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Evolutes of null torus fronts

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Abstract

The main goal of this paper is to characterize evolutes at singular points of curves in hyperbolic plane by analysing evolutes of null torus fronts. We have done some work associated with curves with singular points in Euclidean 2-sphere [H. Yu, D. Pei, X. Cui, J. Nonlinear Sci. Appl., 8 (2015), 678–686]. As a series of this work, we further discuss the relevance between singular points and geodesic vertices of curves and give different characterizations of evolutes in the three pseudo-spheres.

Keywords: Evolute, null torus front, null torus framed curve, hyperbolic plane.
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1. Preliminaries

As a subject closely related to nonlinear sciences, singularity theory [1, 2, 3, 4, 7] has been extensively applied in studying classifications of singularities of submanifolds in Euclidean spaces and semi-Euclidean spaces [11, 12]. However, little information has been got at singular points from the view point of differential geometry. In this paper we characterize the behaviors at singular points of curves in hyperbolic plane.

If a curve has singular points, we can not construct its moving frame. However, we can define a moving frame of a frontal for a framed curve in the unit tangent bundle. Along with the moving frame, we get a pair of smooth functions as the geodesic curvature of a regular curve. It is quite useful to analyse curves with singular points. Because we can get information at singular points through analysing framed curves. We have researched curves with singular points in Euclidean 2-sphere in [13]. In general, one can not define evolutes at singular points of curves on Euclidean 2-sphere, but we define evolutes of fronts under some conditions.

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In this paper, we focus on hyperbolic plane and further discuss the relevance between singular points and geodesic vertices. We show that curves in hyperbolic plane must be spacelike. It is different from the case of Euclidean 2-sphere. We consider a null torus T^4_2 [10], and it is a section of a null cone. It is also a product space of hyperbolic plane and de Sitter sphere, that is $\mathrm{T}^4_2 = \mathrm{H}^2_0 \times \mathrm{S}^2_1$. Spacelike curves with singular points in hyperbolic plane are also null torus frontals. We give the characterizations of evolutes not only in hyperbolic plane but also in de Sitter sphere and nullcone. We point that the evolute at singular points only lies in hyperbolic plane, and the parts in de Sitter sphere must be regular. For the case of Euclidean plane, there are some creative works [5, 6].

Theorem 4.6 describes the evolute of a null torus front by the geodesic curvature. At first we define the evolute of a null torus front. At the regular part, it inherits the definition for a regular curve in hyperbolic plane given by the second author et al. [8], as the locus of the center of its osculating pseudo-spheres. At the singular part, we define it by the limit of the evolutes of its parallel curves. It works, since the evolute of the spacelike curve coincides with the evolute of its parallel curve under some conditions.

On the other hand, we further give some properties. In [8] and [9], the authors point that the four-vertex theorem holds for hyperbolic plane. For a closed curve without inflection points, we discuss the relevance between singular points and geodesic vertices in Theorem 4.13. For a closed curve with inflection points, we discuss the relevance between inflection points and geodesic vertices in Theorem 4.14.

We assume throughout the paper that all manifolds and maps are C^∞ unless explicitly stated otherwise.

2. Regular hyperbolic plane curves

The Minkowski 3-space $(\mathbb{R}^3_1, \langle , \rangle)$ is the vector space $\mathbb{R}^3$ endowed with the metric induced by the pseudo-scalar product

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3$$

for any vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$. The non-zero vector $\boldsymbol{x}$ in $\mathbb{R}^3_1$ is called *spacelike, null or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$, respectively. We have the following pseudo-spheres in $\mathbb{R}^3_1$:

$$\mathrm{Q}^2_\epsilon = \begin{cases} \mathrm{H}^2_0 = \{\boldsymbol{x} \in \mathbb{R}^3_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\} & \text{if } \epsilon = - \\ \mathrm{S}^2_1 = \{\boldsymbol{x} \in \mathbb{R}^3_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\} & \text{if } \epsilon = +. \end{cases}$$

