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Solving nonlinear ϕ-strongly accretive operator equations by a one-step-two-mappings iterative scheme

Safeer Hussain Khan[a,*], Birol Gunduz[b], Sezgin Akbulut[c]

[a] *Department of Mathematics, Statistics and Physics, Qatar University, Doha 2713, Qatar.*
[b] *Department of Mathematics, Faculty of Science and Art, Erzincan University, Erzincan, 24000, Turkey.*
[c] *Department of Mathematics, Faculty of Science, Ataturk University, Erzurum, 25240, Turkey.*

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Abstract

A solution of nonlinear ϕ-strongly accretive operator equations is found in this paper by using a one-step-two-mappings iterative scheme in arbitrary real Banach spaces. We give an example to validate our main theorem. Our results are different from those of Khan *et. al.,* [S. H. Khan, A. Rafiq, N. Hussain, Fixed Point Theory Appl., 2012 (2012), 10 pages] in view of different and independent iterative schemes in the sense that none reduces to the other but extend and improve the results of Ding [X. P. Ding, Computers Math. Appl., 33 (1997), 75–82] and many others.

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2010 MSC: 47H06, 47H10.

1. Introduction

Let K be a nonempty subset of an arbitrary Banach space X and X^* be its dual space. For a single-valued map $T : X \to X$, $x \in X$ is called a fixed point of T iff $T(x) = x$. The symbols $D(T)$, $R(T)$ and

*Corresponding author
Email addresses: safeer@qu.edu.qa (Sezgin Akbulut), birolgndz@gmail.com (Sezgin Akbulut), sezginakbulut@atauni.edu.tr (Sezgin Akbulut)

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$F(T)$, in this paper, stand for the domain, the range and the set of fixed points of T. We denote by J the normalized duality mapping from X to 2^{X^*} defined by

$$J(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\}.$$

Let $T : D(T) \subseteq X \to X$ be an operator. A map T is called demicontinuous if $\{x_n\}$ converging to x in the norm implies that $\{Tx_n\}$ converges weakly to Tx. Recall the following definitions which can be found in [18].

Definition 1.1. T is called *Lipshitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leqslant L \|x - y\|,$$

for all $x, y \in K$. If $L = 1$, then T is called *nonexpansive* and if $0 < L < 1$, T is called *contraction.*

Definition 1.2. (i) T is said to be strongly pseudocontractive if there exists a $t > 1$ such that for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$Re \langle Tx - Ty,\ j(x-y)\rangle \leq \frac{1}{t} \|x - y\|^2 .$$

(ii) T is said to be strictly hemicontractive if $F(T)$ is nonempty and if there exists a $t > 1$ such that for each $x \in D(T)$ and $q \in F(T)$, there exists $j(x-q) \in J(x-q)$ satisfying

$$Re \langle Tx - q,\ j(x-q)\rangle \leq \frac{1}{t} \|x - q\|^2 .$$

(iii) T is said to be ϕ-strongly pseudocontractive if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ satisfying

$$Re \langle Tx - Ty,\ j(x-y)\rangle \leq \|x - y\|^2 - \phi(\|x-y\|) \|x - y\| .$$

(iv) T is said to be ϕ-hemicontractive if $F(T)$ is nonempty and if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x \in D(T)$ and $q \in F(T)$, there exists $j(x-q) \in J(x-q)$ satisfying

$$Re \langle Tx - q,\ j(x-q)\rangle \leq \|x - q\|^2 - \phi(\|x-q\|) \|x - q\| .$$

Clearly, each strictly hemicontractive operator is ϕ-hemicontractive.

Definition 1.3. (i) T is called *accretive* if the inequality

$$||x - y|| \leq ||x - y + s(Tx - Ty)||$$

holds for every $x, y \in D(T)$ and for all $s > 0$.

