Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 719–739
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Fixed point results for various contractions in parametric and fuzzy b-metric spaces

Nawab Hussain[a], Peyman Salimi[b], Vahid Parvaneh[c,*]

[a] *Department of Mathematics, King Abdulaziz University P.O. Box 80203, Jeddah 21589, Saudi Arabia*
[b] *Young Researchers and Elite Club, Rasht Branch, Islamic Azad University, Rasht, Iran*
[c] *Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb, Iran.*

Abstract

The notion of parametric metric spaces being a natural generalization of metric spaces was recently introduced and studied by Hussain et al. [A new approach to fixed point results in triangular intuitionistic fuzzy metric spaces, Abstract and Applied Analysis, Vol. 2014, Article ID 690139, 16 pp]. In this paper we introduce the concept of parametric b-metric space and investigate the existence of fixed points under various contractive conditions in such spaces. As applications, we derive some new fixed point results in triangular partially ordered fuzzy b-metric spaces. Moreover, some examples are provided here to illustrate the usability of the obtained results. ©2015 All rights reserved.

Keywords: Fixed point theorem, fuzzy b-metric spaces, contractions.
2010 MSC: 54H25, 54A40, 54E50.

---

1. Introduction and preliminaries

Fixed point theory has attracted many researchers since 1922 with the admired Banach fixed point theorem. This theorem supplies a method for solving a variety of applied problems in mathematical sciences and engineering. A huge literature on this subject exist and this is a very active area of research at present.

The concept of metric spaces has been generalized in many directions. The notion of a b-metric space was studied by Czerwik in [7, 8] and a lot of fixed point results for single and multivalued mappings by many authors have been obtained in (ordered) b-metric spaces (see, *e.g.*, [2]-[17]). Khmasi and Hussain [21] and Hussain and Shah [19] discussed KKM mappings and related results in b-metric and cone b-metric spaces.

---

*Corresponding author
Email addresses: nhusain@kau.edu.sa (Nawab Hussain), salimipeyman@gmail.com (Peyman Salimi), vahid.parvaneh@kiau.ac.ir (Vahid Parvaneh)

Received 2015-3-30

In this paper, we introduce a new type of generalized metric space, which we call parametric b-metric space, as a generalization of both metric and b-metric spaces. Then, we prove some fixed point theorems under various contractive conditions in parametric b-metric spaces. These contractions include Geraghty-type conditions, conditions using comparison functions and almost generalized weakly contractive conditions. As applications, we derive some new fixed point results in triangular fuzzy b-metric spaces. We illustrate these results by appropriate examples. The notion of a b-metric space was studied by Czerwik in [7, 8].

Definition 1.1 ([7]). Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d: X \times X \to \mathbb{R}^+$ is a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

(b_1) $d(x, y) = 0$ if and only if $x = y$,

(b_2) $d(x, y) = d(y, x)$,

(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

Note that a b-metric is not always a continuous function of its variables (see, e.g., [17, Example 2]), whereas an ordinary metric is.

Hussain et al. [16] defined and studied the concept of parametric metric space.

Definition 1.2. Let X be a nonempty set and $\mathcal{P} : X \times X \times (0, \infty) \to [0, \infty)$ be a function. We say $\mathcal{P}$ is a parametric metric on X if,

(i) $\mathcal{P}(x, y, t) = 0$ for all $t > 0$ if and only if $x = y$;

(ii) $\mathcal{P}(x, y, t) = \mathcal{P}(y, x, t)$ for all $t > 0$;

(iii) $\mathcal{P}(x, y, t) \leq \mathcal{P}(x, z, t) + \mathcal{P}(z, y, t)$ for all $x, y, z \in X$ and all $t > 0$.

and we say the pair $(X, \mathcal{P})$ is a parametric metric space.

Now, we introduce parametric b-metric space, as a generalization of parametric metric space.

Definition 1.3. Let X be a non-empty set, $s \geq 1$ be a real number and let $\mathcal{P} : X^2 \times (0, \infty) \to (0, \infty)$ be a map satisfying the following conditions:

($\mathcal{P}_b1$) $\mathcal{P}(x, y, t) = 0$ for all $t > 0$ if and only if $x = y$,

($\mathcal{P}_b2$) $\mathcal{P}(x, y, t) = \mathcal{P}(y, x, t)$ for all $t > 0$,

($\mathcal{P}_b3$) $\mathcal{P}(x, z, t) \leq s[\mathcal{P}(x, y, t) + \mathcal{P}(y, z, t)]$ for all $t > 0$ where $s \geq 1$.

Then $\mathcal{P}$ is called a parametric b-metric on X and $(X, \mathcal{P})$ is called a parametric b-metric space with parameter s.

Obviously, for $s = 1$, parametric b-metric reduces to parametric metric.

Definition 1.4. Let $\{x_n\}$ be a sequence in a parametric b-metric space $(X, \mathcal{P})$.

1. $\{x_n\}$ is said to be convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, if for all $t > 0$, $\lim_{n\to\infty} \mathcal{P}(x_n, x, t) = 0$.
2. $\{x_n\}$ is said to be a Cauchy sequence in X if for all $t > 0$, $\lim_{n\to\infty} \mathcal{P}(x_n, x_m, t) = 0$.
3. $(X, \mathcal{P})$ is said to be complete if every Cauchy sequence is a convergent sequence.

The following are some easy examples of parametric b-metric spaces.

Example 1.5. Let $X = [0, +\infty)$ and $\mathcal{P}(x, y, t) = t(x - y)^p$. Then $\mathcal{P}$ is a parametric b-metric with constant $s = 2^p$.

Definition 1.6. Let $(X, \mathcal{P}, b)$ be a parametric b-metric space and $T : X \to X$ be a mapping. We say T is a continuous mapping at x in X, if for any sequence $\{x_n\}$ in X such that, $x_n \to x$ as $n \to \infty$ then, $Tx_n \to Tx$ as $n \to \infty$.

In general, a parametric b-metric function for $s > 1$ is not jointly continuous in all its variables. Now, we present an example of a discontinuous parametric b-metric.

Example 1.7. Let $X = \mathbb{N} \cup \{\infty\}$ and let $\mathcal{P} : X^2 \times (0, \infty) \to \mathbb{R}$ be defined by,

$$\mathcal{P}(m, n, t) = \begin{cases} 0, & \text{if } m = n, \\ t\left|\frac{1}{m} - \frac{1}{n}\right|, & \text{if } m, n \text{ are even or } mn = \infty, \\ 5t, & \text{if } m \text{ and } n \text{ are odd and } m \neq n, \\ 2t, & \text{otherwise.} \end{cases}$$

Then it is easy to see that for all $m, n, p \in X$, we have

$$\mathcal{P}(m, p, t) \leq \frac{5}{2}(\mathcal{P}(m, n, t) + \mathcal{P}(n, p, t)).$$

Thus, $(X, \mathcal{P})$ is a parametric b-metric space with $s = \frac{5}{2}$.

Now, we show that $\mathcal{P}$ is not a continuous function. Take $x_n = 2n$ and $y_n = 1$, then we have, $x_n \to \infty$, $y_n \to 1$. Also,

$$\mathcal{P}(2n, \infty, t) = \frac{t}{2n} \to 0,$$

and

$$\mathcal{P}(y_n, 1, t) = 0 \to 0.$$

On the other hand,

$$\mathcal{P}(x_n, y_n, t) = \mathcal{P}(x_n, 1, t) = 2t,$$

and

$$\mathcal{P}(\infty, 1, t) = 1.$$

Hence, $\lim_{n\to\infty} \mathcal{P}(x_n, y_n, t) \neq \mathcal{P}(x, y, t)$.

So, from the above discussion we need the following simple lemma about the convergent sequences in the proof of our main result.

Lemma 1.8. *Let $(X, \mathcal{P}, s)$ be a parametric b-metric space and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to x and y, respectively. Then we have*

$$\frac{1}{s^2}\mathcal{P}(x, y, t) \leq \liminf_{n\to\infty} \mathcal{P}(x_n, y_n, t) \leq \limsup_{n\to\infty} \mathcal{P}(x_n, y_n, t) \leq s^2\mathcal{P}(x, y, t),$$

for all $t \in (0, \infty)$. In particular, if $y_n = y$ is constant, then

$$\frac{1}{s}\mathcal{P}(x, y, t) \leq \liminf_{n\to\infty} \mathcal{P}(x_n, y, t) \leq \limsup_{n\to\infty} \mathcal{P}(x_n, y, t) \leq s\mathcal{P}(x, y, t),$$

for all $t \in (0, \infty)$.

Proof. Using $(\mathcal{P}_b3)$ of Definition 1.3 in the given parametric b-metric space, it is easy to see that

$$\begin{aligned} \mathcal{P}(x, y, t) &\leq s\mathcal{P}(x, x_n, t) + s\mathcal{P}(x_n, y, t) \\ &\leq s\mathcal{P}(x, x_n, t) + s^2\mathcal{P}(x_n, y_n, t) + s^2\mathcal{P}(y_n, y, t) \end{aligned}$$

and

$$\begin{aligned}\mathcal{P}(x_n, y_n, t) &\le s\mathcal{P}(x_n, x, t) + s\mathcal{P}(x, y_n, t)\\ &\le s\mathcal{P}(x_n, x, t) + s^2\mathcal{P}(x, y, t) + s^2\mathcal{P}(y, y_n, t),\end{aligned}$$

for all $t > 0$. Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality we obtain the desired result.

If $y_n = y$, then

$$\mathcal{P}(x, y, t) \le s\mathcal{P}(x, x_n, t) + s\mathcal{P}(x_n, y, t)$$

and

$$\mathcal{P}(x_n, y, t) \le s\mathcal{P}(x_n, x, t) + s\mathcal{P}(x, y, t),$$

for all $t > 0$. □

2. Main results

2.1. Results under Geraghty-type conditions

Fixed point theorems for monotone operators in ordered metric spaces are widely investigated and have found various applications in differential and integral equations (see [1, 15, 20, 24] and references therein). In 1973, M. Geraghty [12] proved a fixed point result, generalizing Banach contraction principle. Several authors proved later various results using Geraghty-type conditions. Fixed point results of this kind in b-metric spaces were obtained by Đukić et al. in [10].