And we take

$$\mathrm{H}^2_+ = \{(x_1, x_2, x_3) \in \mathrm{H}^2_0 | x_1 \geq 1\},$$
$$\mathrm{H}^2_- = \{(x_1, x_2, x_3) \in \mathrm{H}^2_0 | x_1 \leq -1\}$$

and $\mathrm{H}^2_0 = \mathrm{H}^2_+ \bigcup \mathrm{H}^2_-$. We call H^2_0 *a hyperbolic plane* and S^2_1 *a de Sitter sphere.*

We call $\mathrm{H}^2_+ \times \mathrm{S}^2_1$ a *null torus*, denoted by T^4_2, that is

$$\mathrm{T}^4_2 = \mathrm{H}^2_+ \times \mathrm{S}^2_1 = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^6_2 | \boldsymbol{x} \in \mathrm{H}^2_+, \boldsymbol{y} \in \mathrm{S}^2_1\}.$$

For any $\boldsymbol{z} \in \mathrm{T}^4_2$, $\langle \boldsymbol{z}, \boldsymbol{z} \rangle = 0$. So T^4_2 is a part of a null cone.

Let $\boldsymbol{\gamma} : I \to \mathrm{H}^2_+ \subset \mathbb{R}^3_1$ be a regular curve (i.e. $\dot{\boldsymbol{\gamma}}(t) \neq 0$ for any $t \in I$), where I is an open interval. We can show that $\langle \dot{\boldsymbol{\gamma}}(t), \dot{\boldsymbol{\gamma}}(t) \rangle > 0$ for any $t \in I$. We call such a curve a *spacelike curve*. If t_0 is a singular point of $\boldsymbol{\gamma}$, i.e. $\dot{\boldsymbol{\gamma}}(t_0) = 0$, $\boldsymbol{\gamma}$ can also be taken as a spacelike curve. In fact, we can take zero vector as a spacelike vector. The *norm* of the vector $\boldsymbol{x} \in \mathbb{R}^3_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. We take s as the arc-length parameter of $\boldsymbol{\gamma}$ satisfying $\|\boldsymbol{\gamma}'(s)\| = 1$, and t as the general parameter of $\boldsymbol{\gamma}$.

For any $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}^3_1$, the *pseudo vector product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as follows:

$$\boldsymbol{x} \wedge \boldsymbol{y} = \begin{vmatrix} -\boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (-(x_2y_3 - x_3y_2), x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

Obviously we have that $\langle \boldsymbol{x} \wedge \boldsymbol{y}, \boldsymbol{z}\rangle = \det(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$. Hence, $\boldsymbol{x} \wedge \boldsymbol{y}$ is pseudo-orthogonal to $\boldsymbol{x}$, $\boldsymbol{y}$. Let us denote $\boldsymbol{t}(s) = \boldsymbol{\gamma}'(s)$ as the unit tangent vector of $\boldsymbol{\gamma}$ and $\boldsymbol{e}(s) = \boldsymbol{\gamma}(s) \wedge \boldsymbol{t}(s)$ as the unit normal vector. We also have

$$\begin{cases} \boldsymbol{t}(s) \wedge \boldsymbol{e}(s) = -\boldsymbol{\gamma}(s) \\ \boldsymbol{e}(s) \wedge \boldsymbol{\gamma}(s) = \boldsymbol{t}(s). \end{cases}$$

Therefore we obtain a pseudo-orthonormal frame $\{\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{e}(s)\}$ along $\boldsymbol{\gamma}$, where $\boldsymbol{\gamma}(s)$ is a timelike vector, $\boldsymbol{t}(s)$ and $\boldsymbol{e}(s)$ are spacelike vectors.

Proposition 2.1 (see [8]). *The hyperbolic Frenet-Serret formula of $\boldsymbol{\gamma}$ is as follows:*

$$\begin{bmatrix} \boldsymbol{\gamma}'(s) \\ \boldsymbol{t}'(s) \\ \boldsymbol{e}'(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \kappa_g(s) \\ 0 & -\kappa_g(s) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma}(s) \\ \boldsymbol{t}(s) \\ \boldsymbol{e}(s) \end{bmatrix},$$

where $\kappa_g(s) = \det(\boldsymbol{\gamma}(s), \boldsymbol{t}(s), \boldsymbol{t}'(s))$ is the geodesic curvature of $\boldsymbol{\gamma}$ in H^2_+.

