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A new numerical technique for local fractional diffusion equation in fractal heat transfer

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Abstract

In this paper, a new numerical approach, embedding the differential transform (DT) and Laplace transform (LT), is firstly proposed. It is considered in the local fractional derivative operator for obtaining the non-differential solution for diffusion equation in fractal heat transfer.

Keywords: Numerical solution, diffusion equation, differential transform, Laplace transform, fractal heat transfer, local fractional derivative.
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1. Introduction

Fractional calculus and its generalizations started to be considered by many researchers as an optimal tool to describe the dynamics of complex systems [3, 4, 11, 16, 24, 26, 30]. Ordinary differential equations (ODEs) and partial differential equations (PDEs) within the derivative of non-integer orders were used to describe complex phenomena in the time-space sense of differentiability [1, 12, 23, 27, 28, 33] and non-differentiability [7, 17, 37, 38]. Several approaches were developed to find the numerical and analytical solutions for fractional ordinary differential equations (FODEs) and fractional partial differential equations

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(FPDEs), namely the methods of Mellin and Laplace transforms [5], implicit local radial basis function (ILRBF) [19], finite difference (FD) [32], homotopy perturbation (HP)[18], homotopy analysis (HA) [10], heat-balance integral (HBI) [15], variational iteration (VI) [14], generalized differential transform (GDT) [29], and others [9, 22, 25, 31]. There are distinct strategies for solving the local fractional partial differential equations [13, 20, 34, 35], such as the local fractional perspectives to differential transform (DT) [8], variation iteration method (VIM) [36], Fourier transform (FT) [39], Laplace transform (LT) [40], Laplace variation iteration method (LVIM) [21], and functional method (FM) [6].

In this paper, we consider the diffusion equation in fractal heat transfer with local fractional derivative [34, 35, 37, 38]

$$\frac{\partial^{\alpha}\Omega\left(x,t\right)}{\partial t^{\alpha}}-\frac{\partial^{2\alpha}\Omega\left(x,t\right)}{\partial x^{2\alpha}}=0, \tag{1.1}$$

where the local fractional derivative partial derivative operator is defined as [8, 34]

$$\frac{\partial^{\alpha}\Omega\left(x,y\right)}{\partial x^{\alpha}}|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^{\alpha}\left(\Omega\left(x,y\right)-\Omega\left(x_0,y\right)\right)}{\left(x-x_0\right)^{\alpha}},$$

with

$$\Delta^{\alpha}\left(\Omega\left(x,y\right)-\Omega\left(x_0,y\right)\right)\cong\Gamma\left(1+\alpha\right)\Delta\left(\Omega\left(x,y\right)-\Omega\left(x_0,y\right)\right).$$

The classical Laplace differential transform was presented in [2]. However, the Laplace differential transform via local fractional differential operator is not addressed. The aim of the manuscript is to point out the local fractional Laplace differential transform to solve the diffusion equation in fractal heat transfer with local fractional derivative.

The structure of the manuscript is as follows. In Section 2, we introduce the fundamental theory of Laplace and differential transforms via local fractional derivative. In Section 3, we analyze the Laplace differential transform method. In Section 4, we give the numerical solution for diffusion equation in fractal heat transfer. In Section 5, we discuss the error analysis. Finally, in Section 6 we outlines the main conclusions.

2. Fundamental tools

The local fractional derivative of $\Omega\left(x\right)$ is defined as follows [34–36, 40]:

$$\frac{\partial^{\alpha}\Omega\left(x\right)}{\partial x^{\alpha}}|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^{\alpha}\left(\Omega\left(x\right)-\Omega\left(x_0\right)\right)}{\left(x-x_0\right)^{\alpha}},$$

where

$$\Delta^{\alpha}\left(\Omega\left(x\right)-\Omega\left(x_0\right)\right)\cong\Gamma\left(1+\alpha\right)\Delta\left(\Omega\left(x\right)-\Omega\left(x_0\right)\right).$$

