widetilde{\alpha}}(x), \widetilde{o}\right) = 0$, *and by (18) we obtain*

$$\left[\left(\sum_{j=1}^{d} D\left(\frac{\partial}{\partial x_j}, \widetilde{o}\right)\right)^l f(x)\right] = 0, \tag{75}$$

for $l = 1, ..., N$.

So we derive the following special results.

Corollary 41. *Let* $x \in \mathbb{R}^d$ *and* $n \in \mathbb{N}$ *such that* $n \geq \max_{j \in \{1,...,d\}} \left(T_j + |x_j|, T_j^{-\frac{1}{\alpha}}\right)$, $T_j > 0$, $0 < \alpha < 1$. *Let also* $f \in C_{\mathcal{F}}^N\left(\mathbb{R}^d\right)$, $N \in \mathbb{N}$, *such that all of its fuzzy partial derivatives* $f_{\widetilde{\alpha}}$ *of order* N, $\widetilde{\alpha} : |\widetilde{\alpha}| = \sum_{j=1}^{d} \alpha_j = N$, *are fuzzy uniformly continuous or fuzzy continuous and bounded. Assume also (74). Then*

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\frac{d^N}{N!} \sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}}\right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}}\right) \frac{T^*}{n^{1-\alpha}}\right]^N \cdot \tag{76}$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}}\right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}}\right) \frac{T^*}{n^{1-\alpha}}\right]\right).$$

Proof. By (67) and (75). □

Corollary 42. *All as in Corollary 41. Then*

$$D\left(\left(K_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\frac{d^N}{N!} \sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n + \rho_i}\right) + \left(1 + \frac{\rho_i}{n + \rho_i}\right) \frac{T^*}{n^{1-\alpha}}\right]^N \cdot \tag{77}$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n + \rho_i}\right) + \left(1 + \frac{\rho_i}{n + \rho_i}\right) \frac{T^*}{n^{1-\alpha}}\right]\right).$$

Proof. By (72) and (75). □

Corollary 43. *All as in Corollary 41. Then*

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\frac{d^N}{N!} \left(\frac{T^*}{n^{1-\alpha}} + \frac{1}{n}\right)^N \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1^{(\mathcal{F})}\left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} + \frac{1}{n}\right). \tag{78}$$

Proof. By (73) and (75). □

By (76), (77), (78) we get fuzzy pointwise convergence with rates at high speed $\frac{1}{n^{(1-\alpha)(N+1)}}$.
We need

Lemma 44. *([7], p. 131) Let K be a compact subset of the real normed vector space $(V, \|\cdot\|)$ and $f \in C_{\mathcal{F}}(K)$ (space of continuous fuzzy real number valued functions on K). Then f is a fuzzy bounded function.*

Assumption 45. *In the next three corollaries we consider $x \in G = \prod_{j=1}^{d} [-\gamma_j, \gamma_j] \subset \mathbb{R}^d$, $\gamma_j > 0, \gamma^* = \max\{\gamma_1, ..., \gamma_d\}$ and $n \in \mathbb{N} : n \geq \max_{j \in \{1,...,d\}} \left(T_j + \gamma_j, T_j^{-\frac{1}{\alpha}} \right)$, $T_j > 0$, $0 < \alpha < 1$. Let also $f \in C_{\mathcal{F}}^N(\mathbb{R}^d)$, $N \in \mathbb{N}$, such that all of its fuzzy partials $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha} : |\widetilde{\alpha}| = N$, are fuzzy uniformly continuous or fuzzy continuous and bounded.*

Using Lemma 44, Assumption 45 along with (18) and subadditivity of $\|\cdot\|_\infty$, we clearly obtain that

$$\left\| \left(\sum_{j=1}^{d} D\left(\frac{\partial}{\partial x_j}, \widetilde{o} \right) \right)^l f(x) \right\|_{\infty, G} < \infty, \tag{79}$$

for all $l = 1, ..., N$.
We define

$$D^*(f, g)|_G = \sup_{x \in G} D(f(x), g(x)),$$

where $f, g : G \to \mathbb{R}_{\mathcal{F}}$.
We give

Corollary 46. *We suppose Assumption 45. Then*

$$D^*\left(H_n^{\mathcal{F}}(f), f\right)|_G \leq$$

$$\sum_{l=1}^{N} \frac{1}{l!} \left[\sum_{i=1}^{\bar{r}} w_i \left[\left(\frac{\|\nu_i\|_\infty \gamma^* + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right]^l \right]$$

$$\cdot \left\| \left(\sum_{j=1}^{d} D\left(\frac{\partial}{\partial x_j}, \widetilde{o} \right) \right)^l f(x) \right\|_{\infty, G} + \tag{80}$$

$$\frac{d^N}{N!} \sum_{i=1}^{\bar{r}} w_i \left[\left(\frac{\|\nu_i\|_\infty \gamma^* + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right]^N \cdot$$

$$\max_{\widetilde{\alpha} : |\widetilde{\alpha}| = N} \omega_1^{(\mathcal{F})} \left(f_{\widetilde{\alpha}}, \left[\left(\frac{\|\nu_i\|_\infty \gamma^* + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right] \right).$$

