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Generalized fractional integrals involving product of multivariable H-function and a general class of polynomials

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Abstract

A large number of fractional integral formulas involving certain special functions and polynomials have been presented. Here, in this paper, we aim at establishing two fractional integral formulas involving the products of the multivariable H-function and a general class of polynomials by using generalized fractional integration operators given by Saigo and Maeda [M. Saigo, N. Maeda, Varna, Bulgaria, (1996), 386–400]. All the results derived here being of general character, they are seen to yield a number of results (known and new) regarding fractional integrals.

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1. Introduction

Fractional calculus which are derivatives and integrals of arbitrary (real and complex) orders have found many applications in a variety of fields ranging from natural science to social science. In recent years, it has turned out that many phenomena in engineering, physics, chemistry and other sciences can be described very successfully by means of models using mathematical tools deduced from fractional calculus. For example, the

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nonlinear oscillation of earthquakes can be modeled with fractional derivatives and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow. Fractional derivatives are also used in modeling many chemical processes, mathematical biology and many other problems in physics and engineering (see, *e.g.*, [6]–[4], [18], [19], [35]).

Under various fractional calculus operators, the computations of image formulas for special functions of one or more variables are important from the point of view of the usefulness of these results in the evaluation of generalized integrals and the solution of differential and integral equations (see, *e.g.*, [2], [3],[16], [20], [21], [22], [27] and [38] and so on). Motivated essentially by diverse applications of fractional calculus we establish two image formulas for the product of multivariable H-function and general class of polynomials involving left and right sided fractional integral operators of Saigo-Meada [25]. By virtue of the unified nature of our results, a large number of new and known results involving Saigo, Riemann-Liouville and Erdélyi-Kober fractional integral operators and several special functions are shown to follow as special cases of our main results.

The generalized fractional integral operators of arbitrary order involving Appell function F_3 in the kernel were defined and investigated by Saigo and Maeda [25, p. 393, Eqs. (4.12) and (4.13)] as follows:

Let $x > 0$, $\Re(\gamma) > 0$, and $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$. Then

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-t/x,1-x/t\right) f(t)\, dt \tag{1.1}$$

and

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-x/t,1-t/x\right) f(t)\, dt. \tag{1.2}$$

The operators (1.1) and (1.2) are known to reduce to the fractional integral operators introduced by Saigo [24] as follows:

$$I_{0,x}^{\alpha,0,\beta,\beta',\gamma} f(x) = I_{0,x}^{\gamma,\alpha-\gamma,-\beta} f(x) \quad (\gamma \in \mathbb{C}) \tag{1.3}$$

and

$$I_{x,\infty}^{\alpha,0,\beta,\beta',\gamma} f(x) = I_{x,\infty}^{\gamma,\alpha-\gamma,-\beta} f(x) \quad (\gamma \in \mathbb{C}). \tag{1.4}$$

The multivariable H-Function, introduced by Srivastava and Panda (see, [33] and [34]), is an extension of the multivariable G-function. This multivariable H-Function includes Fox's H-function, Meijer's G-function, the generalized Lauricella function of Srivastava and Daoust (see [31]), Apell function, the Whittaker function and so on. The multivariable H-function is defined and represented in the following manner:

$$\begin{aligned} H\,[x_1,\ldots,x_r] &= H_{p,q:\{p_r,q_r\}}^{0,n:\{m_r,n_r\}} \left[\begin{array}{c} x_1 \\ \vdots \\ x_r \end{array} \left| \begin{array}{l} \left(a_j;\alpha'_j,\ldots,\alpha_j^{(r)}\right)_{1,p} : \left\{\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}\right\} \\ \left(b_j;\beta'_j,\ldots,\beta_j^{(r)}\right)_{1,q} : \left\{\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\right\} \end{array} \right. \right] \\ &= \frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} \phi(\xi_1,\ldots,\xi_r) \prod_{i=1}^r \left(\theta_i(\xi_i)\, x_i^{\xi_i}\, d\xi_i\right), \end{aligned} \tag{1.5}$$

where

$$\phi\,(\xi_1,\ldots\xi_r) = \frac{\prod\limits_{j=1}^{n} \Gamma\left(1-a_j+\sum\limits_{i=1}^{r}\alpha_j^{(i)}\xi_i\right)}{\prod\limits_{j=n+1}^{p} \Gamma\left(a_j-\sum\limits_{i=1}^{r}\alpha_j^{(i)}\xi_i\right) \prod\limits_{j=1}^{q} \Gamma\left(1-b_j+\sum\limits_{i=1}^{r}\beta_j^{(i)}\xi_i\right)} \tag{1.6}$$

and

$$\theta_i\,(\xi_i) = \frac{\prod\limits_{j=1}^{n_i} \Gamma\left(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i\right) \prod\limits_{j=1}^{m_i} \Gamma\left(d_j^{(i)}-\delta_j^{(i)}\xi_i\right)}{\prod\limits_{j=n_i+1}^{p_i} \Gamma\left(c_j^{(i)}-\gamma_j^{(i)}\xi_i\right) \prod\limits_{j=m_i+1}^{q_i} \Gamma\left(1-d_j^{(i)}+\delta_j^{(i)}\xi_i\right)} \quad (i \in \{1,2,\ldots,r\}). \tag{1.7}$$

Here, $\{m_r, n_r\}$ stands for $m_1, n_1; ...; m_r, n_r$ and $\left\{\left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,p_r}\right\}$ stands for the sequence of r ordered pairs $\left(c'_j, \gamma'_j\right)_{1,p_1}; ...; \left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,p_r}$.

Remark 1.1. The special case of (1.5) when $r = 2$ reduces to the H-function of two variables.

Also $S_n^m[x]$ occurring in the sequel denotes the general class of polynomials introduced by Srivastava [30]:

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, x^k \quad (m \in \mathbb{N};\ n \in \mathbb{N}_0), \tag{1.8}$$

where $\mathbb{N}$ is the set of positive integers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and the coefficient $A_{n,k}$ $(n,\, k \in \mathbb{N}_0)$ are arbitrary (real or complex) constants. It is noted that, by suitably specializing the coefficients $A_{n,k}$, $S_n^m[x]$ is seen to yield a number of known polynomials (see, *e.g.*, [37, pp. 158-161]).

The following lemma is required in the proof of our main result (see [25]).

Lemma 1.2. *Let* $\alpha,\, \alpha',\, \beta,\, \beta',\, \gamma,\, \rho \in \mathbb{C}$.

