Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 7 (2014), 379–382
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Asymptotic behavior of solutions of a rational system of difference equations

Miron B. Bekker[a], Martin J. Bohner[b,*], Hristo D. Voulov[c]

[a] *University of Pittsburgh at Johnstown, Department of Mathematics, Johnstown, PA, USA.*
[b] *Missouri S&T, Department of Mathematics and Statistics, Rolla, MO, USA.*
[c] *University of Missouri–Kansas City, Department of Mathematics and Statistics, Kansas City, MO, USA.*

Communicated by R. Saadati

Special Issue In Honor of Professor Ravi P. Agarwal

Abstract

We consider a two-dimensional autonomous system of rational difference equations with three positive parameters. It was conjectured by Ladas that every positive solution of this system converges to a finite limit. Here we confirm this conjecture.

Keywords: Systems of rational difference equations, global attractors.
2010 MSC: 39A10, 39A20.

1. Introduction and Preliminaries

Rational systems of first order difference equations in the plane have been studied for a long time. Recently, in [3, 4, 5] (see the references therein), efforts have been made for a more systematic approach. In particular, the rational system

$$x_{n+1} = \frac{\alpha_1 + y_n}{x_n}, \qquad y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \tag{1.1}$$

with nonnegative coefficients and initial conditions was studied in [5]. Along with the results published in [5], there were also posed several conjectures about some nontrivial cases. Our goal here is to confirm one of them, namely for the case when $\alpha_1 = \alpha_2 = \beta_2 = 0$.

*Corresponding author
Email addresses: bekkerm@umr.edu (Miron B. Bekker), bohner@mst.edu (Martin J. Bohner), voulovh@umkc.edu (Hristo D. Voulov)

Received 2014-10-1

Conjecture 1.0. (see [5, Conjecture 2.4, page 1223]) *Let $a, b, c > 0$. Then every positive solution of the system*

$$x_{n+1} = \frac{y_n}{x_n}, \quad y_{n+1} = \frac{cy_n}{a + 2bx_n + y_n}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

converges to a finite limit.

By utilizing the relation

$$y_n = x_n x_{n+1}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

it is easy to see that the x-component of any solution $\{(x_n, y_n)\}_{n\in\mathbb{N}_0}$ of (1.2) must satisfy the difference equation

$$x_{n+2} = \frac{cx_n}{a + 2bx_n + x_n x_{n+1}} = f(x_{n+1}, x_n), \quad n \in \mathbb{N}_0, \tag{1.4}$$

where the function f is decreasing in the first variable and increasing in the second variable. We will need the following theorem, proved in [1] (see also [2, page 11]).

Theorem 1.1. (see [1]) *Let $I \subset \mathbb{R}$ and suppose $F : I \times I \to I$ is decreasing in the first variable and increasing in the second variable. Then, for every solution $\{x_n\}_{n\in\mathbb{N}_0}$ of the difference equation*

$$x_{n+2} = F(x_{n+1}, x_n), \quad n \in \mathbb{N}_0,$$

each of the subsequences $\{x_{2n}\}_{n\in\mathbb{N}_0}$ and $\{x_{2n+1}\}_{n\in\mathbb{N}_0}$ is eventually monotone.

In the next section, we will prove that every positive solution $\{x_n\}_{n\in\mathbb{N}_0}$ of (1.4) converges to a finite limit x^*. Then, every positive solution of (1.2) must converge to $(x^*, (x^*)^2)$, since $\{x_n\}_{n\in\mathbb{N}_0}$ must satisfy (1.4) and (1.3).

2. Main Results

In light of Theorem 1.1, we start with the following auxiliary result about eventually monotone positive solutions of (1.4).

Lemma 2.1. *Let $\{x_n\}_{n\in\mathbb{N}_0}$ be an arbitrary positive solution of* (1.4)*.*

(i) *If $\{x_{2n}\}_{n\in\mathbb{N}_0}$ is eventually increasing, then eventually*

$$a - c \le a - c + 2bx_{2n} \le a - c + 2bx_{2n} + x_{2n}x_{2n+1} \le 0. \tag{2.1}$$

(ii) *If $\{x_{2n}\}_{n\in\mathbb{N}_0}$ is eventually decreasing, then eventually*

$$a - c + 2bx_{2n} + x_{2n}x_{2n+1} \ge 0. \tag{2.2}$$

(iii) *If $\{x_{2n+1}\}_{n\in\mathbb{N}_0}$ is eventually increasing, then eventually*

$$a - c \le a - c + 2bx_{2n+1} \le a - c + 2bx_{2n+1} + x_{2n+1}x_{2n+2} \le 0. \tag{2.3}$$