For the general parameter t, we get $\boldsymbol{t}(t) = \dot{\boldsymbol{\gamma}}(t)/\|\dot{\boldsymbol{\gamma}}(t)\|$ and $\boldsymbol{e}(t) = \boldsymbol{\gamma}(t) \wedge \boldsymbol{t}(t)$.

Proposition 2.2. *We have the following hyperbolic Frenet-Serret formula of $\boldsymbol{\gamma}$:*

$$\begin{bmatrix} \dot{\boldsymbol{\gamma}}(t) \\ \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{e}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \|\dot{\boldsymbol{\gamma}}(t)\| & 0 \\ \|\dot{\boldsymbol{\gamma}}(t)\| & 0 & \|\dot{\boldsymbol{\gamma}}(t)\|\kappa_g(t) \\ 0 & -\|\dot{\boldsymbol{\gamma}}(t)\|\kappa_g(t) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma}(t) \\ \boldsymbol{t}(t) \\ \boldsymbol{e}(t) \end{bmatrix},$$

where $\kappa_g(t) = \det(\boldsymbol{\gamma}(t), \boldsymbol{t}(t), \dot{\boldsymbol{t}}(t))/\|\dot{\boldsymbol{\gamma}}(t)\|$.

The definition of evolute of $\boldsymbol{\gamma}$ in H^2_+ has been given in [8] as follows.

Definition 2.3. Under the assumption $\kappa_g(t) \neq \pm 1$, the evolute of a regular curve $\boldsymbol{\gamma}$ is defined as $E_{\boldsymbol{\gamma}} : I \to \mathrm{Q}^2_\epsilon$ by

$$E_{\boldsymbol{\gamma}}(t) = \frac{1}{\sqrt{\epsilon(1-\kappa_g^2(t))}}(\kappa_g(t)\boldsymbol{\gamma}(t) + \boldsymbol{e}(t)).$$

$E_{\boldsymbol{\gamma}}$ is called hyperbolic evolute or de Sitter evolute of $\boldsymbol{\gamma}$, respectively when $\epsilon = -1$ or $\epsilon = 1$.

Remark 2.4. $E_{\boldsymbol{\gamma}}(t)$ is located in H^2_0 with $\kappa_g^2(t) < 1$, and it is in S^2_1 with $\kappa_g^2(t) > 1$.

Example 2.5. Let $\boldsymbol{\gamma} : I \to \mathrm{H}^2_+$ be a curve,

$$\boldsymbol{\gamma}(t) = (\cosh(t), \sinh(t)\cos(t), \sinh(t)\sin(t)).$$

It is a regular curve, since $\dot{\boldsymbol{\gamma}}(t) \neq \boldsymbol{0}$. We get $\kappa_g(t) = 1 + 1/\cosh^2(t) > 1$. Thus,

$$E_{\boldsymbol{\gamma}}(t) = \frac{(2\cosh(t), \sinh(t)\cos(t) - \cosh(t)\sin(t), \cosh(t)\cos(t) + \sinh(t)\sin(t))}{\sqrt{2\cosh^2(t)+1}}.$$

See Figure 1. The red part is $\boldsymbol{\gamma}$, and the green part is $E_{\boldsymbol{\gamma}}$.

Remark 2.6. If $E_{\boldsymbol{\gamma}}$ has a singular point t_0, then t_0 is just the vertex of $\boldsymbol{\gamma}$. In fact,

$$E'_{\boldsymbol{\gamma}}(s) = -\frac{\kappa'_g(s)(\boldsymbol{\gamma}(s) + \kappa_g(s)\boldsymbol{e}(s))}{(\kappa_g^2(s)-1)^{3/2}}.$$

Thus $E'_{\boldsymbol{\gamma}}(s) = 0$ if and only if $\kappa'_g(s) = 0$.

Note that even if $\boldsymbol{\gamma}$ is a regular curve, $E_{\boldsymbol{\gamma}}$ may have singularities. However we can't consider the evolute of $E_{\boldsymbol{\gamma}}$. In this paper, we give the definition of evolute of a curve with singular points, see § 4. First, we introduce the notions of null torus fronts and null torus frontals in the next section.