Following [10], for a real number $s \ge 1$, let $\mathcal{F}_s$ denote the class of all functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ satisfying the following condition:

$$\beta(t_n) \to \frac{1}{s} \text{ as } n \to \infty \text{ implies } \; t_n \to 0 \text{ as } n \to \infty.$$

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a parametric b-metric $\mathcal{P}$ on X such that $(X, \mathcal{P})$ is a complete parametric b-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$s\mathcal{P}(fx, fy, t) \le \beta(\mathcal{P}(x, y, t))M(x, y, t) \tag{1}$$

for all $t > 0$ and for all comparable elements $x, y \in X$, where

$$M(x, y, t) = \max\left\{\mathcal{P}(x, y, t), \frac{\mathcal{P}(x, fx, t)\mathcal{P}(y, fy, t)}{1 + \mathcal{P}(fx, fy, t)}, \frac{\mathcal{P}(x, fx, t)\mathcal{P}(y, fy, t)}{1 + \mathcal{P}(x, y, t)}\right\}.$$

If f is continuous, then f has a fixed point.

Proof. Starting with the given x_0, put $x_n = f^n x_0$. Since $x_0 \preceq fx_0$ and f is an increasing function we obtain by induction that

$$x_0 \preceq fx_0 \preceq f^2x_0 \preceq \cdots \preceq f^n x_0 \preceq f^{n+1}x_0 \preceq \cdots.$$

Step I: We will show that $\lim_{n\to\infty} \mathcal{P}(x_n, x_{n+1}, t) = 0$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by (1) we have

$$\begin{aligned}s\mathcal{P}(x_n, x_{n+1}, t) = s\mathcal{P}(fx_{n-1}, fx_n, t) &\le \beta(\mathcal{P}(x_{n-1}, x_n, t))M(x_{n-1}, x_n, t)\\ &< \frac{1}{s}\mathcal{P}(x_{n-1}, x_n, t) \le \mathcal{P}(x_{n-1}, x_n, t),\end{aligned} \tag{2}$$

because

$$\begin{aligned}
&M(x_{n-1},x_n,t)\\
&\quad= \max\left\{\mathcal{P}(x_{n-1},x_n,t), \frac{\mathcal{P}(x_{n-1},fx_{n-1},t)\mathcal{P}(x_n,fx_n,t)}{1+\mathcal{P}(fx_{n-1},fx_n,t)}, \frac{\mathcal{P}(x_{n-1},fx_{n-1},t)\mathcal{P}(x_n,fx_n,t)}{1+\mathcal{P}(x_{n-1},x_n,t)}\right\}\\
&\quad= \max\left\{\mathcal{P}(x_{n-1},x_n,t), \frac{\mathcal{P}(x_{n-1},x_n,t)\mathcal{P}(x_n,x_{n+1},t)}{1+\mathcal{P}(x_n,x_{n+1},t)}, \frac{\mathcal{P}(x_{n-1},x_n,t)\mathcal{P}(x_n,x_{n+1},t)}{1+\mathcal{P}(x_{n-1},x_n,t)}\right\}\\
&\quad\le \max\{\mathcal{P}(x_{n-1},x_n,t),\mathcal{P}(x_n,x_{n+1},t)\}.
\end{aligned}$$

If $\max\{\mathcal{P}(x_{n-1},x_n,t),\mathcal{P}(x_n,x_{n+1},t)\} = \mathcal{P}(x_n,x_{n+1},t)$, then from (2) we have,

$$\begin{aligned}
\mathcal{P}(x_n,x_{n+1},t) &\le \beta(\mathcal{P}(x_{n-1},x_n,t))\mathcal{P}(x_n,x_{n+1},t)\\
&< \tfrac{1}{s}\mathcal{P}(x_n,x_{n+1},t)\\
&\le \mathcal{P}(x_n,x_{n+1},t),
\end{aligned} \tag{3}$$

which is a contradiction.

Hence, $\max\{\mathcal{P}(x_{n-1},x_n,t),\mathcal{P}(x_n,x_{n+1},t)\} = \mathcal{P}(x_{n-1},x_n,t)$, so from (3),

$$\mathcal{P}(x_n,x_{n+1},t) \le \beta(\mathcal{P}(x_{n-1},x_n,t))\mathcal{P}(x_{n-1},x_n,t) \le \mathcal{P}(x_{n-1},x_n,t). \tag{4}$$

Therefore, the sequence $\{\mathcal{P}(x_n,x_{n+1},t)\}$ is decreasing, so there exists $r \ge 0$ such that $\lim_{n\to\infty} \mathcal{P}(x_n,x_{n+1},t) = r$. Suppose that $r > 0$. Now, letting $n \to \infty$, from (4) we have

$$\frac{1}{s}r \le r \le \lim_{n\to\infty} \beta(\mathcal{P}(x_{n-1},x_n,t))r \le r.$$

So, we have $\lim_{n\to\infty} \beta(\mathcal{P}(x_{n-1},x_n,t)) \ge \frac{1}{s}$ and since $\beta \in \mathcal{F}_s$ we deduce that $\lim_{n\to\infty} \mathcal{P}(x_{n-1},x_n,t) = 0$ which is a contradiction. Hence, $r = 0$, that is,

$$\lim_{n\to\infty} \mathcal{P}(x_n,x_{n+1},t) = 0. \tag{5}$$

Step II: Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence. Using the triangle inequality and by (1) we have

$$\begin{aligned}
\mathcal{P}(x_n,x_m,t) &\le s\mathcal{P}(x_n,x_{n+1},t) + s^2\mathcal{P}(x_{n+1},x_{m+1},t) + s^2\mathcal{P}(x_{m+1},x_m,t)\\
&\le s\mathcal{P}(x_n,x_{n+1},t) + s^2\mathcal{P}(x_m,x_{m+1},t) + s\beta(\mathcal{P}(x_n,x_m,t))M(x_n,x_m,t).
\end{aligned}$$

Letting $m,n \to \infty$ in the above inequality and applying (5) we have

$$\lim_{m,n\to\infty} \mathcal{P}(x_n,x_m,t) \le s \lim_{m,n\to\infty} \beta(\mathcal{P}(x_n,x_m,t)) \lim_{m,n\to\infty} M(x_n,x_m,t). \tag{6}$$

Here,

$$\begin{aligned}
&\mathcal{P}(x_n,x_m,t) \le M(x_n,x_m,t)\\
&\quad= \max\left\{\mathcal{P}(x_n,x_m,t), \frac{\mathcal{P}(x_n,fx_n,t)\mathcal{P}(x_m,fx_m,t)}{1+\mathcal{P}(fx_n,fx_m,t)}, \frac{\mathcal{P}(x_n,fx_n,t)\mathcal{P}(x_m,fx_m,t)}{1+\mathcal{P}(x_n,x_m,t)}\right\}\\
&\quad= \max\left\{\mathcal{P}(x_n,x_m,t), \frac{\mathcal{P}(x_n,x_{n+1},t)\mathcal{P}(x_m,x_{m+1},t)}{1+\mathcal{P}(x_{n+1},x_{m+1},t)}, \frac{\mathcal{P}(x_n,x_{n+1},t)\mathcal{P}(x_m,x_{m+1},t)}{1+\mathcal{P}(x_n,x_m,t)}\right\}.
\end{aligned}$$

Letting $m,n \to \infty$ in the above inequality we get

$$\lim_{m,n\to\infty} M(x_n,x_m,t) = \lim_{m,n\to\infty} \mathcal{P}(x_n,x_m,t). \tag{7}$$

From (6) and (7), we obtain

$$\lim_{m,n\to\infty} \mathcal{P}(x_n, x_m, t) \leq s \lim_{m,n\to\infty} \beta(\mathcal{P}(x_n, x_m, t)) \lim_{m,n\to\infty} \mathcal{P}(x_n, x_m, t). \tag{8}$$

Now we claim that, $\lim_{m,n\to\infty} \mathcal{P}(x_n, x_m, t) = 0$. On the contrary, if $\lim_{m,n\to\infty} \mathcal{P}(x_n, x_m, t) \neq 0$, then we get

$$\frac{1}{s} \leq \lim_{m,n\to\infty} \beta(\mathcal{P}(x_n, x_m, t)).$$

Since $\beta \in \mathcal{F}_s$ we deduce that

$$\lim_{m,n\to\infty} \mathcal{P}(x_n, x_m, t) = 0. \tag{9}$$

which is a contradiction. Consequently, $\{x_n\}$ is a b-parametric Cauchy sequence in X. Since $(X, \mathcal{P})$ is complete, the sequence $\{x_n\}$ converges to some $z \in X$, that is, $\lim\limits_{n\to\infty} \mathcal{P}(x_n, z, t) = 0$.

Step III: Now, we show that z is a fixed point of f.