(ii) T is called *strongly accretive* if for all $x, y \in D(T)$ there exists a constant $k > 0$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty,\ j(x-y)\rangle \geq k||x - y||^2.$$

(iii) T is called *ϕ-strongly accretive* if there exists $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x, y \in X$,

$$\langle Tx - Ty,\ j(x-y)\rangle \geq \phi(||x-y||)||x - y||.$$

The class of strongly accretive operators is a proper subclass of the class of ϕ-strongly accretive operators, see, for example, [18, 19]. T is called *strongly pseudocontractive* (respectively, *ϕ-strongly pseudocontractive*) if and only if $(I - T)$ is strongly accretive (respectively, ϕ-strongly accretive) where I denotes the identity operator. Mann iterative scheme was used by Chidume [1] in order to approximate fixed points of Lipschitz strongly pseudocontractive operators in L_p (or l_p) spaces for $p \in [2, \infty)$. Chidume and Osilike [2] proved that each strongly pseudocontractive operator with a fixed point is strictly hemicontractive, but the converse is not necessarily true. They also proved that the class of strongly pseudocontractive operators is a proper subclass of the class of ϕ-strongly pseudocontractive operators, and pointed out that the class of ϕ-strongly pseudocontractive operators with a fixed point is a proper subclass of the class of ϕ-hemicontractive operators. These classes of nonlinear operators have been studied by various researchers (see, for example, [3, 5, 6, 7, 9, 12, 16, 15, 13, 17, 18, 19, 20, 21, 22, 4, 10]). Liu *et al.*, [14] proved that under certain conditions a three-step iterative scheme with error terms converges strongly to the unique fixed point of $\phi-$hemi-contractive mappings. Khan *et. al.*, [11] studied strong convergence of three-step iterative scheme with error terms to a common solution of $\phi-$strongly accretive operator equations in a real Banach space.

In this paper, we study a one-step-two-mappings iterative scheme for solving nonlinear ϕ-strongly accretive operator equations in arbitrary real Banach spaces. We give an example to validate our main theorem. Our results are different from those of [11] because of different and independent iterative schemes in the sense that none reduces to the other but extend and improve the results of [5, 15, 18, 19] and many others.

2. Preliminaries

Some useful results are stated below.

Lemma 2.1 ([23])**.** *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be three sequences of nonnegative real numbers with $\sum\limits_{n=1}^{\infty} b_n < \infty$ and $\sum\limits_{n=1}^{\infty} c_n < \infty$. If*

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \ n \geq 1,$$

then the limit $\lim\limits_{n\to\infty} a_n$ exists.

Lemma 2.2 ([8])**.** *Let $x, y \in X$. Then $\|x\| \leq \|x + ry\|$ for every $r > 0$ if and only if there is $f \in J(x)$ such that Re $\langle y, \ f\rangle \geq 0$.*

Lemma 2.3 ([15])**.** *Suppose that X is an arbitrary Banach space and $A : X \to X$ is a continuous ϕ-strongly accretive operator. Then the equation $Ax = f$ has a unique solution for any $f \in X$.*

3. Solving a system of nonlinear operator equations by a one-step-two-mappings iterative scheme

From now onwards, L denotes the Lipschitz constant of $T_1, T_2 : X \to X$, $L^* = (1 + L)$ and $R(T_1)$ and $R(T_2)$ denote the ranges of T_1 and T_2 respectively. Following the techniques of [11] and the references cited therein, we prove our main theorem as follows.

Theorem 3.1. *Let X be an arbitrary real Banach space and $T_1, T_2 : X \to X$ Lipschitz ϕ-strongly accretive operators. Let $f \in R(T_1) \cap R(T_2)$ and generate $\{x_n\}$ from an arbitrary $x_0 \in X$ by*

$$x_{n+1} = a_n x_n + b_n(f + (I - T_1)x_n) + c_n(f + (I - T_2)x_n), \ n \geq 0, \tag{3.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences in $(0, 1)$ satisfying conditions:

(i) $a_n + b_n + c_n = 1$,

(ii) $b_n \in (0, b)$ for some $b \in (0, 1)$,

(iii) $\sum_{n=0}^{\infty} b_n = \infty$,

(iv) $\sum_{n=0}^{\infty} b_n^2 < \infty$,
(v) $\sum_{n=0}^{\infty} c_n < \infty$.
Then the sequence $\{x_n\}$ *converges strongly to the solution of the system* $T_i x = f$; $i = 1, 2$.