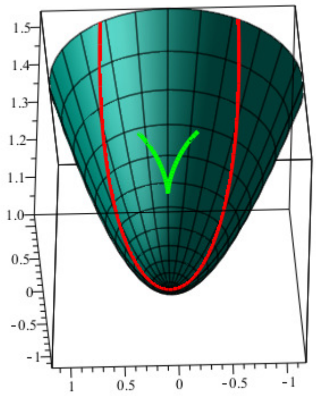

Figure 1: regular curve and its evolute

3. Framed Curves and Framed Immersions

If $\boldsymbol{\gamma}$ has a singular point, we can not construct a moving frame of $\boldsymbol{\gamma}$ in a traditional way. However, we could define a moving frame of a null torus front. First we give the notions.

Definition 3.1. We say that $(\boldsymbol{\gamma}, \nu) : I \to \mathrm{T}_2^4$ is a null torus framed curve, if $\langle \boldsymbol{\gamma}(t), \nu(t) \rangle = 0$ and $\langle \dot{\boldsymbol{\gamma}}(t), \nu(t) \rangle = 0$ for all $t \in I$. Moreover, if $(\boldsymbol{\gamma}, \nu)$ is an immersion, namely, $(\dot{\boldsymbol{\gamma}}(t), \dot{\nu}(t)) \neq (0, 0)$, we call $(\boldsymbol{\gamma}, \nu)$ a null torus framed immersion.

Definition 3.2. We say that $\boldsymbol{\gamma} : I \to \mathrm{H}_+^2$ is a null torus frontal if there exists a smooth mapping $\nu : I \to \mathrm{S}_1^2$ such that $(\boldsymbol{\gamma}, \nu)$ is a null torus framed curve. We also say that $\boldsymbol{\gamma} : I \to \mathrm{H}_+^2$ is a null torus front if there exists a smooth mapping $\nu : I \to \mathrm{S}_1^2$ such that $(\boldsymbol{\gamma}, \nu)$ is a null torus framed immersion.

Throughout the paper, we assume that the pair $(\boldsymbol{\gamma}, \nu)$ is co-orientable, the singular points of $\boldsymbol{\gamma}$ are finite. They can be removed by Remark 4.8 and 4.9, however, we add them for the sake of simplicity.

In order to consider properties of the evolute of a null torus front, we need a moving frame. Let $(\boldsymbol{\gamma}, \nu) : I \to \mathrm{T}_2^4$ be a null torus framed curve. If $\boldsymbol{\gamma}$ is singular at t_0, we can't define a frame in a traditional way. However, ν always exists even if t is a singular point of $\boldsymbol{\gamma}$. We take $\mu = \nu \wedge \boldsymbol{\gamma}$. We call the pair $\{\boldsymbol{\gamma}, \mu, \nu\}$ is a *moving frame* of $\boldsymbol{\gamma}$ and the Hyperbolic Frenet-Serret formula is given by

$$\begin{bmatrix} \dot{\boldsymbol{\gamma}}(t) \\ \dot{\mu}(t) \\ \dot{\nu}(t) \end{bmatrix} = \begin{bmatrix} 0 & \alpha(t) & 0 \\ \alpha(t) & 0 & -\ell(t) \\ 0 & \ell(t) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma}(t) \\ \mu(t) \\ \nu(t) \end{bmatrix},$$

where $\ell(t) = \langle \dot{\nu}(t), \mu(t) \rangle$, $\nu(t)$ and $\mu(t)$ are both unit spacelike vectors. We declare that $(\boldsymbol{\gamma}, -\nu)$ is also a null torus framed curve. In this case, $\ell(t)$ dose not change, but $\alpha(t)$ changes to $-\alpha(t)$. If $(\boldsymbol{\gamma}, \nu)$ is a null torus framed immersion, we have $(\ell(t), \alpha(t)) \neq (0, 0)$ for each $t \in I$. The pair (ℓ, α) is an important pair of functions of the null torus framed curves as the geodesic curvature of a regular curve. We call the pair (ℓ, α) *geodesic curvature* of the null torus framed curve.