Using the triangle inequality, we get

$$\mathcal{P}(fz, z, t) \leq s\mathcal{P}(fz, fx_n, t) + s\mathcal{P}(fx_n, z, t).$$

Letting $n \to \infty$ and using the continuity of f, we have $fz = z$. Thus, z is a fixed point of f. $\square$

Example 2.2. Let $X = [0, \infty)$ be endowed with the parametric b-metric

$$\mathcal{P}(x, y, t) = \begin{cases} t(x+y)^2, & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

for all $x, y \in X$ and all $t > 0$. Define $T : X \to X$ by

$$Tx = \begin{cases} \frac{1}{8}x^2, & \text{if } x \in [0, 1) \\ \frac{1}{8}x, & \text{if } x \in [1, 2) \\ \frac{1}{4} & \text{if } x \in [2, \infty) \end{cases}$$

Also, define, $\beta : [0, \infty) \to [0, \frac{1}{2})$ by $\beta(t) = \frac{1}{4}$. Clearly, $(X, \mathcal{P}, 2)$ is a complete parametric b-metric space, T is a continuous mapping and $\beta \in \mathcal{F}_2$. Now we consider the following cases:

- Let $x, y \in [0, 1)$ with $x \leq y$, then,

$$\begin{aligned} 2\mathcal{P}(Tx, Ty, t) &= 2t(\tfrac{1}{8}x^2 + \tfrac{1}{8}y^2)^2 = \tfrac{1}{32}t(x^2 + y^2)^2 \\ &\leq \tfrac{1}{4}t(x+y)^2 = \tfrac{1}{4}\mathcal{P}(x, y, t) \\ &\leq \tfrac{1}{4}M(x, y, t) = \beta(\mathcal{P}(x, y, t))M(x, y, t) \end{aligned}$$

- Let $x, y \in [1, 2)$ with $x \leq y$, then,

$$\begin{aligned} 2\mathcal{P}(Tx, Ty, t) &= 2t(\tfrac{1}{8}x + \tfrac{1}{8}y)^2 = \tfrac{1}{32}t(x+y)^2 \\ &\leq \tfrac{1}{4}t(x+y)^2 = \tfrac{1}{4}\mathcal{P}(x, y, t) \\ &\leq \tfrac{1}{4}M(x, y, t) = \beta(\mathcal{P}(x, y, t))M(x, y, t) \end{aligned}$$

- Let $x, y \in [2, \infty)$ with $x \leq y$, then,

$$\begin{aligned}
2\mathcal{P}(Tx, Ty, t) &= 2t(\tfrac{1}{4} + \tfrac{1}{4})^2 = \tfrac{1}{2}t \leq t = \tfrac{1}{4}t(1+1)^2 \\
&\leq \tfrac{1}{4}t(x+y)^2 = \tfrac{1}{4}\mathcal{P}(x, y, t) \\
&\leq \tfrac{1}{4}M(x, y, t) = \beta(\mathcal{P}(x, y, t))M(x, y, t)
\end{aligned}$$

- Let $x \in [0, 1)$ and $y \in [1, 2)$ (clearly with $x \leq y$), then,

$$\begin{aligned}
2\mathcal{P}(Tx, Ty, t) &= 2t(\tfrac{1}{8}x^2 + \tfrac{1}{8}y)^2 \leq 2t(\tfrac{1}{8}x + \tfrac{1}{8}y)^2 = \tfrac{1}{32}t(x^2 + y^2)^2 \\
&\leq \tfrac{1}{4}t(x+y)^2 = \tfrac{1}{4}\mathcal{P}(x, y, t) \\
&\leq \tfrac{1}{4}M(x, y, t) = \beta(\mathcal{P}(x, y, t))M(x, y, t)\}
\end{aligned}$$

- Let $x \in [0, 1)$ and $y \in [2, \infty)$ (clearly with $x \leq y$), then,

$$\begin{aligned}
2\mathcal{P}(Tx, Ty, t) &= 2t(\tfrac{1}{8}x^2 + \tfrac{1}{4})^2 \leq 2t(\tfrac{1}{8}x + \tfrac{1}{8}y)^2 = \tfrac{1}{32}t(x+y)^2 \\
&\leq \tfrac{1}{4}t(x+y)^2 = \tfrac{1}{4}\mathcal{P}(x, y, t) \\
&\leq \tfrac{1}{4}M(x, y, t) = \beta(\mathcal{P}(x, y, t))M(x, y, t)\}
\end{aligned}$$

- Let $x \in [1, 2)$ and $y \in [2, \infty)$ (clearly with $x \leq y$), then,

$$\begin{aligned}
2\mathcal{P}(Tx, Ty, t) &= 2t(\tfrac{1}{8}x + \tfrac{1}{4})^2 \leq 2t(\tfrac{1}{8}x + \tfrac{1}{8}y)^2 = \tfrac{1}{32}t(x+y)^2 \\
&\leq \tfrac{1}{4}t(x+y)^2 = \tfrac{1}{4}\mathcal{P}(x, y, t) \\
&\leq \tfrac{1}{4}M(x, y, t) = \beta(\mathcal{P}(x, y, t))M(x, y, t)\}
\end{aligned}$$

Therefore,

$$2\mathcal{P}(Tx, Ty, t) \leq \beta(\mathcal{P}(x, y, t))M(x, y, t)$$

for all $x, y \in X$ with $x \leq y$ and all $t > 0$. Hence, all conditions of Theorem 2.1 holds and T has a unique fixed point.

Note that the continuity of f in Theorem 2.1 is not necessary and can be dropped.

Theorem 2.3. *Under the hypotheses of Theorem 2.1, without the continuity assumption on f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$. Then f has a fixed point.*

Proof. Repeating the proof of Theorem 2.1, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to z \in X$. Using the assumption on X we have $x_n \preceq z$. Now, we show that $z = fz$. By (1) and Lemma 1.8,

$$\begin{aligned}
s\left[\frac{1}{s}\mathcal{P}(z, fz, t)\right] &\leq s \limsup_{n\to\infty} \mathcal{P}(x_{n+1}, fz, t) \\
&\leq \limsup_{n\to\infty} \beta(\mathcal{P}(x_n, z, t)) \limsup_{n\to\infty} M(x_n, z, t),
\end{aligned}$$

where,

$$\begin{aligned}
&\lim_{n\to\infty} M(x_n, z, t)\\
&\quad = \lim_n \max\left\{\mathcal{P}(x_n, z, t), \frac{\mathcal{P}(x_n, fx_n, t)\mathcal{P}(z, fz, t)}{1+\mathcal{P}(fx_n, fz, t)}, \frac{\mathcal{P}(x_n, fx_n, t)\mathcal{P}(z, fz, t)}{1+\mathcal{P}(x_n, z, t)}\right\}\\
&\quad = \lim_n \max\left\{\mathcal{P}(x_n, z, t), \frac{\mathcal{P}(x_n, x_{n+1}, t)\mathcal{P}(z, fz, t)}{1+\mathcal{P}(x_{n+1}, fz, t)}, \frac{\mathcal{P}(x_n, x_{n+1}, t)\mathcal{P}(z, fz, t)}{1+\mathcal{P}(x_n, z, t)}\right\} = 0 \text{ (see (5))}.
\end{aligned}$$

Therefore, we deduce that $\mathcal{P}(z, fz, t) \leq 0$. As t is arbitrary, hence, we have $z = fz$. □

If in the above theorems we take $\beta(t) = r$, where $0 \leq r < \frac{1}{s}$, then we have the following corollary.

Corollary 2.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a parametric b-metric $\mathcal{P}$ on X such that $(X, \mathcal{P})$ is a complete parametric b-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that for some r, with $0 \leq r < \frac{1}{s}$,*

$$s\mathcal{P}(fx, fy, t) \leq rM(x, y, t)$$

holds for each $t > 0$ and all comparable elements $x, y \in X$, where

$$M(x, y, t) = \max\left\{\mathcal{P}(x, y, t), \frac{\mathcal{P}(x, fx, t)\mathcal{P}(y, fy, t)}{1+\mathcal{P}(fx, fy, t)}, \frac{\mathcal{P}(x, fx, t)\mathcal{P}(y, fy, t)}{1+\mathcal{P}(x, y, t)}\right\}.$$

If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

Corollary 2.5. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a parametric b-metric $\mathcal{P}$ on X such that $(X, \mathcal{P})$ is a complete parametric b-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$\mathcal{P}(fx, fy, t) \leq \alpha\mathcal{P}(x, y, t) + \beta\frac{\mathcal{P}(x, fx, t)\mathcal{P}(y, fy, t)}{1+\mathcal{P}(fx, fy, t)} + \gamma\frac{\mathcal{P}(x, fx, t)\mathcal{P}(y, fy, t)}{1+\mathcal{P}(x, y, t)}$$

for each $t > 0$ and all comparable elements $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma \leq \frac{1}{s}$.

If f is continuous, or, for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to u \in X$ one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

2.2. Results using comparison functions

Let Ψ denote the family of all nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that $\lim_n \psi^n(t) = 0$ for all $t > 0$, where ψ^n denotes the n-th iterate of ψ. It is easy to show that, for each $\psi \in \Psi$, the following is satisfied:

(a) $\psi(t) < t$ for all $t > 0$;
(b) $\psi(0) = 0$.

Theorem 2.6. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a parametric b-metric $\mathcal{P}$ on X such that $(X, \mathcal{P})$ is a complete parametric b-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$s\mathcal{P}(fx, fy, t) \leq \psi(N(x, y, t)) \tag{10}$$

where

$$\begin{aligned}
N(x, y, t) = \max\Bigg\{&\mathcal{P}(x, y, t), \frac{\mathcal{P}(x, fx, t)d(x, fy, t) + \mathcal{P}(y, fy, t)\mathcal{P}(y, fx, t)}{1 + s[\mathcal{P}(x, fx, t) + \mathcal{P}(y, fy, t)]},\\
&\frac{\mathcal{P}(x, fx, t)\mathcal{P}(x, fy, t) + \mathcal{P}(y, fy, t)\mathcal{P}(y, fx, t)}{1 + \mathcal{P}(x, fy, t) + \mathcal{P}(y, fx, t)}\Bigg\},
\end{aligned}$$

for some $\psi \in \Psi$ and for all comparable elements $x, y \in X$ and all $t > 0$. If f is continuous, then f has a fixed point.