The local fractional Laplace transform of $\Omega\left(\tau\right)$ can be formulated as follows [21, 36]:

$$\widetilde{Y}_{\alpha}\left\{\Omega\left(\tau\right)\right\}=\Omega_{y}^{\widetilde{Y},\alpha}\left(y\right)=\frac{1}{\Gamma\left(1+\alpha\right)}\int_{0}^{\infty}E_{\alpha}\left(-y^{\alpha}\tau^{\alpha}\right)\Omega\left(\tau\right)\left(d\tau\right)^{\alpha},\ 0<\alpha\leq 1,$$

where the local fractional integral operator of $\theta\left(x\right)$ is defined as follows [6, 21, 34, 36, 40]:

$${}_{a}I_{b}^{(\alpha)}\theta\left(\tau\right)=\frac{1}{\Gamma\left(1+\alpha\right)}\int_{a}^{b}\theta\left(\tau\right)\left(d\tau\right)^{\alpha}=\frac{1}{\Gamma\left(1+\alpha\right)}\lim_{\Delta\tau\to 0}\sum_{j=0}^{j=N-1}\theta\left(\tau\right)\left(\Delta\tau\right)^{\alpha},$$

where $\Delta\tau=t_{j+1}-t_{j}$, $j=0,...,N-1$, $t_0=a$, and $t_N=b$.

The inverse local fractional Laplace transform is defined as [6, 21, 36]:

$$\Omega\left(\tau\right)=\widetilde{Y}_{\alpha}^{-1}\left\{\Omega_{y}^{\widetilde{Y},\alpha}\left(y\right)\right\}=\frac{1}{\left(2\pi\right)^{\alpha}}\int_{\beta-i\infty}^{\beta+i\infty}E_{\alpha}\left(y^{\alpha}\tau^{\alpha}\right)\Omega_{y}^{\widetilde{Y},\alpha}\left(y\right)\left(dy\right)^{\alpha},$$

where $y^{\alpha} = \beta^{\alpha} + i^{\alpha}\infty^{\alpha}$ and $Re\,(y^{\alpha}) = \beta^{\alpha}$.

The properties applied in this paper are given below [21, 36]:

$$\widetilde{Y}_{\alpha}\left\{a\Omega_1(\tau) + b\Omega_2(\tau)\right\} = a\widetilde{Y}_{\alpha}\left\{\Omega_1(\tau)\right\} + b\widetilde{Y}_{\alpha}\left\{\Omega_2(\tau)\right\},$$

$$\widetilde{Y}_{\alpha}\left\{\Omega^{(n\alpha)}(\tau)\right\} = y^{n\alpha}\widetilde{Y}_{\alpha}\left\{\Omega(\tau)\right\} - \sum_{k=1}^{n} y^{(k-1)\alpha}\Omega^{(n-k)\alpha}(0),$$

$$\widetilde{Y}_{\alpha}\left\{E_{\alpha}(x^{\alpha})\right\} = \frac{1}{y^{\alpha}-1},$$

$$\widetilde{Y}_{\alpha}\left\{\frac{\tau^{k\alpha}}{\Gamma(1+k\alpha)}\right\} = \frac{1}{y^{(k+1)\alpha}}.$$

If $g(\tau)$ is a local fractional analytic function [8] in the domain Ω, then the differential transform (the revised differential transform) via local fractional derivative becomes

$$G(\kappa) = \frac{1}{\Gamma(1+\kappa\alpha)}\frac{d^{\kappa\alpha}g(\tau)}{d\tau^{\kappa\alpha}}\Big|_{\tau=\tau_0}, \quad \tau,\tau_0 \in \Omega,$$

where $\kappa = 0, 1, \cdots, n$, $0 < \alpha \leq 1$, and the inverse transform of $G(\kappa)$ in the domain Ω via local fractional derivative is given by:

$$g(\tau) = \sum_{\kappa=0}^{\infty} G(\kappa)(\tau-\tau_0)^{\kappa\alpha}, \quad \kappa \in K. \tag{2.1}$$

The function $G(\kappa)$ is the non-differentiable spectrum of $g(\tau)$. The properties of differential transform via local fractional derivative are listed in Table 1.

Table 1: The fundamental operations of the local fractional DTM (see[8]).