If $\Re(\gamma) > 0$, $\Re(\rho) > \max\{0, \Re(\alpha + \alpha' + \beta - \gamma), \Re(\alpha' - \beta')\}$, *then*

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} = x^{\rho-\alpha-\alpha'+\gamma-1} \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)\Gamma(\rho+\beta')}. \tag{1.9}$$

If $\Re(\gamma) > 0$, $\Re(\rho) < 1 + \min\{\Re(-\beta), \Re(\alpha + \alpha' - \gamma), \Re(\alpha + \beta' - \gamma)\}$, *then*

$$I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} = x^{\rho-\alpha-\alpha'+\gamma-1} \frac{\Gamma(1+\alpha+\alpha'-\gamma-\rho)\Gamma(1+\alpha+\beta'-\gamma-\rho)\Gamma(1-\beta-\rho)}{\Gamma(1-\rho)\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\Gamma(1+\alpha-\beta-\rho)}. \tag{1.10}$$

2. Main Results

In this section, we establish two theorems involving the products of multivariable H-function and a general class of polynomials associated with the Saigo-Maeda fractional integral operators.

Theorem 2.1. *Let* $\alpha,\ \alpha',\ \beta,\ \beta',\ \gamma,\ \mu,\ \eta,\ \delta_j,\ \upsilon_i,\ z_i,\ a,\ b,\ c_j \in \mathbb{C}$ *with* $\Re(\gamma) > 0$, *and* $\lambda_j > 0$, $\sigma_i > 0$ $(i \in \{1, \ldots, r\};\ j \in \{1, \ldots, s\})$. *Then we have*

$$\begin{aligned}
&\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}(b-at)^{-\eta}\prod_{j=1}^{s} S_{n_j}^{m_j}\left[c_j\, t^{\lambda_j}(b-at)^{-\delta_j}\right] \times H\left[z_1\, t^{\sigma_1}(b-at)^{-\upsilon_1}, ..., z_r\, t^{\sigma_r}(b-at)^{-\upsilon_r}\right]\right)\right\}(x)\\
&= b^{-\eta} x^{\mu-\alpha-\alpha'+\gamma-1} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1k_1} \cdots (-n_s)_{m_sk_s}}{k_1! \cdots k_s!} A'_{n_1,m_1} \cdots A^{(s)}_{n_s,m_s} c_1^{k_1} \cdots c_s^{k_s} \times b^{-\sum_{j=1}^{s}\delta_j k_j} x^{\sum_{j=1}^{s}\lambda_j k_j}\\
&\quad \times H_{p+4,q+4:\{p_r,q_r\};0,1}^{0,n+4:\{m_r,n_r\};1,0}\left[\begin{array}{c} z_1 \frac{x^{\sigma_1}}{b^{\upsilon_1}} \\ \vdots \\ z_r \frac{x^{\sigma_r}}{b^{\upsilon_r}} \\ -\frac{a}{b}x \end{array}\left|\begin{array}{l} \left(a_j; A'_j, ..., A_j^{(r)}\right)_{1,p}, E_1 : \left\{\left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,p_r}\right\};\ \ — \\ \left(b_j; B'_j, ..., B_j^{(r)}\right)_{1,q}, E_2 : \left\{\left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,q_r}\right\};\ \ (0,1) \end{array}\right.\right],
\end{aligned} \tag{2.1}$$

where, for simplicity, E_1 *and* E_2 *are denoted by the following arrays:*

$$E_1 := \left(1 - \eta - \sum_{j=1}^{s} \delta_j k_j; \upsilon_1, ..., \upsilon_r, 1\right), \left(1 - \mu - \sum_{j=1}^{s} \lambda_j k_j; \sigma_1, ..., \sigma_r, 1\right),$$

$$\left(1-\mu-\gamma+\alpha+\alpha'+\beta-\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,...,\sigma_r,1\right),\left(1-\mu+\alpha'-\beta'-\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,...,\sigma_r,1\right);$$

and

$$E_2:=\left(1-\eta-\sum_{j=1}^{s}\delta_j k_j;\upsilon_1,...,\upsilon_r,0\right),\left(1-\mu-\gamma+\alpha+\alpha'-\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,...,\sigma_r,1\right),$$
$$\left(1-\mu-\gamma+\alpha'+\beta-\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,...,\sigma_r,1\right),\left(1-\mu-\beta'-\sum_{j=1}^{s}\lambda_j k_j;\sigma_1,...,\sigma_r,1\right).$$

The following conditions are further satisfied:

(i) $|\arg z_i|<\frac{\pi}{2}\Omega_i$, $\Omega_i>0$ $(i=1,...,r)$, *where*

$$\Omega_i=\sum_{j=1}^{n}A_j^{(i)}-\sum_{j=n+1}^{p}A_j^{(i)}-\sum_{j=1}^{q}B_j^{(i)}+\sum_{j=1}^{n_i}\gamma_j^{(i)}-\sum_{j=n_i+1}^{p_i}\gamma_j^{(i)}+\sum_{j=1}^{m_i}\delta_j^{(i)}-\sum_{j=m_i+1}^{q_i}\delta_j^{(i)}>0. \tag{2.2}$$

(ii)

$$\min_{\substack{1\le i\le r\\1\le j\le m_i}}\frac{\sigma_i\Re\left(d_j^{(i)}\right)}{\delta_j^{(i)}}>-\Re(\mu)-\min\left\{\Re(\alpha'),\Re(\beta'),\Re(\gamma-\alpha-\beta)\right\}. \tag{2.3}$$

(iii) $\left|\frac{a}{b}x\right|<1$,

$$\Re(\mu)+\sum_{i=1}^{r}\sigma_i\min_{1\le j\le m_i}\frac{\Re\left(d_j^{(i)}\right)}{\delta_j^{(i)}}>\max\left\{0,\Re\left(\alpha+\alpha'+\beta-\gamma\right),\Re\left(\alpha'-\beta'\right)\right\}$$

and

$$\Re(\eta)+\sum_{i=1}^{r}\upsilon_i\min_{1\le j\le m_i}\frac{\Re\left(d_j^{(i)}\right)}{\delta_j^{(i)}}>\max\left\{0,\Re\left(\alpha+\alpha'+\beta-\gamma\right),\Re\left(\alpha'-\beta'\right)\right\}.$$