Proof. Since $x_0 \preceq fx_0$ and f is an increasing function, we obtain by induction that

$$x_0 \preceq fx_0 \preceq f^2x_0 \preceq \cdots \preceq f^nx_0 \preceq f^{n+1}x_0 \preceq \cdots .$$

Putting $x_n = f^nx_0$, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = fx_{n_0}$ and so we have nothing for prove. Hence, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Step I. We will prove that $\lim\limits_{n\to\infty} \mathcal{P}(x_n, x_{n+1}, t) = 0$. Using condition (39), we obtain

$$\mathcal{P}(x_n, x_{n+1}, t) \leq s\mathcal{P}(x_n, x_{n+1}, t) = s\mathcal{P}(fx_{n-1}, fx_n, t) \leq \psi(N(x_{n-1}, x_n, t)).$$

Here,

$$\begin{aligned} N(x_{n-1}, x_n, t) = \max\{&\mathcal{P}(x_{n-1}, x_n, t), \frac{\mathcal{P}(x_{n-1}, fx_{n-1}, t)\mathcal{P}(x_{n-1}, fx_n, t) + \mathcal{P}(x_n, fx_n, t)\mathcal{P}(x_n, fx_{n-1}, t)}{1 + s[\mathcal{P}(x_{n-1}, fx_{n-1}, t) + \mathcal{P}(x_n, fx_n, t)]}, \\ &\frac{\mathcal{P}(x_{n-1}, fx_{n-1}, t)\mathcal{P}(x_{n-1}, fx_n, t) + \mathcal{P}(x_n, fx_n, t)\mathcal{P}(x_n, fx_{n-1}, t)}{1 + \mathcal{P}(x_{n-1}, fx_n, t) + \mathcal{P}(x_n, fx_{n-1}, t)}\} \end{aligned}$$

$$\begin{aligned} &= \max\{\mathcal{P}(x_{n-1}, x_n, t), \frac{\mathcal{P}(x_{n-1}, x_n, t)\mathcal{P}(x_{n-1}, x_{n+1}, t) + \mathcal{P}(x_n, x_{n+1}, t)\mathcal{P}(x_n, x_n, t)}{1 + s[\mathcal{P}(x_{n-1}, x_n, t) + \mathcal{P}(x_n, x_{n+1}, t)]}, \\ &\qquad \frac{\mathcal{P}(x_{n-1}, x_n, t)\mathcal{P}(x_{n-1}, x_{n+1}, t) + \mathcal{P}(x_n, x_{n+1}, t)\mathcal{P}(x_n, x_n, t)}{1 + \mathcal{P}(x_{n-1}, x_{n+1}, t) + \mathcal{P}(x_n, x_n, t)}\} \\ &= \mathcal{P}(x_{n-1}, x_n, t). \end{aligned}$$

Hence,

$$\mathcal{P}(x_n, x_{n+1}, t) \leq s\mathcal{P}(x_n, x_{n+1}, t) \leq \psi(\mathcal{P}(x_{n-1}, x_n, t)).$$

By induction, we get that

$$\mathcal{P}(x_n, x_{n+1}, t) \leq \psi(\mathcal{P}(x_{n-1}, x_n, t)) \leq \psi^2(\mathcal{P}(x_{n-2}, x_{n-1}, t)) \leq \cdots \leq \psi^n(\mathcal{P}(x_0, x_1, t)).$$

As $\psi \in \Psi$, we conclude that

$$\lim_{n\to\infty} \mathcal{P}(x_n, x_{n+1}, t) = 0. \tag{11}$$

Step II. We will prove that $\{x_n\}$ is a parametric Cauchy sequence. Suppose the contrary. Then there exist $t > 0$ and $\varepsilon > 0$ for them we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ and } \mathcal{P}(x_{m_i}, x_{n_i}, t) \geq \varepsilon. \tag{12}$$

This means that

$$\mathcal{P}(x_{m_i}, x_{n_i-1}, t) < \varepsilon. \tag{13}$$

From (12) and using the triangle inequality, we get

$$\varepsilon \leq \mathcal{P}(x_{m_i}, x_{n_i}, t) \leq s\mathcal{P}(x_{m_i}, x_{m_i+1}, t) + s\mathcal{P}(x_{m_i+1}, x_{n_i}, t).$$

Taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} \mathcal{P}(x_{m_i+1}, x_{n_i}, t). \tag{14}$$

From the definition of $M(x, y, t)$ we have

$$
\begin{aligned}
M(x_{m_i},&x_{n_i-1}, t)\\
&= \max\{\mathcal{P}(x_{m_i}, x_{n_i-1}, t), \frac{\mathcal{P}(x_{m_i}, fx_{m_i}, t)\mathcal{P}(x_{m_i}, fx_{n_i-1}, t) + \mathcal{P}(x_{n_i-1}, fx_{n_i-1}, t)\mathcal{P}(x_{n_i-1}, fx_{m_i}, t)}{1 + s[\mathcal{P}(x_{m_i}, fx_{m_i}, t) + \mathcal{P}(x_{n_i-1}, fx_{n_i-1}, t)]},\\
&\qquad \frac{\mathcal{P}(x_{m_i}, fx_{m_i}, t)\mathcal{P}(x_{m_i}, fx_{n_i-1}, t) + \mathcal{P}(x_{n_i-1}, fx_{n_i-1}, t)\mathcal{P}(x_{n_i-1}, fx_{m_i}, t)}{1 + \mathcal{P}(x_{m_i}, fx_{n_i-1}, t) + \mathcal{P}(x_{n_i-1}, fx_{m_i}, t)}\}\\
&= \max\{\mathcal{P}(x_{m_i}, x_{n_i-1}, t), \frac{\mathcal{P}(x_{m_i}, x_{m_i+1}, t)\mathcal{P}(x_{m_i}, x_{n_i}, t) + \mathcal{P}(x_{n_i-1}, x_{n_i}, t)\mathcal{P}(x_{n_i-1}, x_{m_i+1}, t)}{1 + s[\mathcal{P}(x_{m_i}, x_{m_i+1}, t) + \mathcal{P}(x_{n_i-1}, x_{n_i}, t)]},\\
&\qquad \frac{\mathcal{P}(x_{m_i}, x_{m_i+1}, t)\mathcal{P}(x_{m_i}, x_{n_i}, t) + \mathcal{P}(x_{n_i-1}, x_{n_i}, t)\mathcal{P}(x_{n_i-1}, x_{m_i+1}, t)}{1 + \mathcal{P}(x_{m_i}, x_{n_i}, t) + \mathcal{P}(x_{n_i-1}, x_{m_i+1}, t)}\}
\end{aligned}
$$

and if $i \to \infty$, by (11) and (13) we have

$$\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, t) \le \varepsilon.$$

Now, from (39) we have

$$s\mathcal{P}(x_{m_i+1}, x_{n_i}, t) = s\mathcal{P}(fx_{m_i}, fx_{n_i-1}, t) \le \psi(M(x_{m_i}, x_{n_i-1}, t)).$$

Again, if $i \to \infty$ by (14) we obtain

$$\varepsilon = s \cdot \frac{\varepsilon}{s} \le s \limsup_{i\to\infty} \mathcal{P}(x_{m_i+1}, x_{n_i}, a) \le \psi(\varepsilon) < \varepsilon,$$

which is a contradiction. Consequently, $\{x_n\}$ is a Cauchy sequence in X. Therefore, the sequence $\{x_n\}$ converges to some $z \in X$, that is, $\lim_n \mathcal{P}(x_n, z, t) = 0$ for all $t > 0$.

Step III. Now we show that z is a fixed point of f.

Using the triangle inequality, we get

$$\mathcal{P}(z, fz, t) \le s\mathcal{P}(z, fx_n, t) + s\mathcal{P}(fx_n, fz, t).$$

Letting $n \to \infty$ and using the continuity of f, we get

$$\mathcal{P}(z, fz, t) \le 0.$$

Hence, we have $fz = z$. Thus, z is a fixed point of f. □

Theorem 2.7. *Under the hypotheses of Theorem 2.6, without the continuity assumption on f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$. Then f has a fixed point.*

Proof. Following the proof of Theorem 2.6, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to z \in X$. Using the given assumption on X we have $x_n \preceq z$. Now, we show that $z = fz$. By (39) we have

$$s\mathcal{P}(fz, x_n, t) = s\mathcal{P}(fz, fx_{n-1}, t) \le \psi(M(z, x_{n-1}, t)), \tag{15}$$

where

$$
\begin{aligned}
M(z, x_{n-1}, t) &= \max\{\mathcal{P}(x_{n-1}, z, t), \frac{\mathcal{P}(x_{n-1}, fx_{n-1}, t)\mathcal{P}(x_{n-1}, fz, t) + \mathcal{P}(z, fz, t)\mathcal{P}(z, fx_{n-1}, t)}{1 + s[\mathcal{P}(x_{n-1}, fx_{n-1}, t) + \mathcal{P}(z, fz, t)]},\\
&\qquad \frac{\mathcal{P}(x_{n-1}, fx_{n-1}, t)\mathcal{P}(x_{n-1}, fz, t) + \mathcal{P}(z, fz, t)\mathcal{P}(z, fx_{n-1}, t)}{1 + \mathcal{P}(x_{n-1}, fz, t) + \mathcal{P}(z, fx_{n-1}, t)}\}\\
&= \max\{\mathcal{P}(x_{n-1}, z, t), \frac{\mathcal{P}(x_{n-1}, x_n, t)\mathcal{P}(x_{n-1}, fz, t) + \mathcal{P}(z, fz, t)\mathcal{P}(z, x_n, t)}{1 + s[\mathcal{P}(x_{n-1}, x_n, t) + \mathcal{P}(z, fz, t)]},\\
&\qquad \frac{\mathcal{P}(x_{n-1}, x_n, t)\mathcal{P}(x_{n-1}, fz, t) + \mathcal{P}(z, fz, t)\mathcal{P}(z, x_n, t)}{1 + \mathcal{P}(x_{n-1}, fz, t) + \mathcal{P}(z, x_n, t)}\}.
\end{aligned}
$$

Letting $n \to \infty$ in the above relation, we get

$$\limsup_{n\to\infty} M(z, x_{n-1}, a) = 0. \tag{16}$$

Again, taking the upper limit as $n \to \infty$ in (15) and using Lemma 1.8 and (16) we get

$$\begin{aligned} s\left[\frac{1}{s}\mathcal{P}(z, fz, t)\right] &\le s\limsup_{n\to\infty} \mathcal{P}(x_n, fz, t) \\ &\le \limsup_{n\to\infty} \psi(M(z, x_{n-1}, t)) = 0. \end{aligned}$$

So we get $\mathcal{P}(z, fz, t) = 0$, i.e., $fz = z$. □

Corollary 2.8. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a parametric b-metric $\mathcal{P}$ on X such that $(X, \mathcal{P})$ is a complete parametric b-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$s\mathcal{P}(fx, fy, t) \le rM(x, y, t)$$

where $0 \le r < 1$ and

$$\begin{aligned} N(x, y, t) = \max\Big\{&\mathcal{P}(x, y, t), \frac{\mathcal{P}(x, fx, t)d(x, fy, t) + \mathcal{P}(y, fy, t)\mathcal{P}(y, fx, t)}{1 + s[\mathcal{P}(x, fx, t) + \mathcal{P}(y, fy, t)]}, \\ &\frac{\mathcal{P}(x, fx, t)\mathcal{P}(x, fy, t) + \mathcal{P}(y, fy, t)\mathcal{P}(y, fx, t)}{1 + \mathcal{P}(x, fy, t) + \mathcal{P}(y, fx, t)}\Big\}, \end{aligned}$$

for all comparable elements $x, y \in X$ and all $t > 0$. If f is continuous, or, whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$, then f has a fixed point.