We also have

$$\begin{cases} \mu \wedge \nu = -\boldsymbol{\gamma} \\ \boldsymbol{\gamma} \wedge \mu = \nu. \end{cases}$$

Remark 3.3. (ℓ, α) depends on a parametrisation. In fact, let I and $\tilde{I}$ be intervals. A smooth function $s : \tilde{I} \to I$ is a change of parameter. Let $(\boldsymbol{\gamma}, \nu) : I \to \mathrm{T}_2^4$ and $(\tilde{\boldsymbol{\gamma}}, \tilde{\nu}) : \tilde{I} \to \mathrm{T}_2^4$ be null torus framed curves whose geodesic curvatures are (ℓ, α) and $(\tilde{\ell}, \tilde{\alpha})$ respectively. Suppose $(\boldsymbol{\gamma}, \nu)$ and $(\tilde{\boldsymbol{\gamma}}, \tilde{\nu})$ are parametrically equivalent via the change of parameter $s : \tilde{I} \to I$. Thus $(\tilde{\boldsymbol{\gamma}}(t), \tilde{\nu}(t)) = (\boldsymbol{\gamma}(s(t)), \nu(s(t)))$ for all $t \in \tilde{I}$. By differentiation, we have

$$\tilde{\ell}(t) = \ell(s(t))\dot{s}(t), \qquad \tilde{\alpha}(t) = \alpha(s(t))\dot{s}(t).$$

However, it presents good behaviors if we consider a null torus framed immersion, because its geodesic curvature never depends on a parametrisation. It is just an analogous result to the case of Euclidean plane [5]. We only give its the normalized geodesic curvature. Let $(\boldsymbol{\gamma}, \nu)$ be a null torus framed immersion. Then $(\ell(t), \alpha(t)) \neq (0, 0)$. Setting

$$(\tilde{\ell}(t), \tilde{\alpha}(t)) = (\frac{\ell(t)}{\sqrt{\ell^2(t) + \alpha^2(t)}}, \frac{\alpha(t)}{\sqrt{\ell^2(t) + \alpha^2(t)}}).$$

By Remark 3.3, the normalized curvature $(\tilde{\ell}(t), \tilde{\alpha}(t))$ is independent on the choice of a parametrization.

Next, we give the relationship between $(\ell(t), \alpha(t))$ of the framed curve and $\kappa_g(t)$ if $\boldsymbol{\gamma}$ is a regular curve.

Proposition 3.4. *If $\boldsymbol{\gamma}$ is a regular curve, then $\ell(t) = -|\alpha(t)|\kappa_g(t)$.*

Proof. By a direct calculation, $\|\dot{\boldsymbol{\gamma}}(t)\| = |\alpha(t)|$, $\boldsymbol{\gamma} \wedge \mu = \nu$, and

$$\begin{aligned}\kappa_g(t) &= \frac{\det(\boldsymbol{\gamma}(t), \boldsymbol{t}(t), \dot{\boldsymbol{t}}(t))}{\|\dot{\boldsymbol{\gamma}}(t)\|} \\ &= \frac{\det(\boldsymbol{\gamma}(t), \mu(t), -\ell(t)\nu(t) + \alpha(t)\boldsymbol{\gamma}(t))}{|\alpha(t)|} \\ &= -\frac{\ell(t)}{|\alpha(t)|}.\end{aligned}$$

Therefore we have $\ell(t) = -|\alpha(t)|\kappa_g(t)$. □

Accordingly, it is a nature generalisation. Moreover, for a null torus framed immersion $(\boldsymbol{\gamma}, \nu)$, we say that t_0 is an *inflection point* of the front $\boldsymbol{\gamma}$ if $\ell(t_0) = 0$. Since $\alpha(t_0) \neq 0$ and Proposition 3.4, $\ell(t_0) = 0$ is equivalent to the condition $\kappa_g(t_0) = 0$.