Proof. Let $\mathcal{L}$ be the left-hand side of (2.1). Using (1.8) and (1.5), and by using the generalized binomial theorem, expanding the term $(b-ax)^{-\gamma}$ as follows:

$$(b-ax)^{-\gamma}=b^{-\gamma}\sum_{s=0}^{\infty}\frac{(\gamma)_s}{s!}\left(\frac{ax}{b}\right)^s\quad\left(\left|\frac{ax}{b}\right|<1\right), \tag{2.4}$$

and interchanging the summations and the integrals (which is guaranteed under the given conditions), after a little simplification, we obtain

$$\mathcal{L}=\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\cdots(-n_s)_{m_sk_s}}{k_1!\dots k_s!}A'_{n_1,m_1}\cdots A^{(s)}_{n_s,m_s}\,c_1^{k_1}\cdots c_s^{k_s}\,(b)^{-\sum_{j=1}^{s}\delta_j k_j}$$
$$\times\ \frac{1}{(2\pi i)^{r+1}}\int_{L_1}\cdots\int_{L_r}\phi(\xi_1,...,\xi_r)\prod_{i=1}^{r}\theta_i(\xi_i)\,z_i^{\xi_i}\ (b)^{\sum_{i=1}^{r}(-\upsilon_i\xi_i)}\,d\xi_1\dots d\xi_r$$

$$\times \int_{L_{r+1}} \frac{\Gamma\left(\eta+\sum\limits_{j=1}^{s}\delta_j k_j+\sum\limits_{i=1}^{r}\upsilon_i\xi_i+\xi_{r+1}\right)}{\Gamma\left(\eta+\sum\limits_{j=1}^{s}\delta_j k_j+\sum\limits_{i=1}^{r}\upsilon_i\xi_i\right)\Gamma\left(1+\xi_{r+1}\right)}\left(\frac{-a}{b}\right)^{\xi_{r+1}} d\xi_{r+1}$$
$$\times \left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\, t^{\mu+\sum\limits_{j=1}^{s}\lambda_j k_j+\sum\limits_{i=1}^{r}\sigma_i\xi_i+\xi_{r+1}-1}\right)(x). \tag{2.5}$$

Then applying (1.9) to the last integral in (2.5) and interpreting the involved Mellin-Barnes contour integrals in terms of the multivariable H-function of $r+1$ variables, we readily obtain the right-hand side of (2.1). □

Setting $n=p=q=0$ in (2.1) is seen to break the multivariable H-function in (1.5) into an r-times product of H-functions and yield an interesting result asserted by the following corollary.

Corollary 2.2. *Let $\alpha,\ \alpha',\ \beta,\ \beta',\ \gamma,\ \mu,\ \eta,\ \delta_j,\ \upsilon_i,\ z_i,\ a,\ b,\ c_j \in \mathbb{C}$ with $\Re(\gamma)>0$, and $\lambda_j>0$, $\sigma_i>0$ $(i\in\{1,\dots,r\};\ j\in\{1,\dots,s\})$. Then we obtain*

$$\Bigg\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\Bigg(t^{\mu-1}(b-at)^{-\eta}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right]$$
$$\times\prod_{i=1}^{r}H_{p_i,q_i}^{m_i,n_i}\left[z_i\, t^{\sigma_i}(b-at)^{-\upsilon_i}\,\middle|\,\begin{matrix}\left(c_j^i,C_j^i\right)_{1,p_i}\\ \left(d_j^i,D_j^i\right)_{1,q_i}\end{matrix}\right]\Bigg)\Bigg\}(x)$$
$$= b^{-\eta}x^{\mu-\alpha-\alpha'+\gamma-1}\times\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\dots(-n_s)_{m_sk_s}}{k_1!\dots k_s!}A'_{n_1,m_1}\dots A^{(s)}_{n_s,m_s}c_1^{k_1}\dots c_s^{k_s}$$
$$\times b^{-\sum\limits_{j=1}^{s}\delta_j k_j}x^{\sum\limits_{j=1}^{s}\lambda_j k_j}H^{0,4:m_i,n_i;1,0}_{4,4:p_i,q_i;0,1}\left[\begin{matrix}z_i\frac{x^{\sigma_i}}{b^{\upsilon_i}}\\ \frac{-a}{b}x\end{matrix}\,\middle|\,\begin{matrix}E_1:\left(c_j^{(i)},C_j^{(r)}\right)_{1,p_i};\text{—}\\ E_2:\left(d_j^{(i)},D_j^{(i)}\right)_{1,q_i};(0,1)\end{matrix}\right], \tag{2.6}$$

where E_1 and E_2 are the same arrays in Theorem 2.1, *and $H_{p,q}^{m,n}(\cdot)$ is the familiar Fox's H-function (see* [15]*).*

It is noted that a further special case of (2.6) when $s=1$ and $r=2$ reduces to the known result given by Baleanu *et al.* [5]. Considering the relation (1.3) we are led to the following formula given in Corollary 2.3, concerning the Saigo fractional integral operator, which is seen to be similar to that in Agarwal [1, p. 587, Eq. 18].

Corollary 2.3. *Let $\alpha,\ \beta,\ \gamma,\ \mu,\ \eta,\ \delta_j,\ \upsilon_i,\ z_i,\ a,\ b,\ c_j\in\mathbb{C}$ with $\Re(\alpha)>0,\lambda_j,\ \sigma_i>0$ $(i\in\{1,\dots,r\};$ $j\in\{1,\dots,s\})$. Then we get*

$$\left\{I_{0+}^{\alpha,\beta,\gamma}\left(t^{\mu-1}(b-at)^{-\eta}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right]\times H\left[z_1\,t^{\sigma_1}(b-at)^{-\upsilon_1},\dots,z_r\,t^{\sigma_r}(b-at)^{-\upsilon_r}\right]\right)\right\}(x)$$
$$= b^{-\eta}x^{\mu-\beta-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\dots(-n_s)_{m_sk_s}}{k_1!\dots k_s!}\times A'_{n_1,m_1}\dots A^{(s)}_{n_s,m_s}c_1^{k_1}\dots c_s^{k_s}\,b^{-\sum\limits_{j=1}^{s}\delta_j k_j}x^{\sum\limits_{j=1}^{s}\lambda_j k_j}$$
$$\times H^{0,n+3:\{m_r,n_r\};1,0}_{p+3,q+3:\{p_r,q_r\};0,1}\left[\begin{matrix}z_1\frac{x^{\sigma_1}}{b^{\upsilon_1}}\\ \vdots\\ z_r\frac{x^{\sigma_r}}{b^{\upsilon_r}}\\ -\frac{a}{b}x\end{matrix}\,\middle|\,\begin{matrix}\left(a_j;A'_j,\dots,A_j^{(r)}\right)_{1,p},E'_1:\left\{\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}\right\};\text{—}\\ \left(b_j;B'_j,\dots,B_j^{(r)}\right)_{1,q},E'_2:\left\{\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\right\};(0,1)\end{matrix}\right], \tag{2.7}$$

where E_1' and E_2' are arrays defined by

$$E_1' := \left(1-\eta-\sum_{j=1}^{s}\delta_j k_j; \upsilon_1, ..., \upsilon_r, 1\right), \left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j; \sigma_1, ..., \sigma_r, 1\right),$$
$$\left(1-\mu-\gamma+\beta-\sum_{j=1}^{s}\lambda_j k_j; \sigma_1, ..., \sigma_r, 1\right);$$

and

$$E_2' := \left(1-\eta-\sum_{j=1}^{s}\delta_j k_j; \upsilon_1, ..., \upsilon_r, 0\right), \left(1-\mu+\beta-\sum_{j=1}^{s}\lambda_j k_j; \sigma_1, ..., \sigma_r, 1\right),$$
$$\left(1-\mu-\alpha-\gamma-\sum_{j=1}^{s}\lambda_j k_j; \sigma_1, ..., \sigma_r, 1\right).$$