2.3. Results for almost generalized weakly contractive mappings

Berinde in [5] studied the concept of almost contractions and obtained certain fixed point theorems. Results with similar conditions were obtained, *e.g.*, in [4] and [25]. In this section, we define the notion of almost generalized $(\psi, \varphi)_{s,t}$-contractive mapping and prove our new results. In particular, we extend Theorems 2.1, 2.2 and 2.3 of Ćirić *et al.* in [6] to the setting of b-parametric metric spaces.

Recall that Khan *et al.* introduced in [22] the concept of an altering distance function as follows.

Definition 2.9. A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function, if the following properties hold:

1. φ is continuous and non-decreasing.
2. $\varphi(t) = 0$ if and only if $t = 0$.

Let $(X, \mathcal{P})$ be a parametric b-metric space and let $f : X \to X$ be a mapping. For $x, y \in X$ and for all $t > 0$, set

$$M_t(x, y) = \max\left\{\mathcal{P}(x, y, t), \mathcal{P}(x, fx, t), \mathcal{P}(y, fy, t), \frac{\mathcal{P}(x, fy, t) + \mathcal{P}(y, fx, t)}{2s}\right\}$$

and

$$N_t(x, y) = \min\{\mathcal{P}(x, fx, t), \mathcal{P}(x, fy, t), \mathcal{P}(y, fx, t), \mathcal{P}(y, fy, t)\}.$$

Definition 2.10. Let $(X, \mathcal{P})$ be a parametric b-metric space. We say that a mapping $f : X \to X$ is an almost generalized $(\psi, \varphi)_{s,t}$-contractive mapping if there exist $L \ge 0$ and two altering distance functions ψ and φ such that

$$\psi(s\mathcal{P}(fx, fy, t)) \le \psi(M_t(x, y)) - \varphi(M_t(x, y)) + L\psi(N_t(x, y)) \tag{17}$$

for all $x, y \in X$ and for all $t > 0$.

Now, let us prove our result.

Theorem 2.11. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a parametric b-metric $\mathcal{P}$ on X such that $(X, \mathcal{P})$ is a complete parametric metric space. Let $f : X \to X$ be a continuous non-decreasing mapping with respect to $\preceq$. Suppose that f satisfies condition (17), for all comparable elements $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.*

Proof. Starting with the given x_0, define a sequence $\{x_n\}$ in X such that $x_{n+1} = fx_n$, for all $n \geq 0$. Since $x_0 \preceq fx_0 = x_1$ and f is non-decreasing, we have $x_1 = fx_0 \preceq x_2 = fx_1$, and by induction

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

If $x_n = x_{n+1}$, for some $n \in \mathbb{N}$, then $x_n = fx_n$ and hence x_n is a fixed point of f. So, we may assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. By (17), we have

$$\begin{aligned}
\psi(\mathcal{P}(x_n, x_{n+1}, t)) &\leq \psi(s\mathcal{P}(x_n, x_{n+1}, t)) \\
&= \psi(s\mathcal{P}(fx_{n-1}, fx_n, t)) \\
&\leq \psi(M_t(x_{n-1}, x_n)) - \varphi(M_t(x_{n-1}, x_n)) + L\psi(N_t(x_{n-1}, x_n)),
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
M_t(x_{n-1}, x_n) &= \max\left\{\mathcal{P}(x_{n-1}, x_n, t), \mathcal{P}(x_{n-1}, fx_{n-1}, t), \mathcal{P}(x_n, fx_n, t), \frac{\mathcal{P}(x_{n-1}, fx_n, t) + \mathcal{P}(x_n, fx_{n-1}, t)}{2s}\right\} \\
&= \max\left\{\mathcal{P}(x_{n-1}, x_n, t), \mathcal{P}(x_n, x_{n+1}, t), \frac{\mathcal{P}(x_{n-1}, x_{n+1}, t)}{2s}\right\} \\
&\leq \max\left\{\mathcal{P}(x_{n-1}, x_n, t), \mathcal{P}(x_n, x_{n+1}, t), \frac{\mathcal{P}(x_{n-1}, x_n, t) + \mathcal{P}(x_n, x_{n+1}, t)}{2}\right\}
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
N_t(x_{n-1}, x_n) &= \min\left\{\mathcal{P}(x_{n-1}, fx_{n-1}, t), \mathcal{P}(x_{n-1}, fx_n, t), \mathcal{P}(x_n, fx_{n-1}, t), \mathcal{P}(x_n, fx_n, t)\right\} \\
&= \min\left\{\mathcal{P}(x_{n-1}, x_n, t), \mathcal{P}(x_{n-1}, x_{n+1}, t), 0, \mathcal{P}(x_n, x_{n+1}, t)\right\} = 0.
\end{aligned} \tag{20}$$

From (18)–(20) and the properties of ψ and φ, we get

$$\begin{aligned}
\psi(\mathcal{P}(x_n, x_{n+1}, t)) \leq \psi\bigg(&\max\left\{\mathcal{P}(x_{n-1}, x_n, t), \mathcal{P}(x_n, x_{n+1}, t)\right\}\bigg) \\
&- \varphi\bigg(\max\left\{\mathcal{P}(x_{n-1}, x_n, t), \mathcal{P}(x_n, x_{n+1}, t), \frac{\mathcal{P}(x_{n-1}, x_{n+1}, t)}{2s}\right\}\bigg).
\end{aligned} \tag{21}$$

If

$$\max\left\{\mathcal{P}(x_{n-1}, x_n, t), \mathcal{P}(x_n, x_{n+1}, t)\right\} = \mathcal{P}(x_n, x_{n+1}, t),$$

then by (21) we have

$$\psi(\mathcal{P}(x_n, x_{n+1}, t)) \leq \psi(\mathcal{P}(x_n, x_{n+1}, t)) - \varphi\bigg(\max\left\{\mathcal{P}(x_{n-1}, x_n, t), \mathcal{P}(x_n, x_{n+1}, t), \frac{\mathcal{P}(x_{n-1}, x_{n+1}, t)}{2s}\right\}\bigg),$$

which gives that $x_n = x_{n+1}$, a contradiction.

Thus, $\{\mathcal{P}(x_n, x_{n+1}, t) : n \in \mathbb{N} \cup \{0\}\}$ is a non-increasing sequence of positive numbers. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \mathcal{P}(x_n, x_{n+1}, t) = r.$$

Letting $n \to \infty$ in (21), we get

$$\psi(r) \leq \psi(r) - \varphi\bigg(\max\bigg\{r, r, \lim_{n} \frac{\mathcal{P}(x_{n-1}, x_{n+1}, t)}{2s}\bigg\}\bigg) \leq \psi(r).$$

Therefore,

$$\varphi\bigg(\max\bigg\{r, r, \lim_{n\to\infty} \frac{\mathcal{P}(x_{n-1}, x_{n+1}, t)}{2s}\bigg\}\bigg) = 0,$$

and hence $r = 0$. Thus, we have

$$\lim_{n\to\infty} \mathcal{P}(x_n, x_{n+1}, t) = 0, \tag{22}$$

for each $t > 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in X.

Suppose the contrary, that is, $\{x_n\}$ is not a Cauchy sequence. Then there exist $t > 0$ and $\varepsilon > 0$ for them we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i, \text{and } \mathcal{P}(x_{m_i}, x_{n_i}, t) \geq \varepsilon. \tag{23}$$

This means that

$$\mathcal{P}(x_{m_i}, x_{n_i-1}, t) < \varepsilon. \tag{24}$$

Using (22) and taking the upper limit as $i \to \infty$, we get

$$\limsup_{n\to\infty} \mathcal{P}(x_{m_i}, x_{n_i-1}, t) \leq \varepsilon. \tag{25}$$

On the other hand, we have

$$\mathcal{P}(x_{m_i}, x_{n_i}, t) \leq s\mathcal{P}(x_{m_i}, x_{m_i+1}, t) + s\mathcal{P}(x_{m_i+1}, x_{n_i}, t).$$

Using (22), (24) and taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} \mathcal{P}(x_{m_i+1}, x_{n_i}, t).$$

Again, using the triangular inequality, we have

$$\mathcal{P}(x_{m_i+1}, x_{n_i-1}, t) \leq s\mathcal{P}(x_{m_i+1}, x_{m_i}, t) + s\mathcal{P}(x_{m_i}, x_{n_i-1}, t),$$

and

$$\mathcal{P}(x_{m_i}, x_{n_i}, t) \leq s\mathcal{P}(x_{m_i}, x_{n_i-1}, t) + s\mathcal{P}(x_{n_i-1}, x_{n_i}, t).$$

Taking the upper limit as $i \to \infty$ in the first inequality above, and using (22) and (25) we get

$$\limsup_{i10\to\infty} \mathcal{P}(x_{m_i+1}, x_{n_i-1}, t) \leq \varepsilon s. \tag{26}$$

