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Lagrangians of the (2 + 1)-dimensional KP equation with variable coefficients and cross terms

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Abstract

Zhang constructed a Lagrangian for the (2 + 1)-dimensional KP equation with variable coefficients and cross terms [L. H. Zhang, Appl. Math. Comput., **219** (2013), 4865–4879]. This paper suggests a simple method to construct a needed Lagrangian using the semi-inverse by introducing a simple auxiliary function, the presented method is simpler than Zhang's method to construct a Lagrangian.

Keywords: Variational principle, least square technology, semi-inverse method.
2010 MSC: 47H10, 54H25.

1. Introduction

Zhang studied the following (2 + 1)-dimensional KP equation with variable coefficients and cross terms [17]

$$(u_t + uu_x + u_{xxx})_x + u_{yy} + b(t)u_{xy} + (c_0(t) + c_1(t)y)u_{xx} = 0 \tag{1.1}$$

and obtained a Lagrangian in the form [17]

$$L = v((u_t + uu_x + u_{xxx})_x + u_{yy} + b(t)u_{xy} + (c_0(t) + c_1(t)y)u_{xx}), \tag{1.2}$$

where v is an auxiliary function. The Euler-Lagrange equation of eq. (1.1) with respect to u is

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial u_y}\right) + \frac{\partial^2}{\partial x^2}(\frac{\partial L}{\partial u_{xx}}) + \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial L}{\partial u_{xy}}\right) = 0 \tag{1.3}$$

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or

$$v_{tx} - 2(vu_x)_x + vu_{xx} + (vu)_{xx} + v_{xxxx} + v_{yy} + b(t)v_{xy} + (c_0(t) + c_1(t)y)v_{xx} = 0. \tag{1.4}$$

Simplification of Eq. (1.4) results in

$$v_{tx} + v_{xx}u + v_{xxxx} + v_{yy} + b(t)v_{xy} + (c_0(t) + c_1(t)y)v_{xx} = 0. \tag{1.5}$$

The auxiliary function, v, in Eq. (1.2) must satisfy Eq. (1.5).

Remark 1.1. Equation (1.2) is similar to those by the Galerkin technology [16] which is widely used in the finite element method.

For a general linear equation $Au = 0$, where A is an operator e.g., $A = \frac{d}{dx}$ the Galerkin method is

$$J(u, v) = \int L dt dx dy, \tag{1.6}$$

where v is auxiliary function, L is a Lagrange function defined as

$$L = vAu, \tag{1.7}$$

the Euler-Lagrange equations of Eq. (1.6) are

$$Au = 0 \tag{1.8}$$

and

$$Av = 0. \tag{1.9}$$

So Eq. (1.2) is similar to Galerkin technology.

Remark 1.2. There is an exact Lagrangian for the following equation

$$(u_t + u_{xxx})_x + u_{yy} + b(t)u_{xy} + (c_0(t) + c_1(t)y)u_{xx} = 0. \tag{1.10}$$

By the semi-inverse method [1], [3]–[6], [8], we can obtain the Lagrangian for Eq. (1.10), which reads

$$L = -\frac{1}{2}u_t u_x + \frac{1}{2}(u_{xx})^2 - \frac{1}{2}(u_y)^2 - \frac{b(t)}{2}u_x u_y - \frac{1}{2}(c_0(t) + c_1(t)y)(u_x)^2. \tag{1.11}$$

Remark 1.3. An approximate Lagrangian can be obtained for Eq. (1.1), which is

$$L = -\frac{1}{2}u_t u_x + \frac{1}{2}(u_{xx})^2 - \frac{1}{2}(u_y)^2 - \frac{b(t)}{2}u_x u_y - \frac{1}{2}(c_0(t) + c_1(t)y)(u_x)^2 - wu_x, \tag{1.12}$$

where w is an auxiliary function defined by

$$w = uu_x. \tag{1.13}$$

Remark 1.4. An generalized Lagrangian can obtained by the semi-inverse method [3]–[6], [8], which reads

$$L(u, w) = -\frac{1}{2}u_t u_x + \frac{1}{2}(u_{xx})^2 - \frac{1}{2}(u_y)^2 - \frac{b(t)}{2}u_x u_y - \frac{1}{2}(c_0(t) + c_1(t)y)(u_x)^2 - wu_x + \lambda(w - uu_x)^2, \tag{1.14}$$

where $\lambda \gg 1$ is a nonzero constant.

Proof. The Euler-Lagrange equations of Eq. (1.14) with respect to u and w are

$$(u_t + u_{xxx})_x + u_{yy} + b(t)u_{xy} + (c_0(t) + c_1(t)y)u_{xx} + w_x - 2\lambda(w_x - uu_x)u_x + 2\lambda((w_x - uu_x)u)_x = 0, \tag{1.15}$$

$$-u_x + 2\lambda(w - uu_x) = 0. \tag{1.16}$$

Considering $\lambda \gg 1$, saying $\lambda = 10^{10}$, Eq. (1.16) is approximated as

$$w - uu_x = 0. \tag{1.17}$$

Submitting Eq. (1.17) into Eq. (1.15) results in Eq. (1.1). □

Similar results can be obtained for the Burgers equation [17] by the semi-inverse method [3]–[6], [8]. Some illustrating examples for construction of Lagrangian of a nonlinear equation are available in Refs [2, 7, 9, 10, 11, 12, 13, 14, 15, 18].

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