Proof. Let $T_1, T_2 : X \to X$ be two Lipschitz ϕ-strongly accretive operators with strictly increasing functions $\phi_1, \phi_2 : [0, \infty) \to [0, \infty)$ satisfying $\phi_1(0) = \phi_2(0) = 0$. It follows from [15] that the system $T_i x = f; i = 1, 2$ has the unique solution, say $x^* \in X$. Define $V_i : X \to X$ by $V_i x = f + (I - T_i)x$; $i = 1, 2$; then each V_i is demicontinuous and x^* is the unique fixed point of V_i; $i = 1, 2$. Furthermore, for all $x, y \in X$, we have

$$\begin{aligned}\langle (I - V_i)x - (I - V_i)y, j(x - y)\rangle &\geq \phi_i(||x - y||)||x - y|| \\ &\geq \frac{\phi_i(||x - y||)}{(1 + \phi_i(||x - y||) + ||x - y||)}||x - y||^2 \\ &= \psi_i(x, y)||x - y||^2,\end{aligned}$$

where $\psi_i(x, y) = \frac{\phi_i(||x-y||)}{(1+\phi_i(||x-y||)+||x-y||)} \in [0, 1)$ for all $x, y \in X$; $i = 1, 2$. Let $x^* \in \bigcap_{i=1}^{2} F(V_i)$ be the fixed point set of V_i and let $\psi(x, y) = \inf \min_i\{\psi_i(x, y)\} \in [0, 1]$. Then

$$\langle (I - V_i)x - (I - V_i)y, j(x - y)\rangle \geq \psi(x, y)||x - y||^2;\ i = 1, 2, \tag{3.2}$$

and it follows from Lemma 2.2 and inequality (3.2) that

$$||x - y|| \leq ||x - y + \lambda[(I - V_i)x - \psi(x, y)x - ((I - V_i)y - \psi(x, y)y)]||, \tag{3.3}$$

for all $x, y \in X$ and for all $\lambda > 0; i = 1, 2$.

Using definition of V_i, (3.1) can be rewritten

$$\begin{aligned}x_{n+1} &= a_n x_n + b_n V_1 x_n + c_n V_2 x_n \\ &= (1 - b_n - c_n)x_n + (b_n + c_n) V_1 x_n + c_n(V_2 x_n - V_1 x_n) \\ &= (1 - \alpha_n)x_n + \alpha_n V_1 x_n + c_n(V_2 x_n - V_1 x_n),\end{aligned} \tag{3.4}$$

where $\alpha_n = b_n + c_n$. From (3.4) we have

$$\begin{aligned}x_n &= x_{n+1} + \alpha_n x_n - \alpha_n V_1 x_n - c_n(V_2 x_n - V_1 x_n) \\ &= (1 + \alpha_n)x_{n+1} + \alpha_n x_{n+1} - 2\alpha_n((1 - \alpha_n)x_n + \alpha_n V_1 x_n \\ &\quad + c_n(V_2 x_n - V_1 x_n)) + \alpha_n x_n - \alpha_n V_1 x_n - c_n(V_2 x_n - V_1 x_n) \\ &\quad + \alpha_n V_1 x_{n+1} - \alpha_n V_1 x_{n+1} \\ &\quad + \alpha_n \psi(x_{n+1}, x^*)(-x_{n+1} + (1 - \alpha_n)x_n + \alpha_n V_1 x_n + c_n(V_2 x_n - V_1 x_n)) \\ &= (1 + \alpha_n)x_{n+1} + \alpha_n[(I - V_1)x_{n+1} - \psi(x_{n+1}, x^*)x_{n+1}] \\ &\quad - (1 - \psi(x_{n+1}, x^*))\alpha_n x_n + (2 - \psi(x_{n+1}, x^*))\alpha_n^2(x_n - V_1 x_n) \\ &\quad + \alpha_n(V_1 x_{n+1} - V_1 x_n) - [1 + (2 - \psi(x_{n+1}, x^*))\alpha_n]c_n(V_2 x_n - V_1 x_n).\end{aligned}$$