Original functions	**Transformed functions**
$s(\tau) = g(\tau) + h(\tau)$	$S(\kappa) = G(\kappa) + H(\kappa)$
$s(\tau) = ag(\tau)$	$S(\kappa) = aG(\kappa)$, a is a constant
$s(\tau) = g(\tau)h(\tau)$	$S(\kappa) = \sum\limits_{l=0}^{\kappa} G(l)H(\kappa-l)$
$s(\tau) = E_{\alpha}(\tau^{\alpha})$	$S(\kappa) = \frac{1}{\Gamma(1+\kappa\alpha)}$
$s(\tau) = D^{n\alpha}g(\tau)$	$S(\kappa) = \frac{\Gamma(1+(\kappa+n)\alpha)}{\Gamma(1+\kappa\alpha)}G(\kappa+n)$

3. The method of the Laplace differential transform via local fractional derivative

In order to present the Laplace differential transform method via local fractional derivative, we consider the initial-boundary conditions of Eq. (1.1) as

$$\Omega(0,t) = \varphi_1(t),$$

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}\Omega(0,t) = \varphi_2(t),$$

$$\Omega(x,0) = \phi(x).$$

We take the local fractional Laplace transform of Eq. (1.1) with respect to t, namely,

$$y^{\alpha}\Omega(x,y) - \phi(x) - \frac{\partial^{2\alpha}\Omega(x,y)}{\partial x^{2\alpha}} = 0,$$

and the initial conditions are presented as follows:

$$\frac{\partial^\alpha}{\partial x^\alpha}\Omega(0,y)=\varphi_2(y),\qquad \Omega(0,y)=\varphi_1(y). \tag{3.1}$$

Using the LFDT to Eq. (1.1) with respect to x, we have

$$\Omega(x,y)=\sum_{\kappa=0}^{\infty}\Omega(\kappa,y)x^{\kappa\alpha},\ \kappa\in K,$$

so that

$$y^\alpha\Omega(\kappa,y)-\phi(\kappa)-\frac{\Gamma(1+(\kappa+2)\alpha)}{\Gamma(1+\kappa\alpha)}\Omega(\kappa+2,y)=0.$$

Therefore, after taking the inverse local fractional Laplace transform, we obtain

$$\Omega(x,t)=\widetilde{Y}_\alpha^{-1}\{\Omega(x,y)\}=\widetilde{Y}_\alpha^{-1}\left\{\sum_{\kappa=0}^{\infty}\Omega(\kappa,y)x^{\kappa\alpha}\right\}=\sum_{\kappa=0}^{\infty}\Omega(\kappa,t)x^{\kappa\alpha},$$

which, using the inverse local fractional Laplace transform of Eq. (2.1), leads to the solution of Eq. (1.1).

4. Application to diffusion equation in fractal heat transfer

In this section we consider a typical application to diffusion equation in fractal heat transfer. We now suggest

$$\varphi_1(t)=E_\alpha(t^\alpha),\ \varphi_2(t)=E_\alpha(t^\alpha),\ \phi(x)=E_\alpha(x^\alpha).$$

From (3.1), we obtain

$$y^\alpha\Omega(\kappa,y)-\frac{1}{\Gamma(1+\kappa\alpha)}-\frac{\Gamma(1+(\kappa+2)\alpha)}{\Gamma(1+\kappa\alpha)}\Omega(\kappa+2,y)=0,\quad \kappa=0,1,2,..., \tag{4.1}$$

where

$$\frac{\partial^\alpha}{\partial x^\alpha}\Omega(0,y)=\frac{1}{y^\alpha-1},\ \Omega(0,y)=\frac{1}{y^\alpha-1}. \tag{4.2}$$

Submitting Eq. (4.2) into the iterative relationship (4.1) leads to the simulation results for the components $\Omega(\kappa,y)$ listed in Table 2.

Table 2: The simulation values for the components $\Omega(\kappa,y)$.