The following conditions are further satisfied:

(i) $|\arg z_i| < \frac{\pi}{2}\Omega_i,\ \Omega_i > 0\ (i = 1, \ldots, r)$, *where*

$$\Omega_i = -\sum_{j=n+1}^{p} A_j^{(i)} - \sum_{j=1}^{q} B_j^{(i)} + \sum_{j=1}^{n_i}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i}\gamma_j^{(i)} + \sum_{j=1}^{m_i}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i}\delta_j^{(i)} > 0;$$

(ii) $\left|\frac{a}{b}x\right| < 1$,

$$\Re(\mu) + \sum_{i=1}^{r}\sigma_i \min_{1\le j\le m_i}\frac{\Re\left(d_j^{(i)}\right)}{\delta_j^{(i)}} > \max\{0, \Re(\beta-\gamma)\}$$

and

$$\Re(\eta) + \sum_{i=1}^{r}\upsilon_i \min_{1\le j\le m_i}\frac{\Re\left(d_j^{(i)}\right)}{\delta_j^{(i)}} > \max\{0,\ \Re(\beta-\gamma)\}.$$

It is noted that the corresponding results associated with the Riemann-Liouville and Erdélyi-Kober fractional integral operators can be easily obtained by setting $\beta = -\alpha$ and $\beta = 0$, respectively, in Corollary 2.3.

Theorem 2.4. *Let $\alpha,\ \alpha',\ \beta,\ \beta',\ \gamma,\ \mu,\ \eta,\ \delta_j,\ \upsilon_i,\ z_i,\ a,\ b,\ c_j \in \mathbb{C}$ with $\Re(\gamma) > 0$, and $\lambda_j > 0,\ \sigma_i > 0$ $(i \in \{1, \ldots, r\};\ j \in \{1, \ldots, s\})$. Then we have*

$$\left\{I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}(b-at)^{-\eta}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right] \times H\left[z_1 t^{\sigma_1}(b-at)^{-\upsilon_1}, ..., z_r t^{\sigma_r}(b-at)^{-\upsilon_r}\right]\right)\right\}(x)$$
$$= b^{-\eta}x^{\mu-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\cdots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!} \times A'_{n_1,m_1}\cdots A^{(s)}_{n_s,m_s}c_1^{k_1}\cdots c_s^{k_s}\, b^{-\sum_{j=1}^{s}\delta_j k_j}x^{\sum_{j=1}^{s}\lambda_j k_j}$$
$$\times H^{0,n+4:\{m_r,n_r\};1,0}_{p+4,q+4:\{p_r,q_r\};0,1}\left[\begin{array}{c} z_1\frac{x^{\sigma_1}}{b^{\upsilon_1}} \\ \vdots \\ z_r\frac{x^{\sigma_r}}{b^{\upsilon_r}} \\ -\frac{a}{b}x \end{array}\left|\begin{array}{l} \left(a_j; A'_j, ..., A_j^{(r)}\right)_{1,p}, F_1 : \left\{\left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,p_r}\right\}; — \\ \left(b_j; B'_j, ..., B_j^{(r)}\right)_{1,q}, F_2 : \left\{\left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,q_r}\right\}; (0,1) \end{array}\right.\right], \tag{2.8}$$

where F_1 and F_2 are arrays defined by

$$F_1 := \left(1-\eta-\sum_{j=1}^{s}\delta_j k_j; \upsilon_1, ..., \upsilon_r, 1\right), \left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j; \sigma_1, ..., \sigma_r, 1\right),$$
$$\left(1+\alpha+\alpha'+\beta'-\gamma-\mu-\sum_{j=1}^{s}\lambda_j k_j; \sigma_1, ..., \sigma_r, 1\right), \left(1+\alpha-\beta-\mu-\sum_{j=1}^{s}\lambda_j k_j; \sigma_1, ..., \sigma_r, 1\right);$$

and

$$F_2 := \left(1-\eta-\sum_{j=1}^{s}\delta_j k_j; \upsilon_1, ..., \upsilon_r, 0\right), \left(1+\alpha+\alpha'-\gamma-\mu-\sum_{j=1}^{s}\lambda_j k_j; \sigma_1, ..., \sigma_r, 1\right),$$
$$\left(1+\alpha+\beta'-\gamma-\mu-\sum_{j=1}^{s}\lambda_j k_j; \sigma_1, ..., \sigma_r, 1\right), \left(1-\beta-\mu-\sum_{j=1}^{s}\lambda_j k_j; \sigma_1, ..., \sigma_r, 1\right),$$

$$\Re(\mu) - \sum_{i=1}^{r}\sigma_i \min_{1\le j\le m_i}\left[\frac{\Re\left(d_j^{(i)}\right)}{\delta_j^{(i)}}\right] < 1 + \min\left\{\Re(-\beta), \Re\left(\alpha+\alpha'-\gamma\right), \Re\left(\alpha+\beta'-\gamma\right)\right\},$$
$$\Re(\eta) - \sum_{i=1}^{r}\upsilon_i \min_{1\le j\le m_i}\left[\frac{\Re\left(d_j^{(i)}\right)}{\delta_j^{(i)}}\right] < 1 + \min\left\{\Re(-\beta), \Re\left(\alpha+\alpha'-\gamma\right), \Re\left(\alpha+\beta'-\gamma\right)\right\},$$

and the conditions (i)-(iii) *in* Theorem 2.1 *are also satisfied.*

Proof. The same argument as in the proof of Theorem 2.1 will establish the result in Theorem 2.4. So its proof details are omitted. □

Setting $n = p = q = 0$ in the result in Theorem 2.4, we obtain the following formula.