Observe that

$$x^* = (1 + \alpha_n)x^* + \alpha_n[(I - V_1)x^* - \psi(x_{n+1}, x^*)x^*] - (1 - \psi(x_{n+1}, x^*))\alpha_n x^*,$$

so that

$$\begin{aligned}x_n - x^* &= (1 + \alpha_n)(x_{n+1} - x^*) + \alpha_n[(I - V_1)x_{n+1} - \psi(x_{n+1}, x^*)x_{n+1} \\ &\quad - ((I - V_1)x^* - \psi(x_{n+1}, x^*)x^*)] \\ &\quad - (1 - \psi(x_{n+1}, x^*))\alpha_n(x_n - x^*) + (2 - \psi(x_{n+1}, x^*))\alpha_n^2(x_n - V_1 x_n) \\ &\quad + \alpha_n(V_1 x_{n+1} - V_1 x_n) - [1 + (2 - \psi(x_{n+1}, x^*))\alpha_n]c_n(V_2 x_n - V_1 x_n).\end{aligned}$$

Hence

$$
\begin{aligned}
||x_n - x^*|| \geq{} & (1+\alpha_n)||x_{n+1} - x^* + \frac{\alpha_n}{(1+\alpha_n)}[(I-V_1)x_{n+1} - \psi(x_{n+1},x^*)x_{n+1} \\
& - ((I-V_1)x^* - \psi(x_{n+1},x^*)x^*)]|| \\
& - (1-\psi(x_{n+1},x^*))\alpha_n||x_n - x^*|| - (2-\psi(x_{n+1},x^*))\alpha_n^2||x_n - V_1x_n|| \\
& - \alpha_n||V_1x_{n+1} - V_1x_n|| - [1+(2-\psi(x_{n+1},x^*))\alpha_n]c_n||V_2x_n - V_1x_n|| \\
\geq{} & (1+\alpha_n)||x_{n+1} - x^*|| - (1-\psi(x_{n+1},x^*))\alpha_n||x_n - x^*|| \\
& - (2-\psi(x_{n+1},x^*))\alpha_n^2||x_n - V_1x_n|| - \alpha_n||V_1x_{n+1} - V_1x_n|| \\
& - [1+(2-\psi(x_{n+1},x^*))\alpha_n]c_n||V_2x_n - V_1x_n||.
\end{aligned}
$$

so that

$$
\begin{aligned}
||x_{n+1} - x^*|| \leq{} & \frac{[1+(1-\psi(x_{n+1},x^*))\alpha_n]}{(1+\alpha_n)}||x_n - x^*|| + 2\alpha_n^2||x_n - V_1x_n|| \\
& + \alpha_n||V_1x_{n+1} - V_1x_n|| + [1+(2-\psi(x_{n+1},x^*))\alpha_n]c_n||V_2x_n - V_1x_n|| \\
\leq{} & [1+(1-\psi(x_{n+1},x^*))\alpha_n][1-\alpha_n+\alpha_n^2]||x_n - x^*|| \\
& + 2\alpha_n^2||x_n - V_1x_n|| + \alpha_n||V_1x_{n+1} - V_1x_n|| + 3c_n||V_2x_n - V_1x_n|| \\
\leq{} & [1-\psi(x_{n+1},x^*)\alpha_n + \alpha_n^2]||x_n - x^*|| + 2\alpha_n^2||x_n - V_1x_n|| \\
& + \alpha_n||V_1x_{n+1} - V_1x_n|| + 3c_n||V_2x_n - V_1x_n||.
\end{aligned} \tag{3.5}
$$

Furthermore, we have the following estimates

$$||x_n - V_1x_n|| \leq ||x_n - x^*|| + ||V_1x_n - x^*|| = (1+L^*)||x_n - x^*||, \tag{3.6}$$

$$||V_2x_n - V_1x_n|| \leq ||V_2x_n - x^*|| + ||V_1x_n - x^*|| \leq 2L^*||x_n - x^*||. \tag{3.7}$$

Using (3.6) and (3.7),

$$
\begin{aligned}
||V_1x_{n+1} - V_1x_n|| &\leq L^*||x_{n+1} - x_n|| \\
&= L^*\left\|\alpha_n(V_1x_n - x_n) + c_n(V_2x_n - V_1x_n)\right\| \\
&\leq L^*[\alpha_n\left\|V_1x_n - x_n\right\| + c_n\left\|V_2x_n - V_1x_n\right\|] \\
&\leq L^*[(1+L^*)\alpha_n||x_n - x^*|| + 2L^*c_n\left\|x_n - x^*\right\|] \\
&= L^*(1+L^*)\alpha_n||x_n - x^*|| + 2\left(L^*\right)^2 c_n\left\|x_n - x^*\right\|.
\end{aligned} \tag{3.8}
$$