Similarly, taking the upper limit as $i \to \infty$ in the second inequality above, and using (22) and (24), we get

$$\limsup_{i\to\infty} \mathcal{P}(x_{m_i}, x_{n_i}, t) \leq \varepsilon s. \tag{27}$$

From (17), we have

$$\begin{aligned}\psi(s\mathcal{P}(x_{m_i+1}, x_{n_i}, t)) &= \psi(s\mathcal{P}(fx_{m_i}, fx_{n_i-1}, t)) \\ &\leq \psi(M_t(x_{m_i}, x_{n_i-1})) - \varphi(M_t(x_{m_i}, x_{n_i-1})) + L\psi(N_t(x_{m_i}, x_{n_i-1})),\end{aligned} \tag{28}$$

where

$$\begin{aligned}
&M_t(x_{m_i}, x_{n_i-1})\\
&= \max\left\{\mathcal{P}(x_{m_i}, x_{n_i-1}, t), \mathcal{P}(x_{m_i}, fx_{m_i}, t), \mathcal{P}(x_{n_i-1}, fx_{n_i-1}, t), \frac{\mathcal{P}(x_{m_i}, fx_{n_i-1}, t) + \mathcal{P}(fx_{m_i}, x_{n_i-1}, t)}{2s}\right\}\\
&= \max\left\{\mathcal{P}(x_{m_i}, x_{n_i-1}, t), \mathcal{P}(x_{m_i}, x_{m_i+1}, t), \mathcal{P}(x_{n_i-1}, x_{n_i}, t), \frac{\mathcal{P}(x_{m_i}, x_{n_i}, t) + \mathcal{P}(x_{m_i+1}, x_{n_i-1}, t)}{2s}\right\}, \qquad (29)
\end{aligned}$$

and

$$\begin{aligned}
N_t(x_{m_i}, x_{n_i-1}) &= \min\left\{\mathcal{P}(x_{m_i}, fx_{m_i}, t), \mathcal{P}(x_{m_i}, fx_{n_i-1}, t), \mathcal{P}(x_{n_i-1}, fx_{m_i}, t), \mathcal{P}(x_{n_i-1}, fx_{n_i-1}, t)\right\}\\
&= \min\left\{\mathcal{P}(x_{m_i}, x_{m_i+1}, t), \mathcal{P}(x_{m_i}, x_{n_i}, t), \mathcal{P}(x_{n_i-1}, x_{m_i+1}, t), \mathcal{P}(x_{n_i-1}, x_{n_i}, t)\right\}. \qquad (30)
\end{aligned}$$

Taking the upper limit as $i \to \infty$ in (29) and (30) and using (22), (25), (26) and (27), we get

$$\begin{aligned}
\limsup_{i\to\infty} M_t(x_{m_i-1}, x_{n_i-1}) &= \max\Bigg\{\limsup_{i\to\infty} \mathcal{P}(x_{m_i}, x_{n_i-1}, t), 0, 0,\\
&\qquad \frac{\limsup_{i\to\infty} \mathcal{P}(x_{m_i}, x_{n_i}, t) + \limsup_{n\to\infty} \mathcal{P}(x_{m_i+1}, x_{n_i-1}, t)}{2s}\Bigg\}\\
&\le \max\left\{\varepsilon, \frac{\varepsilon s + \varepsilon s}{2s}\right\} = \varepsilon. \qquad (31)
\end{aligned}$$

So, we have

$$\limsup_{i\to\infty} M_t(x_{m_i-1}, x_{n_i-1}) \le \varepsilon, \qquad (32)$$

and

$$\limsup_{i\to\infty} N_t(x_{m_i}, x_{n_i-1}) = 0. \qquad (33)$$

Now, taking the upper limit as $i \to \infty$ in (28) and using (23), (32) and (33) we have

$$\begin{aligned}
\psi\left(s \cdot \frac{\varepsilon}{s}\right) &\le \psi\big(s \limsup_{i\to\infty} \mathcal{P}(x_{m_i+1}, x_{n_i}, t)\big)\\
&\le \psi\big(\limsup_{i\to\infty} M_t(x_{m_i}, x_{n_i-1}, t)\big) - \liminf_{i\to\infty} \varphi(M_t(x_{m_i}, x_{n_i-1}))\\
&\le \psi(\varepsilon) - \varphi\big(\liminf_{i\to\infty} M_t(x_{m_i}, x_{n_i-1})\big),
\end{aligned}$$

which further implies that

$$\varphi\big(\liminf_{i\to\infty} M_t(x_{m_i}, x_{n_i-1})\big) = 0,$$

so $\liminf_{i\to\infty} M_t(x_{m_i}, x_{n_i-1}) = 0$, a contradiction to (31). Thus, $\{x_{n+1} = fx_n\}$ is a Cauchy sequence in X.

As X is a complete space, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = u.$$

Now, suppose that f is continuous. Using the triangular inequality, we get

$$\mathcal{P}(u, fu, t) \le s\mathcal{P}(u, fx_n, t) + s\mathcal{P}(fx_n, fu, t).$$

Letting $n \to \infty$, we get

$$\mathcal{P}(u, fu, t) \le s \lim_{n\to\infty} \mathcal{P}(u, fx_n, t) + s \lim_{n\to\infty} \mathcal{P}(fx_n, fu, t).$$

So, we have $fu = u$. Thus, u is a fixed point of f. □

Note that the continuity of f in Theorem 2.11 is not necessary and can be dropped.

Theorem 2.12. *Under the hypotheses of Theorem 2.11, without the continuity assumption on f, assume that whenever $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x \in X$, one has $x_n \preceq x$, for all $n \in \mathbb{N}$. Then f has a fixed point in X.*

Proof. Following similar arguments to those given in the proof of Theorem 2.11, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to u$, for some $u \in X$. Using the assumption on X, we have that $x_n \preceq u$, for all $n \in \mathbb{N}$. Now, we show that $fu = u$. By (17), we have

$$\begin{aligned}\psi(s\mathcal{P}(x_{n+1}, fu, t)) &= \psi(s\mathcal{P}(fx_n, fu, t)) \\ &\leq \psi(M_t(x_n, u)) - \varphi(M_t(x_n, u)) + L\psi(N_t(x_n, u)),\end{aligned} \tag{34}$$

where

$$\begin{aligned}M_t(x_n, u) &= \max\left\{\mathcal{P}(x_n, u, t), \mathcal{P}(x_n, fx_n, t), \mathcal{P}(u, fu, t), \frac{\mathcal{P}(x_n, fu, t) + \mathcal{P}(fx_n, u, t)}{2s}\right\} \\ &= \max\left\{\mathcal{P}(x_n, u, t), \mathcal{P}(x_n, x_{n+1}, t), \mathcal{P}(u, fu, t), \frac{\mathcal{P}(x_n, fu, t) + \mathcal{P}(x_{n+1}, u, t)}{2s}\right\}\end{aligned} \tag{35}$$

and

$$\begin{aligned}N_t(x_n, u) &= \min\{\mathcal{P}(x_n, fx_n, t), \mathcal{P}(x_n, fu, t), \mathcal{P}(u, fx_n, t), \mathcal{P}(u, fu, t)\} \\ &= \min\{\mathcal{P}(x_n, x_{n+1}, t), \mathcal{P}(x_n, fu, t), \mathcal{P}(u, x_{n+1}, t), \mathcal{P}(u, fu, t)\}.\end{aligned} \tag{36}$$

Letting $n \to \infty$ in (35) and (36) and using Lemma 1.8, we get

$$\frac{\frac{1}{s}\mathcal{P}(u, fu, t)}{2s} \liminf_{n\to\infty} M_t(x_n, u) \leq \limsup_{n\to\infty} M_t(x_n, u) \leq \max\left\{\mathcal{P}(u, fu, t), \frac{s\mathcal{P}(u, fu, t)}{2s}\right\} = \mathcal{P}(u, fu, t), \tag{37}$$

and

$$N_t(x_n, u) \to 0.$$

Again, taking the upper limit as $i \to \infty$ in (34) and using Lemma 1.8 and (37) we get

$$\begin{aligned}\psi(\mathcal{P}(u, fu, t) &= \psi(s \cdot \frac{1}{s}\mathcal{P}(u, fu, t)) \leq \psi\big(s \limsup_{n\to\infty} \mathcal{P}(x_{n+1}, fu, t)\big) \\ &\leq \psi\big(\limsup_{n\to\infty} M_t(x_n, u)\big) - \liminf_{n\to\infty} \varphi(M_t(x_n, u)) \\ &\leq \psi(\mathcal{P}(u, fu, t)) - \varphi\big(\liminf_{n\to\infty} M_t(x_n, u)\big).\end{aligned}$$

Therefore, $\varphi\big(\liminf_{n\to\infty} M_t(x_n, u)\big) \leq 0$, equivalently, $\liminf_{n\to\infty} M_t(x_n, u) = 0$. Thus, from (37) we get $u = fu$ and hence u is a fixed point of f. □

Corollary 2.13. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-parametric metric $\mathcal{P}$ on X such that $(X, \mathcal{P})$ is a complete parametric b-metric space. Let $f : X \to X$ be a non-decreasing continuous mapping with respect to $\preceq$. Suppose that there exist $k \in [0, 1)$ and $L \geq 0$ such that*

$$\begin{aligned}\mathcal{P}(fx, fy, t) \leq \frac{k}{s} &\max\left\{\mathcal{P}(x, y, t), \mathcal{P}(x, fx, t), \mathcal{P}(y, fy, t), \frac{\mathcal{P}(x, fy, t) + \mathcal{P}(y, fx, t)}{2s}\right\} \\ &+ \frac{L}{s}\min\{\mathcal{P}(x, fx, t), \mathcal{P}(y, fx, t)\},\end{aligned}$$

for all comparable elements $x, y \in X$ and all $t > 0$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. Follows from Theorem 2.11 by taking $\psi(t) = t$ and $\varphi(t) = (1 - k)t$, for all $t \in [0, +\infty)$. □

Corollary 2.14. *Under the hypotheses of Corollary 2.13, without the continuity assumption of f, let for any non-decreasing sequence $\{x_n\}$ in X such that $x_n \to x \in X$ we have $x_n \preceq x$, for all $n \in \mathbb{N}$. Then, f has a fixed point in X.*

3. Fuzzy b-metric spaces

In 1988, Grabiec [14] defined contractive mappings on a fuzzy metric space and extended fixed point theorems of Banach and Edelstein in such spaces. Successively, George and Veeramani [11] slightly modified the notion of a fuzzy metric space introduced by Kramosil and Michálek and then defined a Hausdorff and first countable topology on it. Since then, the notion of a complete fuzzy metric space presented by George and Veeramani has emerged as another characterization of completeness, and many fixed point theorems have also been proved (see for more details [9, 3, 13, 16, 23, 18] and the references therein). In this section we develop an important relation between parametric b-metric and fuzzy b-metric and deduce certain new fixed point results in triangular partially ordered fuzzy b-metric space.