$\Omega(0,y)$	$\Omega(1,y)$	$\Omega(2,y)$	$\Omega(3,y)$	$\Omega(4,y)$
$\frac{1}{y^\alpha-1}$	$\frac{1}{y^\alpha-1}\frac{1}{\Gamma(1+\alpha)}$	$\frac{1}{y^\alpha-1}\frac{1}{\Gamma(1+2\alpha)}$	$\frac{1}{y^\alpha-1}\frac{1}{\Gamma(1+3\alpha)}$	$\frac{1}{y^\alpha-1}\frac{1}{\Gamma(1+4\alpha)}$

Therefore, we obtain

$$\Omega(\kappa,y)=\sum_{\kappa=0}^{4}\frac{1}{y^\alpha-1}\frac{1}{\Gamma(1+\kappa\alpha)}.$$

Making use of the inverse local fractional Laplace transform, it yields:

$$\Omega(\kappa,t)=E_\alpha(t^\alpha)\sum_{\kappa=0}^{4}\frac{1}{\Gamma(1+\kappa\alpha)}. \tag{4.3}$$

Adopting the inverse local fractional Laplace transform to Eq. (4.3), the simulation value of the solution becomes:

$$\Omega(\kappa,t)=E_\alpha(t^\alpha)\sum_{\kappa=0}^{4}\frac{x^{\kappa\alpha}}{\Gamma(1+\kappa\alpha)}. \tag{4.4}$$

The plot of Eq. (4.4) is shown in Figure 1 when $\alpha=\ln 2/\ln 3$.

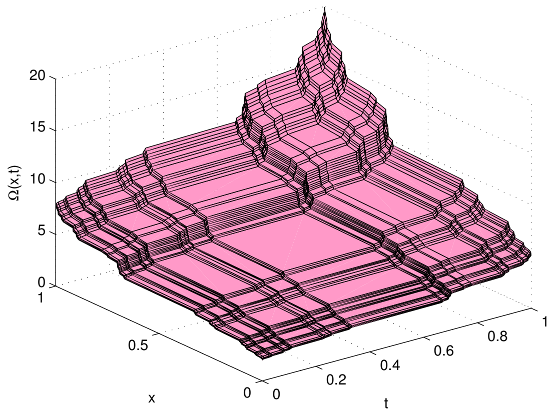

Figure 1: The simulation value of Eq. (4.4) when $\alpha = \ln 2/\ln 3$.

5. Error analysis

In this section we present the error analysis of the local fractional Laplace differential transform.

Taking the successively iterative process, we have

$$\Omega(x,t) = E_\alpha(t^\alpha) \lim_{n\to\infty} \sum_{\kappa=0}^{n} \frac{x^{\kappa\alpha}}{\Gamma(1+\kappa\alpha)}, \tag{5.1}$$

so that the exact solution of Eq. (1.1) takes the form:

$$\Omega^*(x,t) = E_\alpha(t^\alpha) \lim_{n\to\infty} \sum_{\kappa=0}^{n} \frac{x^{\kappa\alpha}}{\Gamma(1+\kappa\alpha)} = E_\alpha(t^\alpha) E_\alpha(x^\alpha). \tag{5.2}$$

From Eqs. (5.1) and (5.2) the error value can be written as:

$$\Re(x,t) = |\Omega^*(x,t) - \Omega(x,t)| = E_\alpha(t^\alpha) \sum_{\kappa=n+1}^{\infty} \frac{x^{\kappa\alpha}}{\Gamma(1+\kappa\alpha)}.$$

The comparison between the numerical solution when $n = 4$ and exact solution of Eq. (1.1) is displayed in Figure 2. The components of the term $\widetilde{Y}_\alpha^{-1}\{\Omega(\kappa,y)\}$, when $\kappa = 0,1,2,3,4$, are listed in Table 3. The plots of the components of the term $\widetilde{Y}_\alpha^{-1}\{\Omega(\kappa,y)\}$, when $\kappa = 0,1,2,3,4$, are illustrated in Figure 3. The numerical solutions of Eq. (1.1), when $n = 0,1,2,3,4$, are listed in Table 4. The plots of numerical solutions of Eq. (1.1), when $n = 0,1,2,3,4$, are demonstrated in Figure 4.

Table 3: The components of the term $\widetilde{Y}_\alpha^{-1}\{\Omega(\kappa,y)\}$ when $\kappa = 0,1,2,3,4$.