Substituting (3.6), (3.7) and (3.8) in (3.5), we obtain

$$
\begin{aligned}
||x_{n+1} - x^*|| \leq{} & (1+\alpha_n^2)||x_n - x^*|| - \psi(x_{n+1},x^*))\alpha_n||x_n - x^*|| \\
& + 2(1+L^*)\alpha_n^2||x_n - x^*|| + \alpha_n[L^*(1+L^*)\alpha_n||x_n - x^*|| \\
& + 2\left(L^*\right)^2 c_n\left\|x_n - x^*\right\|] + 6L^*c_n||x_n - x^*|| \\
={} & [1+(1+2(1+L^*)+L^*(1+L^*))\alpha_n^2 \\
& + 2\left(L^*\right)^2\alpha_n c_n + 6L^*c_n]||x_n - x^*|| - \psi(x_{n+1},x^*))\alpha_n||x_n - x^*|| \\
\leq{} & (1+\delta_n)||x_n - x^*||,
\end{aligned} \tag{3.9}
$$

where

$$
\begin{aligned}
\delta_n &= [1+(1+2(1+L^*)+L^*(1+L^*))\alpha_n^2 + 2\left(L^*\right)^2\alpha_n c_n + 6L^*c_n] \\
&= [1+(1+2(1+L^*)+L^*(1+L^*))(b_n+c_n)^2 + 2\left(L^*\right)^2(b_n+c_n)c_n + 6L^*c_n] \\
&\leq [1+(1+2(1+L^*)+L^*(1+L^*))(b_n+c_n)^2 + 2\left(L^*\right)^2(bc_n+c_n^2) + 6L^*c_n].
\end{aligned}
$$

Since $c_n \in (0,1), c_n \geq c_n^2$ and $\sum_{n=0}^{\infty} c_n < \infty$, the Comparison Test implies that $\sum_{n=0}^{\infty} c_n^2 < \infty$. Hence $\sum\limits_{n=0}^{\infty} \delta_n < \infty$. It then follows from Lemma 2.1 that $\lim\limits_{n\to\infty} ||x_n - x^*||$ exists and so $\{||x_n - x^*||\}$ bounded. Let $\lim\limits_{n\to\infty} ||x_n - x^*|| = \delta \geq 0$. We now prove that $\delta = 0$. Assume that $\delta > 0$. Then there exists a positive integer N_0 such that $||x_n - x^*|| \geq \frac{\delta}{2}$ for all $n \geq N_0$. Since $\lim\limits_{n\to\infty} ||x_n - x^*||$ exists, there exists a real number D such that $||x_{n+1} - x^*|| \leq D$. Thus

$$\begin{aligned}\psi(x_{n+1}, x^*)||x_n - x^*|| &= \frac{\phi(||x_{n+1} - x^*||)}{1 + \phi(||x_{n+1} - x^*||) + ||x_{n+1} - x^*||}||x_n - x^*|| \\ &\geq \frac{\phi(\frac{\delta}{2})\delta}{2(1 + \phi(D) + D)},\end{aligned}$$

for all $n \geq N_0$, it follows from (3.9) that

$$||x_{n+1} - x^*|| \leq ||x_n - x^*|| - \frac{\phi(\frac{\delta}{2})\delta}{2(1 + \phi(D) + D)}\alpha_n \text{ for all } n \geq N_0,$$

or

$$\frac{\phi(\frac{\delta}{2})\delta}{2(1 + \phi(D) + D)}\alpha_n \leq ||x_n - x^*|| - ||x_{n+1} - x^*|| \text{ for all } n \geq N_0.$$

This implies that

$$\frac{\phi(\frac{\delta}{2})\delta}{2(1 + \phi(D) + D)} \sum_{j=N_0}^{n} \alpha_j \leq ||x_{N_0} - x^*||.$$

Since $b_n \leq \alpha_n$, so

$$\frac{\phi(\frac{\delta}{2})\delta}{2(1 + \phi(D) + D)} \sum_{j=N_0}^{n} b_j \leq ||x_{N_0} - x^*||,$$

yields $\sum\limits_{n=0}^{\infty} b_n < \infty$, contradicting the fact that $\sum\limits_{n=0}^{\infty} b_n = \infty$. Hence $\lim\limits_{n\to\infty} ||x_n - x^*|| = 0$ and the proof is complete. □