Definition 3.1. (Schweizer and Sklar [26]) A binary operation $\star : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if it satisfies the following assertions:

(T1) $\star$ is commutative and associative;

(T2) $\star$ is continuous;

(T3) $a \star 1 = a$ for all $a \in [0, 1]$;

(T4) $a \star b \leq c \star d$ when $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0, 1]$.

Definition 3.2. A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $t, s > 0$,

(i) $M(x, y, t) > 0$;

(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(iii) $M(x, y, t) = M(y, x, t)$;

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(v) $M(x, y, .) : (0, \infty) \to [0, 1]$ is continuous;

The function $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Definition 3.3. A fuzzy b-metric space is an ordered triple $(X, B, \star)$ such that X is a nonempty set, $\star$ is a continuous t-norm and B is a fuzzy set on $X \times X \times (0, +\infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $t, s > 0$:

(F1) $B(x, y, t) > 0$;

(F2) $B(x, y, t) = 1$ if and only if $x = y$;

(F3) $B(x, y, t) = B(y, x, t)$;

(F4) $B(x, y, t) \star B(y, z, s) \leq B(x, z, b(t + s))$ where $b \geq 1$;

(F5) $B(x, y, \cdot) : (0, +\infty) \to (0, 1]$ is left continuous.

Definition 3.4. Let $(X, B, \star)$ be a fuzzy b-metric space. Then

(i) a sequence $\{x_n\}$ converges to $x \in X$, if and only if $\lim_{n\to+\infty} B(x_n, x, t) = 1$ for all $t > 0$;

(ii) a sequence $\{x_n\}$ in X is a Cauchy sequence if and only if for all $\epsilon \in (0, 1)$ and $t > 0$, there exists n_0 such that $B(x_n, x_m, t) > 1 - \epsilon$ for all $m, n \geq n_0$;

(iii) the fuzzy b-metric space is called complete if every Cauchy sequence converges to some $x \in X$.

Definition 3.5. Let $(X, B, *, b)$ be a fuzzy b-metric space. The fuzzy b-metric B is called triangular whenever,

$$\frac{1}{B(x, y, t)} - 1 \leq b\Big[\frac{1}{B(x, z, t)} - 1 + \frac{1}{B(z, y, t)} - 1)\Big]$$

for all $x, y, z \in X$ and all $t > 0$.

Example 3.6. Let (X, d, s) be a b-metric space. Define $B : X \times X \times (0, \infty) \to [0, \infty)$ by $B(x, y, t) = \frac{t}{t+d(x,y)}$. Also suppose $a * b = \min\{a, b\}$. Then $(X, B, *)$ is a fuzzy b-metric spaces with constant $b = s$. Further B is a triangular fuzzy B-metric.

Remark 3.7. Notice that $\mathcal{P}(x, y, t) = \frac{1}{B(x,y,t)} - 1$ is a parametric b-metric whenever B is a triangular fuzzy b-metric.

As an applications of Remark 3.7 and the results established in section 2, we can deduce the following results in ordered fuzzy b-metric spaces.

Theorem 3.8. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a triangular fuzzy b-metric B on X such that $(X, B, *, b)$ is a complete fuzzy b-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$b\Big[\frac{1}{B(fx, fy, t))} - 1\Big] \leq \beta\Big(\frac{1}{B(x, y, t)} - 1\Big)\mathcal{M}(x, y, t) \tag{38}$$

for all $t > 0$ and for all comparable elements $x, y \in X$, where

$$\mathcal{M}(x, y, t) = \max\left\{\frac{1}{B(x, y, t)} - 1, \frac{[\frac{1}{B(x,fx,t)} - 1][\frac{1}{B(y,fy,t)} - 1]}{\frac{1}{B(fx,fy,t)}}, \frac{[\frac{1}{B(x,fx,t)} - 1][\frac{1}{B(y,fy,t)} - 1]}{\frac{1}{B(x,y,t)}}\right\}$$

If f is continuous, then f has a fixed point.

Theorem 3.9. *Under the hypotheses of Theorem 3.8, without the continuity assumption on f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$. Then f has a fixed point.*

Theorem 3.10. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a triangular fuzzy b-metric B on X such that $(X, B, *, b)$ is a complete fuzzy b-metric space. Let $f : X \to X$ be a continuous non-decreasing mapping with respect to $\preceq$. Also suppose that there exist $L \geq 0$ and two altering distance functions ψ and φ such that*

$$\psi\Big(b\Big[\frac{1}{B(fx, fy, t))} - 1\Big]\Big) \leq \psi(\mathcal{M}_t(x, y)) - \varphi(\mathcal{M}_t(x, y)) + L\psi(\mathcal{N}_t(x, y))$$

for all comparable elements $x, y \in X$ where,

$$\mathcal{M}_t(x, y) = \max\left\{\frac{1}{B(x, y, t)} - 1, \frac{1}{B(x, fx, t)} - 1, \frac{1}{B(y, fy, t)} - 1, \frac{1}{2b}\Big[\frac{1}{B(x, fy, t)} + \frac{1}{B(y, fx, t)} - 2\Big]\right\}$$

and

$$\mathcal{N}_t(x,y) = \min\{\frac{1}{B(x,fx,t)} - 1, \frac{1}{B(y,fy,t)} - 1, \frac{1}{B(y,fx,t)} - 1, \frac{1}{B(x,fy,t)} - 1\}.$$

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Theorem 3.11. *Under the hypotheses of Theorem 3.10, without the continuity assumption on f, assume that whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$. Then f has a fixed point.*

Theorem 3.12. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a triangular fuzzy b-metric B on X such that $(X, B, *, b)$ is a complete fuzzy b-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq fx_0$. Suppose that*

$$b[\frac{1}{B(fx,fy,t))} - 1] \le \psi(\mathcal{N}(x,y,t)) \tag{39}$$

where

$$\begin{aligned}\mathcal{N}(x,y,t) \quad = \quad & \max\Big\{\frac{1}{B(x,y,t))} - 1, \frac{[\frac{1}{B(x,fx,t)} - 1][\frac{1}{B(x,fy,t)} - 1] + [\frac{1}{B(y,fy,t)} - 1][\frac{1}{B(y,fx,t)} - 1]}{1 + b[\frac{1}{B(x,fx,t)} + \frac{1}{B(y,fy,t)} - 2]}, \\ & \frac{[\frac{1}{B(x,fx,t)} - 1][\frac{1}{B(x,fy,t)} - 1] + [\frac{1}{B(y,fy,t)} - 1][\frac{1}{B(y,fx,t)} - 1]}{\frac{1}{B(x,fy,t)} + \frac{1}{B(y,fx,t)} - 1}\end{aligned}$$

for some $\psi \in \Psi$ and for all comparable elements $x, y \in X$ and all $t > 0$. If f is continuous, then f has a fixed point.

4. Application to existence of solutions of integral equations

Let $X = C([0,T], \mathbb{R})$ be the set of real continuous functions defined on $[0,T]$ and $\mathcal{P} : X \times X \times (0,\infty) \to [0,+\infty)$ be defined by $\mathcal{P}(x,y,\alpha) = \sup_{t\in[0,T]} e^{-\alpha t}|x(t) - y(t)|^2$ for all $x, y \in X$ and all $t > 0$. Then $(X, \mathcal{P}, 2)$ is a complete parametric $b-$metric space. Let $\preceq$ be the partial order on X defined by $x \preceq y$ if and only if $x(t) \le y(t)$ for all $t \in [0,T]$. Then $(X, d_\alpha, \preceq)$ is a complete partially ordered metric space. Consider the following integral equation

$$x(t) = p(t) + \int_0^T S(t,s)f(s,x(s))ds \tag{40}$$

where

(A) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous,

(B) $p : [0,T] \to \mathbb{R}$ is continuous,

(C) $S : [0,T] \times [0,T] \to [0,+\infty)$ is continuous and

$$\sup_{t\in[0,T]} e^{-\alpha t}(\int_0^T S(t,s)ds)^2 \le 1,$$

(D) there exist $k \in [0,1)$ and $L \ge 0$ such that

$$\begin{aligned}0 \le f(s,y(s)) - f(s,x(s)) \le \Big(&\frac{ke^{-\alpha s}}{2} \max\Big\{|x(s) - y(s)|, |x(s) - Hx(s)|, |y(s) - Hy(s)|, \\ &\frac{|x(s) - Hy(s)| + |y(s) - Hx(s)|}{4}\Big\} \\ &+ \frac{Le^{-\alpha s}}{2} \min\{|x(s), Hx(s)|, |y(s) - Hx(s)|\}\Big)^{\frac{1}{2}}\end{aligned}$$

for all $x, y \in X$ with $x \preceq y$, $s \in [0,T]$ and $\alpha > 0$ where

$$Hx(t) = p(t) + \int_0^T S(t,s)f(s,x(s))ds, \quad t \in [0,T], \quad \text{for all} \quad x \in X.$$

(E) there exist $x_0 \in X$ such that

$$x_0(t) \leq p(t) + \int_0^T S(t,s)f(s,x_0(s))ds.$$

We have the following result of existence of solutions for integral equations.