Corollary 2.5. *Let α, α', β, β', γ, μ, η, δ_j, υ_i, z_i, a, b, $c_j \in \mathbb{C}$ with $\Re(\gamma) > 0$, and $\lambda_j > 0$, $\sigma_i > 0$ $(i \in \{1, \ldots, r\};\ j \in \{1, \ldots, s\})$. Then we get*

$$\left\{I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}(b-at)^{-\eta}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right]\right.\right.$$
$$\left.\left.\times \prod_{i=1}^{r}H_{p_i,q_i}^{m_i,n_i}\left[z_i\, t^{\sigma_i}(b-at)^{-\upsilon_i}\left|\begin{matrix}\left(c_j^i, C_j^i\right)_{1,p_i}\\ \left(d_j^i, D_j^i\right)_{1,q_i}\end{matrix}\right.\right]\right)\right\}(x)$$
$$= b^{-\eta}x^{\mu-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}$$
$$\times A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}c_1^{k_1}\ldots c_s^{k_s}\, b^{-\sum_{j=1}^{s}\delta_j k_j}x^{\sum_{j=1}^{s}\lambda_j k_j}$$

$$\times\ H^{0,4:m_i,n_i;1,0}_{4,4:p_i,q_i;0,1}\left[\begin{array}{c|c} z_i\frac{x^{\sigma_i}}{b^{\upsilon_i}} & F_1:\left(c_j^{(i)},C_j^{(r)}\right)_{1,p_i};\text{—} \\ \frac{-a}{b}x & F_2:\left(d_j^{(i)},D_j^{(i)}\right)_{1,q_i};(0,1)\end{array}\right], \tag{2.9}$$

where F_1 and F_2 are the same arrays in Theorem 2.4, *and $H^{m,n}_{p,q}(\cdot)$ is the familiar H-function.*

It is noted that a further special case of (2.9) when $s=1$ and $r=2$ reduces to the known result given by Baleanu *et al.* [5]. Considering the relation (1.4) we are led to the following formula given in Corollary 2.6, concerning the Saigo fractional integral operator, which is seen to be similar to that in Agarwal [1, p. 590, Eq. 25].

Corollary 2.6. *Let α, β, γ, μ, η, δ_j, υ_i, z_i, a, b, $c_j\in\mathbb{C}$ with $\Re(\alpha)>0$, λ_j, $\sigma_i>0$ $(i\in\{1,\ldots,r\};\ j\in\{1,\ldots,s\})$. Then we obtain*

$$\left\{I_-^{\alpha,\beta,\gamma}\left(t^{\mu-1}(b-at)^{-\eta}\prod_{j=1}^{s}S^{m_j}_{n_j}\left[c_jt^{\lambda_j}(b-at)^{-\delta_j}\right]\times H\left[z_1t^{\sigma_1}(b-at)^{-\upsilon_1},...,z_rt^{\sigma_r}(b-at)^{-\upsilon_r}\right]\right)\right\}(x)$$
$$=b^{-\eta}x^{\mu-\beta-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\ldots(-n_s)_{m_sk_s}}{k_1!\ldots k_s!}A'_{n_1,m_1}\ldots A^{(s)}_{n_s,m_s}c_1^{k_1}\ldots c_s^{k_s}\,b^{-\sum_{j=1}^{s}\delta_jk_j}x^{\sum_{j=1}^{s}\lambda_jk_j}$$
$$\times\ H^{0,n+3:\{m_r,n_r\};1,0}_{p+3,q+3:\{p_r,q_r\};0,1}\left[\begin{array}{c|c} z_1\frac{x^{\sigma_1}}{b^{\upsilon_1}} & \left(a_j;A'_j,...,A_j^{(r)}\right)_{1,p},F'_1:\left\{\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}\right\};\text{—} \\ \vdots & \\ z_r\frac{x^{\sigma_r}}{b^{\upsilon_r}} & \left(b_j;B'_j,...,B_j^{(r)}\right)_{1,q},F'_2:\left\{\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\right\};(0,1) \\ -\frac{a}{b}x & \end{array}\right], \tag{2.10}$$

where

$$F'_1=\left(1-\eta-\sum_{j=1}^{s}\delta_jk_j;\upsilon_1,...,\upsilon_r,1\right),\ \left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j;\sigma_1,...,\sigma_r,1\right),$$
$$\left(1+\alpha+\beta-\gamma-\mu-\sum_{j=1}^{s}\lambda_jk_j;\sigma_1,...,\sigma_r,1\right);$$

and

$$F'_2=\left(1-\eta-\sum_{j=1}^{s}\delta_jk_j;\upsilon_1,...,\upsilon_r,0\right),\ \left(1+\gamma-\mu-\sum_{j=1}^{s}\lambda_jk_j;\sigma_1,...,\sigma_r,1\right),$$
$$\left(1+\beta-\mu-\sum_{j=1}^{s}\lambda_jk_j;\sigma_1,...,\sigma_r,1\right),$$

The following conditions are also satisfied:

(i) $|\arg z_i|<\frac{\pi}{2}\Omega_i,\ \Omega_i>0\ (i=1,...,r)$,

$$\Omega_i:=-\sum_{j=n+1}^{p}A_j^{(i)}-\sum_{j=1}^{q}B_j^{(i)}+\sum_{j=1}^{n_i}\gamma_j^{(i)}-\sum_{j=n_i+1}^{p_i}\gamma_j^{(i)}+\sum_{j=1}^{m_i}\delta_j^{(i)}-\sum_{j=m_i+1}^{q_i}\delta_j^{(i)}>0,$$

(ii)

$$\Re(\mu) - \sum_{i=1}^{r} \sigma_i \min_{1 \le j \le m_i} \frac{\Re\left(d_j^{(i)}\right)}{\delta_j^{(i)}} < 1 + \min\left\{\Re(\beta), \Re(\gamma)\right\},$$
$$\Re(\eta) - \sum_{i=1}^{r} \upsilon_i \min_{1 \le j \le m_i} \frac{\Re\left(d_j^{(i)}\right)}{\delta_j^{(i)}} < 1 + \min\left\{\Re(\beta), \Re(\gamma)\right\}.$$

It is noted that the corresponding results associated with the Riemann-Liouville and Erdélyi-Kober fractional integral operators can be easily obtained by setting $\beta = -\alpha$ and $\beta = 0$, respectively, in Corollary 2.6.