Corollary 3.2. *Let X be an arbitrary real Banach space and $T_1, T_2 : X \to X$ be two Lipschitz ϕ-strongly accretive operators, where ϕ, in addition, is continuous. Suppose $\liminf\limits_{r\to\infty} \phi(r) > 0$ or $||T_i x|| \to \infty$ as $||x|| \to \infty$; $i = 1, 2$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{x_n\}$ be as in Theorem 3.1. Then for any given $f \in X$, the sequence $\{x_n\}$ converges strongly to the solution of the system $T_i x = f$; $i = 1, 2$.*

Proof. The existence of a unique solution to the system $T_i x = f; i = 1, 2$ follows from [15] and the result follows from Theorem 3.1. □

Theorem 3.3. *Let X be a real Banach space and K a nonempty closed convex subset of X. Let $T_1, T_2 : K \to K$ be two Lipschitz ϕ-strong pseudocontractions with a nonempty fixed-point set. Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be as in Theorem 3.1. Let $\{x_n\}$ be the sequence generated iteratively from an arbitrary $x_0 \in K$ by*

$$x_{n+1} = a_n x_n + b_n T_1 x_n + c_n T_2 x_n, \ n \geq 0.$$

Then $\{x_n\}$ converges strongly to the common fixed point of T_1 and T_2.

Proof. As in the proof of Theorem 3.1, set $\alpha_n = b_n + c_n$ to obtain

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1 x_n + c_n(T_2 x_n - T_1 x_n), \ n \geq 0.$$

Since each $T_i; i = 1, 2$ is a ϕ-strong pseudocontraction, $(I-T_i)$ is ϕ-strongly accretive so that for all $x, y \in X$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : (0,\infty) \to (0,\infty)$ with $\phi(0) = 0$ such that

$$\langle (I-T_i)x - (I-T_i)y, j(x-y)\rangle \geq \phi(||x-y||)||x-y|| \geq \psi(x,y)||x-y||^2;\ i = 1, 2.$$

The rest of the argument is now essentially the same as in the proof of Theorem 3.1 and is therefore omitted. □

Remark 3.4. Example given in [2] shows that the class of ϕ-strongly pseudocontractive operators with nonempty fixed-point sets is a proper subclass of the class of ϕ-hemicontractive operators. It is easy to see that Theorem 3.1 easily extends to the class of ϕ-hemicontractive operators.

Remark 3.5. Let $\{\alpha_n\}$ be a real sequence satisfying the following conditions:

$$\text{(i) } 0 \leq \alpha_n \leq 1,\ n \geq 0,\ \text{(ii) } \lim_{n\to\infty} \alpha_n = 0,\ \text{(iii) } \sum_{n=0}^{\infty} \alpha_n = \infty \text{ and (iv) } \sum_{n=0}^{\infty} \alpha_n^2 < \infty.$$

If we set $a_n = (1-\alpha_n)$, $b_n = \alpha_n$, $c_n = 0$ for all $n \geq 0$ in Theorems 3.1 and 3.3, we obtain the corresponding convergence theorems for the original Mann iterative scheme.

Remark 3.6. All the results proved in this paper can also be proved for the iterative scheme with error terms or a finite family of $\phi-$strongly accretive operators. In these cases our main iterative scheme (3.1) looks like

$$x_{n+1} = a_n x_n + b_n(f + (I-T_1)x_n) + c_n(f + (I-T_2)x_n) + d_n u_n,\ n \geq 0, \tag{3.10}$$

where $\{u_n\}$ is a bounded sequence and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$ are sequences in $[0,1]$ such that $a_n + b_n + c_n + d_n = 1$ and

$$x_{n+1} = \alpha_{0n} x_n + \sum_{i=1}^{m} \alpha_{in}(f + (I-T_i)x_n),\ n \geq 0,$$

where $\{\alpha_{0n}\}, \{\alpha_{1n}\}, \ldots, \{\alpha_{mn}\}$ are $m+1$ real sequences in $[0,1]$ satisfying $\sum\limits_{i=0}^{m} \alpha_{in} = 1$ respectively.