Example 3.5. Let $\boldsymbol{\gamma} : I \to \mathrm{H}^2_+$ be a curve,

$$\boldsymbol{\gamma}(t) = \left(\cosh(t^2), \sinh(t^2)\cos(t^3), \sinh(t^2)\sin(t^3)\right).$$

We get

$$\dot{\boldsymbol{\gamma}}(t) = (2t\sinh(t^2), 2t\cosh(t^2)\cos(t^3) - 3t^2\sinh(t^2)\sin(t^3), 2t\cosh(t^2)\sin(t^3) + 3t^2\sinh(t^2)\cos(t^3)).$$

So $\boldsymbol{\gamma}$ is singular at $t = 0$. Take $\nu = (\nu_1, \nu_2, \nu_3)$, where

$$\begin{aligned}\nu_1(t) &= \frac{3t\sinh^2(t^2)}{\sqrt{4 + 9t^2\sinh^2(t^2)}}, \\ \nu_2(t) &= \frac{2\sin(t^3 + 3t\sinh(t^2)\cosh(t^2)\cos(t^3))}{\sqrt{4 + 9t^2\sinh^2(t^2)}}, \\ \nu_3(t) &= \frac{-2\cos(t^3 + 3t\sinh(t^2)\cosh(t^2)\sin(t^3))}{\sqrt{4 + 9t^2\sinh^2(t^2)}}.\end{aligned}$$

It satisfies $\langle \boldsymbol{\gamma}, \nu \rangle = \langle \dot{\boldsymbol{\gamma}}, \nu \rangle = 0$ and $\langle \nu, \nu \rangle = 1$. Hence, $(\boldsymbol{\gamma}, \nu)$ is a null torus framed curve. See the left one in Figure 2. In general, if

$$\boldsymbol{\gamma}(t) = (\cosh(t^m), \sinh(t^m)\cos(t^n), \sinh(t^m)\sin(t^n)),$$

it is also a null torus framed curve. When $m = 3$, $n = 4$, see the right one in Figure 2.

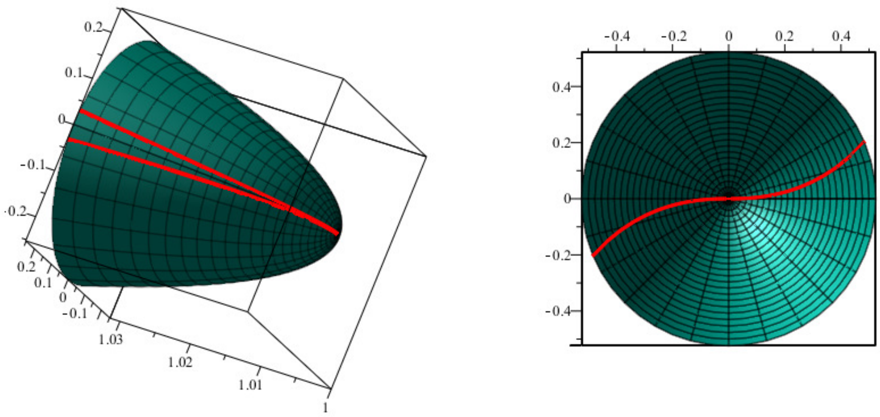

Figure 2: null torus framed curve with m=2, n=3 and projection of a null torus framed curve with m=3, n=4 along the timelike axis

4. Evolute of Null Torus Fronts

In this section, we give the definition of the evolute of a null torus front and further focus on its properties. First, we introduce the parallel curves.

Let $(\boldsymbol{\gamma}, \nu) : I \to \mathrm{T}_2^4$ be a null torus framed curve with the geodesic curvature (ℓ, α). We define a *parallel curve* $\boldsymbol{\gamma}_\lambda : I \to Q_\epsilon^2$ of $\boldsymbol{\gamma}$ by

$$\boldsymbol{\gamma}_\lambda(t) = \frac{\boldsymbol{\gamma}(t) + \lambda\nu(t)}{\sqrt{\epsilon(\lambda^2 - 1)}},$$

where $\lambda \neq \pm 1$.

Through a direct calculation, we have the following.