Theorem 4.1. *Under assumptions* $(A) - (E)$*, the integral equation* (40) *has a unique solution in* $X = C([0,T],R)$.

Proof. Let $H : X \to X$ be defined by

$$Hx(t) = p(t) + \int_0^T S(t,s)f(s,x(s))ds, \quad t \in [0,T], \quad \text{for all} \quad x \in X.$$

First, we will prove that H is a non-decreasing mapping with respect to $\preceq$. Let $x \preceq y$ then by (D) we have $0 \leq f(s,y(s)) - f(s,x(s))$ for all $s \in [0,T]$. On the other hand by definition of H we have

$$Hy - Hx = \int_0^T S(t,s)[f(s,y(s)) - f(s,x(s))]ds \geq 0 \quad \text{for all} \quad t \in [0,T].$$

Then $Hx \preceq Hy$, that is, H is a non-decreasing mapping with respect to $\preceq$. Now suppose that $x, y \in X$ with $x \preceq y$. Then by (C), (D) and the definition of H we get

$$\begin{aligned}
\mathcal{P}(Hx,Hy,\alpha) &= \sup_{t\in[0,T]} e^{-\alpha t}|Hx(t) - Hy(t)|^2 \\
&= \sup_{t\in[0,T]} e^{-\alpha t}\Big|\int_0^T S(t,s)[f(s,x(s)) - f(s,y(s))]ds\Big|^2 \\
&\leq \sup_{t\in[0,T]} e^{-\alpha t}\Big(\int_0^T S(t,s)|f(s,x(s)) - f(s,y(s))|ds\Big)^2 \\
&\leq \sup_{t\in[0,T]} e^{-\alpha t}\Big(\int_0^T S(t,s)\Big(\frac{ke^{-\alpha s}}{2}\max\Big\{|x(s)-y(s)|, \\
&\qquad |x(s)-Hx(s)|, |y(s)-Hy(s)|, \frac{|x(s)-Hy(s)| + |y(s)-Hx(s)|}{4}\Big\} \\
&\qquad + \frac{Le^{-\alpha s}}{2}\min\{|x(s), Hx(s)|, |y(s)-Hx(s)|\}\Big)^{\frac{1}{2}}ds\Big)^2 \\
&\leq \sup_{t\in[0,T]} e^{-\alpha t}\Big(\int_0^T S(t,s)\Big(\frac{k}{2}\max\Big\{\sup_{s\in[0,T]} e^{-\alpha s}|x(s)-y(s)|, \sup_{s\in[0,T]} e^{-\alpha s}|x(s)-Hx(s)|, \\
&\qquad \sup_{s\in[0,T]} e^{-\alpha s}|y(s)-Hy(s)|, \frac{\sup_{s\in[0,T]} e^{-\alpha s}|x(s)-Hy(s)| + \sup_{s\in[0,T]} e^{-\alpha s}|y(s)-Hx(s)|}{4}\Big\} \\
&\qquad + \frac{L}{2}\min\{\sup_{s\in[0,T]} e^{-\alpha s}|x(s), Hx(s)|, \sup_{s\in[0,T]} e^{-\alpha s}|y(s)-Hx(s)|\}\Big)^{\frac{1}{2}}ds\Big)^2 \\
&= \sup_{t\in[0,T]} e^{-\alpha t}\Big(\int_0^T S(t,s)\Big(\frac{k}{2}\max\Big\{\mathcal{P}(x,y,\alpha), \mathcal{P}(x,Hx,\alpha), \mathcal{P}(y,Hy,\alpha), \\
&\qquad \frac{\mathcal{P}(x,Hy,\alpha) + \mathcal{P}(y,Hx,\alpha)}{4}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{L}{2}\min\{\mathcal{P}(x,Hx,\alpha),\mathcal{P}(y,Hx,\alpha)\}\Big)^{\frac{1}{2}}ds\Big)^2\\
=\;&\Big(\sup_{t\in[0,T]} e^{-\alpha t}\big(\int_0^T S(t,s)ds\big)^2\Big)\Big(\frac{k}{2}\max\Big\{\mathcal{P}(x,y,\alpha),\mathcal{P}(x,Hx,\alpha),\\
&\mathcal{P}(y,Hy,\alpha),\frac{\mathcal{P}(x,Hy,\alpha)+\mathcal{P}(y,Hx,\alpha)}{4}\Big\}+\frac{L}{2}\min\{\mathcal{P}(x,Hx,\alpha),\mathcal{P}(y,Hx,\alpha)\}\Big)\\
\le\;&\frac{k}{2}\max\Big\{\mathcal{P}(x,y,\alpha),\mathcal{P}(x,Hx,\alpha),\mathcal{P}(y,Hy,\alpha),\frac{\mathcal{P}(x,Hy,\alpha)+\mathcal{P}(y,Hx,\alpha)}{4}\Big\}\\
&+\frac{L}{2}\min\{\mathcal{P}(x,Hx,\alpha),\mathcal{P}(y,Hx,\alpha)\}
\end{aligned}
$$

Now, by (E) there exists $x_0 \in X$ such that $x_0 \preceq Hx_0$. Then, the conditions of Corollary 2.13 are satisfied and hence the integral equation (40) has a unique solution in $X = C([0,T],\mathbb{R})$. □

Acknowledgement

This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors acknowledge with thanks DSR, KAU for financial support.

References

[1] R. P. Agarwal, N. Hussain, M. A. Taoudi, *Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations*, Abstr. Appl. Anal., **2012** (2012), 15 pages. 2.1

[2] M. A. Alghamdi, N. Hussain, P. Salimi, *Fixed point and coupled fixed point theorems on b-metric-like spaces*, J. Inequal. Appl., **2013** (2013), 25 pages. 1

[3] I. Altun, D. Turkoglu, *Some fixed point theorems on fuzzy metric spaces with implicit relations*, Commun. Korean Math. Soc., **23** (2008), 111–124. 3

[4] G. V. R. Babu, M. L. Sandhya, M. V. R. Kameswari, *A note on a fixed point theorem of Berinde on weak contractions*, Carpathian J. Math., **24** (2008), 8–12. 2.3

[5] V. Berinde, *General contrcactive fixed point theorems for Ćirić-type almost contraction in metric spaces*, Carpathian J. Math., **24** (2008), 10–19. 2.3

[6] L. Ćirić, M. Abbas, R. Saadati, N. Hussain, *Common fixed points of almost generalized contractive mappings in ordered metric spaces*, Appl. Math. Comput., **217** (2011), 5784–5789. 2.3

[7] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inf. Univ. Ostravensis, **1** (1993), 5–11. 1, 1.1

[8] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric spaces*, Atti Sem. Mat. Fis. Univ. Modena, **46** (1998), 263–276. 1

[9] C. Di Bari, C. Vetro, *A fixed point theorem for a family of mappings in a fuzzy metric space*, Rend. Circ. Math. Palermo, **52** (2003), 315–321. 3

[10] D. Đukić, Z. Kadelburg, S. Radenović, *Fixed points of Geraghty-type mappings in various generalized metric spaces*, Abstr. Appl. Anal., **2011** (2011), 13 pages. 2.1

[11] A. George, P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, **64** (1994), 395–399. 3

[12] M. A. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc., **40** (1973), 604–608. 2.1

[13] D. Gopal, M. Imdad, C. Vetro, M. Hasan, *Fixed point theory for cyclic weak ϕ-contraction in fuzzy metric space*, J. Nonlinear Anal. Appl., **2012** (2012), 11 pages. 3

[14] M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems, **27** (1988), 385–389. 3

[15] N. Hussain, S. Al-Mezel, P. Salimi, *Fixed points for ψ-graphic contractions with application to integral equations*, Abstr. Appl. Anal., **2013** (2013), 11 pages. 2.1

[16] N. Hussain, S. Khaleghizadeh, P. Salimi, A. A. N. Abdou, *A new approach to fixed point results in triangular intuitionistic fuzzy metric spaces*, Abstr. Appl. Anal., **2014** (2014), 16 pages. 1, 3

[17] N. Hussain, V. Parvaneh, J. R. Roshan, Z. Kadelburg, *Fixed points of cyclic weakly (ψ,φ,L,A,B)-contractive mappings in ordered b-metric spaces with applications*, Fixed Point Theory Appl., **2013** (2013), 18pages. 1, 1

[18] N. Hussain, P. Salimi, *Implicit contractive mappings in Modular metric and Fuzzy Metric Spaces*, The Sci. World J., **2014** (2014), 13 pages. 3

[19] N. Hussain, M. H. Shah, *KKM mappings in cone b-metric spaces*, Comput. Math. Appl., **62** (2011), 1677–1684. 1

[20] N. Hussain, M. A. Taoudi, *Krasnosel'skii-type fixed point theorems with applications to Volterra integral equations*, Fixed Point Theory Appl., **2013** (2013), 16 pages. 2.1

[21] M. A. Khamsi, N. Hussain, *KKM mappings in metric type spaces*, Nonlinear Anal., **73** (2010), 3123–3129. 1

[22] M. S. Khan, M. Swaleh, S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Austral. Math. Soc., **30** (1984), 1–9. 2.3

[23] M. A. Kutbi, J. Ahmad, A. Azam, N. Hussain, *On fuzzy fixed points for fuzzy maps with generalized weak property*, J. Appl. Math., **2014** (2014), 12 pages. 3

[24] J. J. Nieto, R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22** (2005), 223-229. 2.1

[25] J. R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei, W. Shatanawi, *Common fixed points of almost generalized* $(\psi,\varphi)_s$*-contractive mappings in ordered b-metric spaces*, Fixed Point Theory Appl., **2013** (2013), 23 pages. 2.3

[26] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Math., **10** (1960), 314–334. 3.1