Remark 3.7. Since the iterative scheme (3.1) and (3.10) are computationally simpler than Ishikawa iterative scheme used by Osilike [19] and Ishikawa iterative scheme with error terms used by Ding [5] respectively, therefore our results are better.

Example 3.8. Let $X = (-\infty, +\infty)$ with the usual norm $|\cdot|$. Define $T_1, T_2 : X \to X$ and $\phi : [0,+\infty) \to [0,+\infty)$ by $T_1 x = \begin{cases} \frac{2x^2}{1+2x} \text{ for } x \in [0,+\infty) \\ x \text{ for } x \in (-\infty, 0) \end{cases}$, $T_2 x = \frac{4}{5}x$ for $x \in (-\infty,+\infty)$ and $\phi(t) = \frac{t^2}{1+2t}$ for $t \in [0,+\infty)$, respectively. Set

$$a_n = 1 - (2+n)^{-1} - (3+n^2)^{-1},\ b_n = (2+n)^{-1},\ c_n = (3+n^2)^{-1},\ \text{for all } n \geq 0.$$

Clearly (i) $a_n + b_n + c_n = 1$ (ii) $b_n \in (0,b)$ for $b = \frac{1}{2} \in (0,1)$ (iii) To prove that $\sum_{n=0}^{\infty} b_n = \infty$, take $b'_n = \frac{1}{n}$. Since $\lim \frac{b'_n}{b_n} = \lim \frac{n}{2+n} = 1 > 0$ and $\sum_{n=0}^{\infty} b'_n = \infty$, therefore $\sum_{n=0}^{\infty} b_n = \infty$ by using Limit Comparison Test. (iv) $\sum_{n=0}^{\infty} b_n^2 < \infty$ and (v) $\sum_{n=0}^{\infty} c_n < \infty$ can be proved in a similar way.

In order to prove that T_1 is ϕ-strongly accretive, we have to consider the following four possible cases:

Case 1. $x, y \in [0,+\infty)$. It follows that

$$\begin{aligned} &\langle Tx - Ty,\ x-y\rangle - \phi(|x-y|)\,|x-y| \\ &= \left(\frac{2x^2}{1+2x} - \frac{2y^2}{1+2y}\right)(x-y) - \frac{(x-y)^2}{1+2\,|x-y|}\,|x-y| \\ &= (x-y)^2\left(\frac{2x+2y+4xy}{(1+2x)(1+2y)} - \frac{|x-y|}{1+2\,|x-y|}\right) \end{aligned}$$

$$
\begin{aligned}
&= (x-y)^2 \left(\frac{2x+2y+4xy+|x-y|\,(2x+2y+4xy)-|x-y|}{(1+2x)(1+2y)(1+2\,|x-y|)} \right) \\
&= (x-y)^2 \left(\frac{(1+2x+2y+4xy)\,|x-y|+2x+2y+4xy-2\,|x-y|}{(1+2x)(1+2y)(1+2\,|x-y|)} \right) \\
&= (x-y)^2 \left(\frac{|x-y|}{(1+2\,|x-y|)} + \frac{2x+2y+4xy-2\,|x-y|}{(1+2x)(1+2y)(1+2\,|x-y|)} \right) \\
&\geq 0.
\end{aligned}
$$

which implies that

$$\langle Tx - Ty,\ x-y \rangle \geq \phi(|x-y|)\,|x-y|\,. \tag{3.11}$$

Case 2. $x, y \in (-\infty, 0)$. It is easy to verify that

$$\langle Tx - Ty,\ x-y \rangle = (x-y)^2 \geq \frac{|x-y|^3}{1+2\,|x-y|} = \phi(|x-y|)\,|x-y|\,.$$

Case 3. $x \in [0, +\infty)$ and $y \in (-\infty, 0)$. Then

$$
\begin{aligned}
&\langle Tx - Ty,\ x-y \rangle - \phi(|x-y|)\,|x-y| \\
&\qquad = \left(\frac{2x^2}{1+2x} - y \right)(x-y) - \frac{(x-y)^3}{1+2(x-y)} \\
&\qquad = (x-y) \left(\frac{2x^2-2xy-y}{1+2x} - \frac{(x-y)^2}{1+2(x-y)} \right) \\
&\qquad = (x-y) \left(\frac{x^2-2xy+y^2+2x^3-4x^2y+2xy^2-y}{(1+2x)\,(1+2(x-y))} \right) \\
&\qquad = (x-y) \left(\frac{(x-y)^2+(x-y)(2x^2-2xy)-y}{(1+2x)\,(1+2(x-y))} \right) \\
&\qquad = (x-y) \left(\frac{(x-y)^2(1+2x)-y}{(1+2x)\,(1+2(x-y))} \right) \\
&\qquad \geq 0,
\end{aligned}
$$

which means that (3.11) holds.