Proof. By (67) and (79). □

Corollary 47. *We suppose Assumption 45. Then*

$$D^*\left(K_n^{\mathcal{F}}(f), f\right)|_G \leq$$

$$\sum_{l=1}^{N} \frac{1}{l!} \left[\sum_{i=1}^{\bar{r}} w_i \left[\left(\frac{\rho_i \gamma^* + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right]^l \right]$$

$$\cdot \left\| \left(\sum_{j=1}^{d} D\left(\frac{\partial}{\partial x_j}, \widetilde{o} \right) \right)^{l} f(x) \right\|_{\infty, G} + \tag{81}$$

$$\frac{d^N}{N!} \sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\rho_i \gamma^* + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right]^N \cdot$$

$$\max_{\widetilde{\alpha}: |\widetilde{\alpha}| = N} \omega_1^{(\mathcal{F})} \left(f_{\widetilde{\alpha}}, \left[\left(\frac{\rho_i \gamma^* + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right] \right).$$

Proof. By (72) and (79). □

Corollary 48. *We suppose Assumption 45. Then*

$$D^* \left(M_n^{\mathcal{F}}(f), f \right) |_G \leq$$

$$\sum_{i=1}^{N} \frac{1}{l!} \left\| \left(\sum_{j=1}^{d} D\left(\frac{\partial}{\partial x_j}, \widetilde{o} \right) \right)^{l} f(x) \right\|_{\infty, G} \left(\frac{T^*}{n^{1-\alpha}} + \frac{1}{n} \right)^{l} +$$

$$\frac{d^N}{N!} \left(\frac{T^*}{n^{1-\alpha}} + \frac{1}{n} \right)^N \max_{\widetilde{\alpha}: |\widetilde{\alpha}| = N} \omega_1^{(\mathcal{F})} \left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} + \frac{1}{n} \right). \tag{82}$$

Proof. By (73) and (79). □

Note 49. *Inequalities (80), (81), (82) imply the uniform convergence on G with rates of $H_n^{\mathcal{F}}(f) \overset{D^*}{\to} f$, $K_n^{\mathcal{F}}(f) \overset{D^*}{\to} f$ and $M_n^{\mathcal{F}}(f) \overset{D^*}{\to} f$, as $n \to \infty$, at speed $\frac{1}{n^{1-\alpha}}$.*

We continue with

Corollary 50. *(to Theorem 36) Case of $N = 1$. It holds*

$$D\left(\left(H_n^{\mathcal{F}}(f) \right)(x), f(x) \right) \leq$$

$$\left[\sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right] \right] \tag{83}$$

$$\cdot \left(\sum_{j=1}^{d} D\left(\frac{\partial f}{\partial x_j}, \widetilde{o} \right) \right) +$$

$$d \left\{ \sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right] \cdot \right.$$

$$\left. \max_{j \in \{1, \ldots, d\}} \omega_1^{(\mathcal{F})} \left(\frac{\partial f}{\partial x_j}, \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right] \right) \right\}.$$

Proof. By (67). □

Corollary 51. *(to Theorem 37) Case of $N = 1$. Then*

$$D\left(\left(K_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\left(\sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n+\rho_i}\right) + \left(1 + \frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]\right) \tag{84}$$

$$\cdot \left(\sum_{j=1}^{d} D\left(\frac{\partial f}{\partial x_j}, \widetilde{o}\right)\right) +$$

$$d\left\{\sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n+\rho_i}\right) + \left(1 + \frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right] \cdot\right.$$

$$\left.\max_{j\in\{1,\ldots,d\}} \omega_1^{(\mathcal{F})}\left(\frac{\partial f}{\partial x_j}, \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n+\rho_i}\right) + \left(1 + \frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]\right)\right\}.$$

Proof. By (72). □

We finish with

Corollary 52. *(to Theorem 38) Case $N = 1$. Then*

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq \tag{85}$$

$$\left[\sum_{j=1}^{d} D\left(\frac{\partial f}{\partial x_j}, \widetilde{o}\right) + d \max_{j\in\{1,\ldots,d\}} \omega_1^{(\mathcal{F})}\left(\frac{\partial f}{\partial x_j}, \frac{T^*}{n^{1-\alpha}} + \frac{1}{n}\right)\right]\left(\frac{T^*}{n^{1-\alpha}} + \frac{1}{n}\right).$$

Proof. By (73). □

Note 53. *Inequalities (83), (84) and (85) imply the pointwise convergence with rates of $H\left(_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, $\left(K_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$ and $\left(M_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, as $n \to \infty$, at speed $\frac{1}{n^{1-\alpha}}$.*

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