$\kappa=0$	$\kappa=1$	$\kappa=2$	$\kappa=3$	$\kappa=4$
$E_\delta(t^\delta)$	$\frac{E_\alpha(t^\alpha)x^\alpha}{\Gamma(1+\alpha)}$	$\frac{E_\alpha(t^\alpha)x^{2\alpha}}{\Gamma(1+2\alpha)}$	$\frac{E_\alpha(t^\alpha)x^{3\alpha}}{\Gamma(1+3\alpha)}$	$\frac{E_\alpha(t^\alpha)x^{4\alpha}}{\Gamma(1+4\alpha)}$

Table 4: The numerical solutions of Eq.(1.1) when $n = 0,1,2,3,4$.

$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
$E_\delta(t^\delta)$	$\sum_{\kappa=0}^{1}\frac{E_\alpha(t^\alpha)x^{\kappa\alpha}}{\Gamma(1+\kappa\alpha)}$	$\sum_{\kappa=0}^{2}\frac{E_\alpha(t^\alpha)x^{\kappa\alpha}}{\Gamma(1+\kappa\alpha)}$	$\sum_{\kappa=0}^{3}\frac{E_\alpha(t^\alpha)x^{\kappa\alpha}}{\Gamma(1+\kappa\alpha)}$	$\sum_{\kappa=0}^{4}\frac{E_\alpha(t^\alpha)x^{\kappa\alpha}}{\Gamma(1+\kappa\alpha)}$

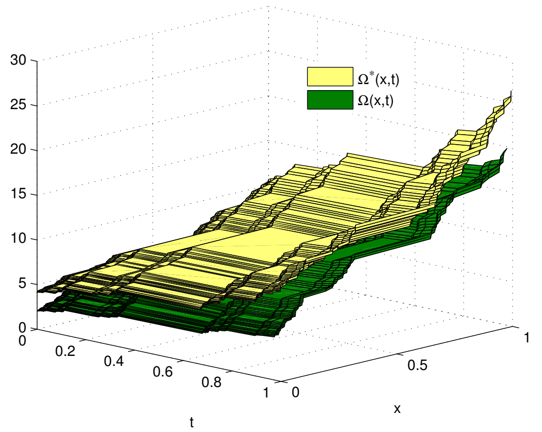

Figure 2: The comparison between numerical solution and exact solution of Eq. (1.1).

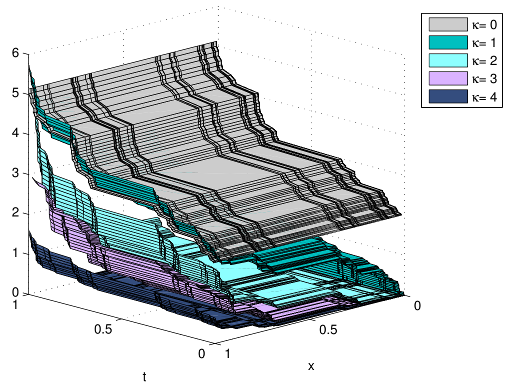

Figure 3: The components of the term $\widetilde{Y}_{\alpha}^{-1}\{\Omega(\kappa, y)\}$ when $\kappa = 0, 1, 2, 3, 4$.

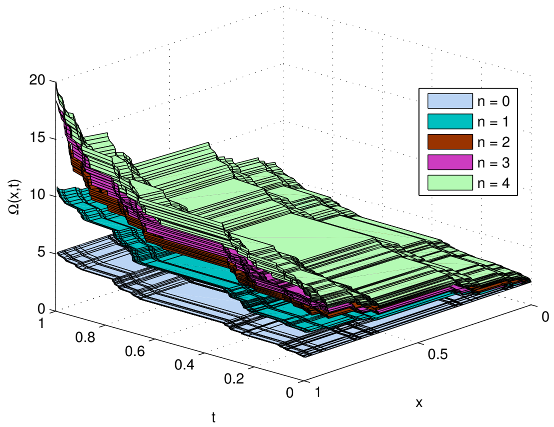

Figure 4: The numerical solutions of Eq.(1.1) when $n = 0, 1, 2, 3, 4$.

6. Conclusions

The local fractional Laplace differential transform was firstly proposed based upon local fractional calculus (LFC). We adopted the method to solve the diffusion equation in fractal heat transfer in the presence of the local fractional derivative. The obtained result strongly illustrates that the method is easy, simple, efficient and accurate for a class of local fractional partial differential equations.

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