3. Special Cases and Applications

We begin by remarking the following facts:

(i) The generalized fractional integral operators used in Theorem 2.1 and 2.4 are unified ones in nature.
(ii) The product of the general class of polynomials given in Theorem 2.1 and 2.4 reduce to a large spectrum of known polynomials as illustrated in Srivastava and Singh [37].
(iii) The multivariable H-functions occurring in Theorem 2.1 and 2.4 can be suitably specialized to give a large number of useful functions, for example, the generalized Mittag-Leffler function, Bessel functions of one variable, generalized Wright hypergeometric functions, generalized Lauricella function and so on.

Here, among a remarkably large number of possible special examples of the results in Theorem 2.1 and 2.4, we consider only the following five examples.

Example 1. Let us reduce the multivariable H-function in Theorem 2.1 to the product of two Fox's H-functions, one of which reduces to an exponential function by taking $\sigma_1 = 1$ and $\upsilon_1 \to 0$. Then, after a little simplification, we obtain the following result:

$$\begin{aligned}
&\Bigg\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \Bigg(t^{\mu-1}(b-at)^{-\eta} \prod_{j=1}^{s} S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right] e^{-z_1 t} \\
&\times H_{p_2,q_2}^{m_2,n_2}\left[z_2 t^{\sigma_2}(b-at)^{-\upsilon_2} \,\middle|\, \begin{matrix} (c_j, C_j)_{1,p_2} \\ (d_j, D_j)_{1,q_2} \end{matrix} \right] \Bigg) \Bigg\}(x) \\
&= b^{-\eta} x^{\mu-\alpha-\alpha'+\gamma-1} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1 k_1} \ldots (-n_s)_{m_s k_s}}{k_1! \ldots k_s!} \\
&\times A'_{n_1,m_1} \ldots A^{(s)}_{n_s,m_s} c_1^{k_1} \ldots c_s^{k_s} \, b^{-\sum_{j=1}^{s} \delta_j k_j} x^{\sum_{j=1}^{s} \lambda_j k_j} \\
&\times H_{4,4:0,1;\,p_2,q_2;0,1}^{0,4:1,0;\,m_2,n_2;1,0} \left[\begin{matrix} z_1 x \\ z_2 \frac{x^{\sigma_2}}{b^{\upsilon_2}} \\ -\frac{a}{b} x \end{matrix} \,\middle|\, \begin{matrix} \left(1-\eta-\sum_{j=1}^{s}\delta_j k_j;1,\upsilon_2,1\right), \left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right), \\ \left(1-\eta-\sum_{j=1}^{s}\delta_j k_j;1,\upsilon_2,0\right), \left(1-\mu-\gamma+\alpha+\alpha'-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right), \end{matrix} \right. \\
&\left. \begin{matrix} \left(1-\mu-\gamma+\alpha+\alpha'+\beta-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right), \left(1-\mu+\alpha'-\beta'-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right): -;(c_j,C_j)_{1,p_2};- \\ \left(1-\mu-\gamma+\alpha'+\beta-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right), \left(1-\mu-\beta'-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right):(0,1);(d_j,D_j)_{1,q_2};(0,1) \end{matrix} \right],
\end{aligned} \tag{3.1}$$

whose validity conditions are easily modified and given from those in Theorem 2.1.

Remark 3.1. First, setting $S_{n_j}^{m_j}=1$, $\eta=0$, $\upsilon_2=0$ and $z_1=0$ in (3.1) and making suitable adjustment of the involved parameters is seen to yield the known result given by Ram and Kumar [22]. Secondly, reducing the Saigo-Maeda fractional integral operator into the Saigo fractional integral operator, set $S_{n_j}^{m_j}=1$, $\eta=0$, $\upsilon_2=0$ $z_1=0$ in (3.1) and making suitable adjustment of the involved parameters is seen to give the known result due to Gupta *et al.* [8, p. 209, Eq. (25)]. Thirdly, if we reduce the Saigo-Maeda fractional integral operator into the Riemann-Liouville fractional integral operator, set $S_{n_j}^{m_j}=1$, $\eta=0$, $\upsilon_2=0$ and arrange the involved parameters suitably, we obtain the known result given by Kilbas and Saigo [10, p. 52, Eq.(2.7.9)]. Lastly, if we put $S_{n_j}^{m_j}=1$, $\eta=0$, $\upsilon_2=0$, $z_2=1/4$, $\sigma_2=2$ and reduce the Fox's H-function to the Bessel function of the first kind, we obtain the known result given by Kilbas and Sebastain [11, p. 873, Eqs. (25)-(29)].

Example 2. If we take $z_2=1$, $\sigma_2=1$ and $\upsilon_2=0$ in (3.1) and reduce the H-function of one variable to the generalized Mittag-Leffler function [17] in the result in Theorem 2.1, after a little simplification, we can easily obtain the following result:

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}(b-at)^{-\eta}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right]e^{-z_1 t}E_{\omega,\xi}^{\vartheta}[t]\right)\right\}(x)$$
$$=\frac{b^{-\eta}}{\Gamma(\vartheta)}x^{\mu-\alpha-\alpha'+\gamma-1}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_s=0}^{[n_s/m_s]}\frac{(-n_1)_{m_1k_1}\dots(-n_s)_{m_sk_s}}{k_1!\dots k_s!}$$
$$\times A'_{n_1,m_1}\dots A^{(s)}_{n_s,m_s}c_1^{k_1}\dots c_s^{k_s}\,(b)^{-\sum_{j=1}^{s}\delta_jk_j}(-x)^{\sum_{j=1}^{s}\lambda_j k_j}$$
$$\times H^{0,4:1,0;\,1,1;1,0}_{4,3:0,1;\,1,3;0,1}\left[\begin{array}{c} z_1x\\ x\\ -\frac{a}{b}x\end{array}\left|\begin{array}{l}\left(1-\eta-\sum_{j=1}^{s}\delta_jk_j;1,0,1\right),\left(1-\mu-\sum_{j=1}^{s}\lambda_jk_j;1,1,1\right),\\ \left(1-\mu-\gamma+\alpha+\alpha'+\beta-\sum_{j=1}^{s}\lambda_jk_j;1,1,1\right),\left(1-\mu+\alpha'-\beta'-\sum_{j=1}^{s}\lambda_jk_j;1,1,1\right):-;(1-\vartheta,1);-\\ \left(1-\eta-\sum_{j=1}^{s}\delta_jk_j;1,0,0\right),\left(1-\mu-\gamma+\alpha+\alpha'-\sum_{j=1}^{s}\lambda_jk_j;1,1,1\right),\\ \left(1-\mu-\gamma+\alpha'+\beta-\sum_{j=1}^{s}\lambda_jk_j;1,1,1\right),\left(1-\mu-\beta'-\sum_{j=1}^{s}\lambda_jk_j;1,1,1\right):(0,1);(0,1),(1-\eta;0),(1-\xi;\omega);(0,1)\end{array}\right.\right], \tag{3.2}$$

whose validity conditions are easily modified and given from those in Theorem 2.1.