(iv) *If $\{x_{2n+1}\}_{n\in\mathbb{N}_0}$ is eventually decreasing, then eventually*

$$a - c + 2bx_{2n+1} + x_{2n+1}x_{2n+2} \ge 0. \tag{2.4}$$

Proof. First suppose $\{x_{2n}\}_{n\in\mathbb{N}_0}$ is eventually increasing. Hence, we have eventually

$$x_{2n} \le x_{2n+2} = \frac{cx_{2n}}{a + 2bx_{2n} + x_{2n}x_{2n+1}}$$

and thus eventually

$$(a - c + 2bx_{2n} + x_{2n}x_{2n+1})x_{2n} \le 0$$

so that (2.1) follows. Next suppose $\{x_{2n}\}_{n\in\mathbb{N}_0}$ is eventually decreasing. Hence, we have eventually

$$\begin{aligned} x_{2n} &\geq x_{2n+2} \\ &= \frac{cx_{2n}}{a+2bx_{2n}+x_{2n}x_{2n+1}} \end{aligned}$$

and thus eventually

$$(a-c+2bx_{2n}+x_{2n}x_{2n+1})x_{2n} \geq 0$$

so that (2.2) follows. Now suppose $\{x_{2n+1}\}_{n\in\mathbb{N}_0}$ is eventually increasing. Hence, we have eventually

$$\begin{aligned} x_{2n+1} &\leq x_{2n+3} \\ &= \frac{cx_{2n+1}}{a+2bx_{2n+1}+x_{2n+1}x_{2n+2}} \end{aligned}$$

and thus eventually

$$(a-c+2bx_{2n+1}+x_{2n+1}x_{2n+2})x_{2n+1} \leq 0$$

so that (2.3) follows. Finally suppose $\{x_{2n+1}\}_{n\in\mathbb{N}_0}$ is eventually decreasing. Hence, we have eventually

$$\begin{aligned} x_{2n+1} &\geq x_{2n+3} \\ &= \frac{cx_{2n+1}}{a+2bx_{2n+1}+x_{2n+1}x_{2n+2}} \end{aligned}$$

and thus eventually

$$(a-c+2bx_{2n+1}+x_{2n+1}x_{2n+2})x_{2n+1} \geq 0$$

so that (2.4) follows. □

Corollary 2.2. *Let $\{x_n\}_{n\in\mathbb{N}_0}$ be an arbitrary positive solution of* (1.4).

(i) *If $\{x_{2n}\}_{n\in\mathbb{N}_0}$ is eventually monotone, then it converges to a finite nonnegative limit.*

(ii) *If $\{x_{2n+1}\}_{n\in\mathbb{N}_0}$ is eventually monotone, then it converges to a finite nonnegative limit.*

Proof. First suppose $\{x_{2n}\}_{n\in\mathbb{N}_0}$ is eventually increasing. By (2.1), we have eventually

$$x_{2n} \leq \frac{c-a}{2b}$$

so that $\{x_{2n}\}_{n\in\mathbb{N}_0}$ is bounded above and hence converges to a finite (nonnegative) limit.

If $\{x_{2n}\}_{n\in\mathbb{N}_0}$ is eventually decreasing, then, being bounded below by zero, it also converges to a nonnegative limit. Next suppose $\{x_{2n+1}\}_{n\in\mathbb{N}_0}$ is eventually increasing. By (2.3), we have eventually

$$x_{2n+1} \leq \frac{c-a}{2b}$$

so that $\{x_{2n+1}\}_{n\in\mathbb{N}_0}$ is bounded above and hence converges to a finite (nonnegative) limit. If $\{x_{2n+1}\}_{n\in\mathbb{N}_0}$ is eventually decreasing, then, being bounded below by zero, it also converges to a nonnegative limit. □

We now state and prove our main result, which confirms Conjecture 1.0.

Theorem 2.3. *Let $a, b, c > 0$. Let $\{x_n\}_{n\in\mathbb{N}_0}$ be an arbitrary positive solution of* (1.4). *Then $\{x_n\}_{n\in\mathbb{N}_0}$ converges to a finite nonnegative limit; more precisely:*

$$\lim_{n\to\infty} x_n = x^* := \begin{cases} 0 & \text{if} \quad c \leq a, \\ \sqrt{b^2+c-a}-b > 0 & \text{if} \quad c > a. \end{cases}$$