Case 4. $x \in (-\infty, 0)$ and $y \in [0, +\infty)$. As in the proof of Case 3, we conclude that (3.11) holds.
Next we assert that T_1 is Lipschitzian mapping with $L = 1$. We consider the following possible cases:
Case 1. $x, y \in [0, +\infty)$. Then

$$
\begin{aligned}
|T_1x - T_1y| &= \left| \frac{2x^2}{1+2x} - \frac{2y^2}{1+2y} \right| \\
&= \left| (x-y) \left(\frac{2x+2y+4xy}{(1+2x)\,(1+2y)} \right) \right| \\
&\leq |x-y|\,.
\end{aligned}
$$

Case 2. $x, y \in (-\infty, 0)$. It is clear that

$$|T_1x - T_1y| \leq |x-y|\,.$$

Case 3. $x \in [0, +\infty)$ and $y \in (-\infty, 0)$. It follows that

$$|T_1x - T_1y| - |x-y| = \frac{2x^2}{1+2x} - y - (x-y) = \frac{-x}{1+2x} \leq 0,$$

that is,

$$|T_1x - T_1y| \leq |x - y| . \tag{3.12}$$

Case 4. $x \in (-\infty, 0)$ and $y \in [0, +\infty)$. As in the proof of Case 3, deduce that (3.12) holds.

Clearly, T_2 is a Lipschitz ϕ-strongly accretive operator with Lipschitz constant $L = \frac{4}{5}$ and $F = F(T_1) \cap F(T_2) = \{0\} \neq \emptyset$.

We take $\frac{8}{5} = f \in R(T_1) \cap R(T_2)$. Then solution of the system $T_ix = \frac{8}{5}$; $i = 1, 2$ is 2. Now we show that $\{x_n\}$ converges strongly to 2 which is solution of the system $T_ix = \frac{8}{5}$; $i = 1, 2$. By taking $n = 0$ and $x_0 \in (-\infty, +\infty)$, we get $a_0 = \frac{1}{6}, b_0 = \frac{1}{2}, c_0 = \frac{1}{3}$ and find x_1 from

$$x_1 = a_0x_0 + b_0(\frac{8}{5} + (I - T_1)x_0) + c_0(\frac{8}{5} + (I - T_2)x_0).$$

Similarly, $x_2, x_3, \ldots, x_n, \ldots$. We obtain the first 100 terms of $\{x_n\}$ as in following table for initial value $x_0 = -1$, $x_0 = 0$ and $x_0 = 3$, respectively.

No. of Iterations	$x_0 = -1$	$x_0 = 0$	$x_0 = 3$
n	x_n	x_n	x_n
1	1.100000000	1.333333333	2.247619048
2	1.561250000	1.676767677	2.247619048
3	1.715758950	1.790804297	2.076501948
4	1.788990932	1.844763423	2.056695272
5	1.831507297	1.876069036	2.056695272
10	1.914411300	1.937072451	2.022932785
15	1.942009195	1.957368784	2.015528882
20	1.955964577	1.967629983	2.011788345
25	1.964426111	1.973850983	2.009521452
30	1.970117561	1.978035119	2.007997145
35	1.974213714	1.981046318	2.006900331
40	1.977305872	1.983319378	2.006072477
45	1.979724589	1.985097363	2.005424999
50	1.981669323	1.986526861	2.004904455
75	1.987657138	1.990826026	2.003326049
100	1.990652252	1.993063486	2.002524633

From the table above, we see that the sequence $\{x_n\}$ converges strongly to 2 which is the solution of the system $T_ix = \frac{8}{5}$;$i = 1, 2$. This means that Theorem 3.1 is applicable.

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