Remark 3.2. First, setting $S_{n_j}^{m_j}=1$, $\eta=0$ in (3.2) and making suitable adjustment of the involved parameters is seen to yield the known result given by Ram and Kumar [22]. Secondly, reducing the Saigo-Maeda fractional integral operator into the Saigo fractional integral operator, putting $S_{n_j}^{m_j}=1$, $\eta=0$ and making suitable adjustment of in the involed parameters in (3.2) gives the known result given by Gupta *et al.* [8, p. 210, Eq. (29)]. Lastly, if we reduce the Saigo-Maeda fractional integral operator to the Riemann-Liouville fractional integral operator, set $S_{n_j}^{m_j}=1$, $\eta=0$, $\upsilon_2=0$ and make suitable adjustment of the involved parameters in (3.2), we arrive at the known result given by Saxena *et al.* [28, p. 168, Eq. (2.1)].

Example 3. If we reduce the H-function of one variable to the generalized Wright hypergeometric function [32, p. 19, Eq. (2.6.11)] in (3.1), we arrive at the following result:

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\mu-1}(b-at)^{-\eta}\prod_{j=1}^{s}S_{n_j}^{m_j}\left[c_j t^{\lambda_j}(b-at)^{-\delta_j}\right]\right.\right.$$
$$\left.\left.\times\, e^{-z_1t}_{p_2}\psi_{q_2}\left[-z_2t^{\sigma_2}(b-at)^{-\upsilon_2}\left|\begin{array}{c}(1-c_j,C_j)_{1,p_2}\\ (0,1),\,(1-d_j,D_j)_{1,q_2}\end{array}\right.\right]\right)\right\}(x)$$

$$= b^{-\eta} x^{\mu-\alpha-\alpha'+\gamma-1} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1k_1} \dots (-n_s)_{m_sk_s}}{k_1! \dots k_s!}$$

$$\times A'_{n_1,m_1} \dots A^{(s)}_{n_s,m_s} c_1^{k_1} \dots c_s^{k_s} \, b^{-\sum_{j=1}^{s} \delta_j k_j} x^{\sum_{j=1}^{s} \lambda_j k_j}$$

$$\times H^{0,4:1,0;\,1,p_2;1,0}_{4,4:0,1;\,p_2,q_2;0,1} \left[\begin{array}{c} z_1 x \\ -z_2 \frac{x^{\sigma_2}}{b^{\upsilon_2}} \\ -\frac{a}{b} x \end{array} \middle| \begin{array}{l} \left(1-\eta-\sum_{j=1}^{s}\delta_j k_j;1,\upsilon_2,1\right), \left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right), \\ \left(1-\eta-\sum_{j=1}^{s}\delta_j k_j;1,\upsilon_2,0\right), \left(1-\mu-\gamma+\alpha+\alpha'-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right), \end{array} \right.$$

$$\left. \begin{array}{l} \left(1-\mu-\gamma+\alpha+\alpha'+\beta-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right), \left(1-\mu+\alpha'-\beta'-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right): -;(c_j,C_j)_{1,p_2}; - \\ \left(1-\mu-\gamma+\alpha'+\beta-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right), \left(1-\mu-\beta'-\sum_{j=1}^{s}\lambda_j k_j;1,\sigma_2,1\right):(0,1);(d_j,D_j)_{1,q_2};(0,1) \end{array} \right], \quad (3.3)$$

whose validity conditions are easily modified and given from those in Theorem 2.1.

Remark 3.3. Setting $S_{n_j}^{m_j} = 1$, $\eta = 0$, $\upsilon_2 = 0$, $z_1 = 0$ in (3.3) and making suitable adjustment in the involved parameters is seen to yield the known result given by Ram and Kumar [22]. If we set $S_{n_j}^{m_j} = 1$, $\eta = 0$, υ_2, $z_1 = 0$ in (3.3) and arrange the involved parameters suitably, we obtain the known result due to Gupta *et al.* [8, p. 210, Eq. (27)].

Example 4. In Theorem 2.1, if we reduce the multivariable H-function to the product of r different Whittaker functions [32, p. 18, Eq. (2.6.7)] and take $\upsilon_i \to 0$ and $\sigma_i = 1$, we are led to the following result:

$$\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\mu-1} (b-at)^{-\eta} \prod_{j=1}^{s} S_{n_j}^{m_j} \left[c_j \, t^{\lambda_j} (b-at)^{-\delta_j} \right] \prod_{i=1}^{r} e^{\frac{-z_i t}{2}} W_{\lambda_i,\mu_i}(z_i t) \right) \right\} (x)$$

$$= b^{-\eta} x^{\mu-\alpha-\alpha'+\gamma-1} \sum_{k_1=0}^{[n_1/m_1]} \cdots \sum_{k_s=0}^{[n_s/m_s]} \frac{(-n_1)_{m_1k_1} \dots (-n_s)_{m_sk_s}}{k_1! \dots k_s!}$$

$$\times A'_{n_1,m_1} \dots A^{(s)}_{n_s,m_s} c_1^{k_1} \dots c_s^{k_s} \, b^{-\sum_{j=1}^{s} \delta_j k_j} x^{\sum_{j=1}^{s} \lambda_j k_j}$$

$$\times H^{0,4:2,0;\dots;2,0;1,0}_{4,4:1,2;\dots;1,2;0,1} \left[\begin{array}{c} z_1 x \\ \vdots \\ z_r x \\ -\frac{a}{b} x \end{array} \middle| \begin{array}{l} \left(1-\eta-\sum_{j=1}^{s}\delta_j k_j;\underbrace{1,\dots 1}_{r \text{ times}},1\right), \left(1-\mu-\sum_{j=1}^{s}\lambda_j k_j;\underbrace{1,\dots 1}_{r \text{ times}},1\right), \\ \left(1-\eta-\sum_{j=1}^{s}\delta_j k_j;\underbrace{1,\dots 1}_{r \text{ times}},0\right), \left(1-\mu-\gamma+\alpha+\alpha'-\sum_{j=1}^{s}\lambda_j k_j;\underbrace{1,\dots 1}_{r \text{ times}},1\right), \end{array} \right.$$

$$\left. \begin{array}{l} \left(1-\mu-\gamma+\alpha+\alpha'+\beta-\sum_{j=1}^{s}\lambda_j k_j;\underbrace{1,\dots 1}_{r \text{ times}},1\right), \left(1-\mu+\alpha'-\beta'-\sum_{j=1}^{s}\lambda_j k_j;\underbrace{1,\dots 1}_{r \text{ times}},1\right): -;(1-\lambda_1,1);\dots;(1-\lambda_r,1); - \\ \left(1-\mu-\gamma+\alpha'+\beta-\sum_{j=1}^{s}\lambda_j k_j;\underbrace{1,\dots 1}_{r \text{ times}},1\right), \left(1-\mu-\beta'-\sum_{j=1}^{s}\lambda_j k_j;\underbrace{1,\dots 1}_{r \text{ times}},1\right):\left(\tfrac{1}{2}\pm\mu_1,1\right);\dots;\left(\tfrac{1}{2}\pm\mu_r,1\right);(0,1) \end{array} \right], \quad (3.4)$$

whose validity conditions are easily modified and given from those in Theorem 2.1.