Proof. By Theorem 1.1, both $\{x_{2n}\}_{n\in\mathbb{N}_0}$ and $\{x_{2n+1}\}_{n\in\mathbb{N}_0}$ are eventually monotone. By Corollary 2.2, $\{x_{2n}\}_{n\in\mathbb{N}_0}$ and $\{x_{2n+1}\}_{n\in\mathbb{N}_0}$ both converge to finite nonnegative limits. Put

$$x_{\mathrm{e}} := \lim_{n\to\infty} x_{2n} \quad \text{and} \quad x_{\mathrm{o}} := \lim_{n\to\infty} x_{2n+1} \quad \text{so that} \quad x_{\mathrm{e}}, x_{\mathrm{o}} \in [0,\infty).$$

By (1.4), we have

$$(a + 2bx_{2n} + x_{2n}x_{2n+1})x_{2n+2} = cx_{2n} \tag{2.5}$$

and

$$(a + 2bx_{2n+1} + x_{2n+1}x_{2n+2})x_{2n+3} = cx_{2n+1} \tag{2.6}$$

for all $n \in \mathbb{N}_0$. By letting $n \to \infty$ in (2.5) and (2.6), we obtain

$$(a - c + 2bx_{\mathrm{e}} + x_{\mathrm{e}}x_{\mathrm{o}})x_{\mathrm{e}} = 0 \quad \text{and} \quad (a - c + 2bx_{\mathrm{o}} + x_{\mathrm{o}}x_{\mathrm{e}})x_{\mathrm{o}} = 0. \tag{2.7}$$

From the first equation in (2.7), we must have $x_{\mathrm{e}} = 0$ or

$$a - c + 2bx_{\mathrm{e}} + x_{\mathrm{e}}x_{\mathrm{o}} = 0. \tag{2.8}$$

First, if $x_{\mathrm{e}} = 0$, then, since in this case $\{x_{2n}\}_{n\in\mathbb{N}_0}$ is eventually decreasing, $a-c \geq 0$ by taking limits in (2.2). But then, by the second equation in (2.7), $(a - c + 2bx_{\mathrm{o}})x_{\mathrm{o}} = 0$ implies $x_{\mathrm{o}} = 0$ since $a \geq c$. In summary, if $x_{\mathrm{e}} = 0$, then $x_{\mathrm{o}} = 0$, and then $a \geq c$ and $x^* = 0$. Second, we assume $x_{\mathrm{e}} > 0$ so that (2.8) holds. From the second equation in (2.7), we must have $x_{\mathrm{o}} = 0$ or

$$a - c + 2bx_{\mathrm{o}} + x_{\mathrm{o}}x_{\mathrm{e}} = 0. \tag{2.9}$$

If $x_{\mathrm{o}} = 0$, then, since in this case $\{x_{2n+1}\}_{n\in\mathbb{N}_0}$ is eventually decreasing, $a - c \geq 0$ by taking limits in (2.4). But then, by (2.8), $a - c + 2bx_{\mathrm{e}} = 0$ implies $x_{\mathrm{e}} = 0$ since $a \geq c$, a contradiction. Hence, $x_{\mathrm{o}} > 0$ and thus (2.9) holds. Now subtracting (2.9) from (2.8) yields $2b(x_{\mathrm{e}} - x_{\mathrm{o}}) = 0$, i.e., $x^* := x_{\mathrm{e}} = x_{\mathrm{o}} > 0$. By (2.9),

$$(x^*)^2 + 2bx^* + a - c = 0$$

and therefore $x^* = \sqrt{b^2 + c - a} - b$. The proof is complete. □

References

[1] E. Camouzis, G. Ladas, *When does local asymptotic stability imply global attractivity in rational equations?*, J. Difference Equ. Appl., **12** (2006), no. 8, 863–885. 1, 1.1

[2] E. Camouzis, G. Ladas, *Dynamics of third-order rational difference equations with open problems and conjectures*, Advances in Discrete Mathematics and Applications, 5. Chapman & Hall CRC, Boca Raton, FL, (2008). 1

[3] E. Camouzis, M. R. S. Kulenović, G. Ladas, O. Merino, *Rational systems in the plane*, J. Difference Equ. Appl., **15** (2009), no. 3, 303–323. 1

[4] E. Camouzis, G. Ladas, *Global results on rational systems in the plane, part 1*, J. Difference Equ. Appl., **16** (2010), no. 8, 975–1013. 1

[5] E. Camouzis, C. M. Kent, G. Ladas, C. D. Lynd, *On the global character of solutions of the system: $x_{n+1} = \frac{\alpha_1+y_n}{x_n}$ and $y_{n+1} = \frac{\alpha_2+\beta_2x_n+\gamma_2y_n}{A_2+B_2x_n+C_2y_n}$*, J. Difference Equ. Appl., **18** (2012), no. 7, 1205–1252. 1, 1, 1.0