Remark 3.4. First, if we set $S_{n_j}^{m_j} = 1$, $r = 1$, $\eta = 0$ in (3.4) and make suitable adjustment of the involved parameters, then we arrive at the known result given by Ram and Kumar [22]. Secondly, if we reduce the Saigo-Maeda fractional integral operator into Saigo fractional integral operator, put $S_{n_j}^{m_j} = 1$, $r = 1$, $\eta = 0$ in (3.4) and make a suitable adjustment of the involved parameters, we obtain the known result given by Gupta *et al.* [8, p. 211, Eq. (31)]. Lastly, reducing the Saigo-Maeda fractional integral operator to the

Riemann-Liouville fractional integral operator, set $S_{n_j}^{m_j} = 1$, $\eta = 0$, $\upsilon_2 = 0$ in (3.4) and making a suitable adjustment of the involved parameters is seen to yield the known result due to Kilbas [9, p. 117, Eq. (11)].

Example 5. If we reduce the multivariable H-function to the product of two Fox's H-functions in Theorem 2.1, one of which further reduces to the exponential function by setting $\sigma_1 = 0$, $\lambda_j = 0$, $\upsilon_1 = 0$, $\delta_j = 0$, $c_j = 1$, $s = 1$, reducing S_n^m to the Hermite polynomial [15, 36] and set $S_n^2[x] = x^{n/2} H_n\left[\frac{1}{2\sqrt{x}}\right]$, in which case $m = 2$, $A_{n,k} = (-1)^k$, we obtain the following interesting result:

$$\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\mu-1} (b-at)^{-\eta} t^{n/2} H_n\left[\frac{1}{2\sqrt{t}}\right] e^{-\upsilon_1 t} H_{p_2,q_2}^{m_2,n_2}\left[z_2 t^{\sigma_2} (b-at)^{-\upsilon_2} \left| \begin{matrix} (c_j,C_j)_{1,p_2} \\ (d_j,D_j)_{1,q_2} \end{matrix} \right. \right] \right) \right\}(x)$$
$$= b^{-\eta} x^{\mu-\alpha-\alpha'+\gamma-1} \sum_{k=0}^{[n/2]} \frac{(-n)_{2k}}{k!} (-x)^k$$
$$\times H_{4,3:0,1;p_2,q_2+1;0,1}^{0,4:1,0;m_2,n_2;1,0} \left[\begin{matrix} z_1 x \\ z_2 x^{\sigma_2} \\ -\frac{a}{b} x \end{matrix} \left| \begin{matrix} \chi_1 : \text{—}; (c_j, C_j)_{1,p_2}; \text{—} \\ \chi_2 : (0,1); (d_j, D_j)_{1,q_2}, (1-\eta, \upsilon_2); (0,1) \end{matrix} \right. \right], \tag{3.5}$$

where $\chi_1 = (1-\eta; 0, \upsilon_2, 1), (1-\mu-k; 1, \sigma_2, 1), (1-\mu-\gamma+\alpha+\alpha'+\beta-k; 1, \sigma_2, 1)$, $(1-\mu+\alpha'-\beta'-k; 1, \sigma_2, 1)$;
and $\chi_2 = (1-\mu-\gamma+\alpha+\alpha'-k; 1, \sigma_2, 1), (1-\mu-\gamma+\alpha'+\beta-k; 1, \sigma_2, 1)$, $(1-\mu-\beta'-k; 1, \sigma_2, 1)$, whose validity conditions are easily modified and given from those in Theorem 2.1.

Remark 3.5. Setting $z_1 = 0$ and $\upsilon_2 = 0$ in (3.5) yields the following formula:

$$\left\{ I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} \left(t^{\mu-1} (b-at)^{-\eta} t^{n/2} H_n\left[\frac{1}{2\sqrt{t}}\right] H_{p_2,q_2}^{m_2,n_2}\left[z_2 t^{\sigma_2} \left| \begin{matrix} (c_j,C_j)_{1,p_2} \\ (d_j,D_j)_{1,q_2} \end{matrix} \right. \right] \right) \right\}(x)$$
$$= b^{-\eta} x^{\mu-\alpha-\alpha'+\gamma-1} \sum_{k=0}^{[n/2]} \frac{(-n)_{2k}}{k!} (-x)^k$$
$$\times H_{4,3:0,1;p_2,q_2+1;0,1}^{0,4:1,0;m_2,n_2;1,0} \left[\begin{matrix} z_2 x^{\sigma_2} \\ -\frac{a}{b} x \end{matrix} \left| \begin{matrix} (1-\eta; 0, 1), (1-\mu-k; \sigma_2, 1), \\ (1-\mu-\gamma+\alpha+\alpha'-k; \sigma_2, 1), (1-\mu-\beta'-k; \sigma_2, 1) \end{matrix} \right. \right.$$
$$\left. \begin{matrix} (1-\mu-\gamma+\alpha+\alpha'+\beta-k; \sigma_2, 1), (1-\mu+\alpha'-\beta'-k; \sigma_2, 1) : \text{—}; (c_j, C_j)_{1,p_2}; \text{—} \\ (1-\mu-\gamma+\alpha'+\beta-k; \sigma_2, 1) : (0,1); (d_j, D_j)_{1,q_2}, (1-\eta, 0); (0,1) \end{matrix} \right]. \tag{3.6}$$

4. Conclusion

Here we presented two very generalized and unified theorems associated with the generalized fractional integral operators given by Saigo-Maeda. The main fractional integrals whose integrands being the products of multivariable H-functions and a general class of polynomials, as shown in Section 3, can be specialized to yield a large number of simpler results. The main results may find potentially useful applications in a variety of areas.

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