Lemma 4.1. *If $\boldsymbol{\gamma}_\lambda$ is a regular curve, then*

$$\ell(t) + \lambda\alpha(t) = -|\alpha(t) + \lambda\ell(t)|\kappa_{g\lambda}(t).$$

Lemma 4.2. *For a null torus framed immersion $(\boldsymbol{\gamma}, \nu) : I \to \mathrm{T}_2^4$, the parallel curve $\boldsymbol{\gamma}_\lambda : I \to \mathrm{Q}_\epsilon^2$ is a null torus front for each $\lambda \neq \pm 1$.*

Proof. We consider the case of $\epsilon = -1$. It means the parallel curve is located in H_+^2. Take

$$\nu_\lambda(t) = \frac{\lambda\boldsymbol{\gamma}(t) + \nu(t)}{\sqrt{1 - \lambda^2}} \in \mathrm{S}_1^2.$$

Since

$$\boldsymbol{\gamma}_\lambda(t) = \frac{\boldsymbol{\gamma}(t) + \lambda\nu(t)}{\sqrt{1 - \lambda^2}} \quad \text{and} \quad \dot{\boldsymbol{\gamma}}_\lambda(t) = \frac{\dot{\boldsymbol{\gamma}}(t) + \lambda\dot{\nu}(t)}{\sqrt{1 - \lambda^2}},$$

we have $\langle \boldsymbol{\gamma}_\lambda(t), \nu_\lambda \rangle = \langle \dot{\boldsymbol{\gamma}}_\lambda(t), \nu_\lambda \rangle = 0$. We make a hypothesis that $\boldsymbol{\gamma}_\lambda$ is not a front. There exists $t_0 \in I$, such that $(\dot{\boldsymbol{\gamma}}_\lambda(t_0), \dot{\nu}_\lambda(t_0)) = (0, 0)$. It means

$$\dot{\boldsymbol{\gamma}}(t_0) + \lambda\dot{\nu}(t_0) = \lambda\dot{\boldsymbol{\gamma}}(t_0) + \dot{\nu}(t_0) = 0.$$

Accordingly, $(\dot{\boldsymbol{\gamma}}(t_0), \dot{\nu}(t_0)) = (0, 0)$. This contradicts with the fact that $(\boldsymbol{\gamma}, \nu)$ is an immersion.

When $\epsilon = 1$, we consider timelike parallel curves located in S_1^2. Proof is similar, that we omitted here. $\square$

Proposition 4.3. *Let $(\boldsymbol{\gamma}, \nu)$ be a null torus framed curve. If $\boldsymbol{\gamma}$ is a regular curve and $\lambda \neq 1/\kappa_g(t)$, then a parallel curve γ_λ is also a regular curve and*

$$E_{\boldsymbol{\gamma}_\lambda}(t) = -\epsilon E_{\boldsymbol{\gamma}}(t).$$

Proof. Since

$$\boldsymbol{\gamma}_\lambda(t) = \frac{\boldsymbol{\gamma}(t) + \lambda \boldsymbol{e}(t)}{\sqrt{\epsilon(\lambda^2 - 1)}} \quad \text{and} \quad \dot{\boldsymbol{\gamma}}_\lambda(t) = \frac{|\dot{\boldsymbol{\gamma}}(t)|(1 - \lambda\kappa_g(t))\boldsymbol{t}}{\sqrt{\epsilon(\lambda^2 - 1)}}$$

and $\lambda \neq 1/\kappa_g(t)$, $\boldsymbol{\gamma}_\lambda$ is a regular curve. By a direct calculation, we have

$$\begin{aligned}
\kappa_{g\lambda}(t) &= \frac{\kappa_g(t) - \lambda}{|1 - \lambda\kappa_g(t)|}, \\
\boldsymbol{e}_\lambda(t) &= \frac{1 - \lambda\kappa_g(t)}{|1 - \lambda\kappa_g(t)|} \frac{1}{\sqrt{\varepsilon(\lambda^2 - 1)}} (\boldsymbol{e}(t) + \lambda\boldsymbol{\gamma}(t)).
\end{aligned}$$

Hence

$$\begin{aligned}
E_{\boldsymbol{\gamma}_\lambda}(t) =& \frac{1}{\sqrt{\epsilon(1 - \kappa_{g\lambda}^2)}} (\kappa_{g\lambda}(t)\boldsymbol{\gamma}_\lambda(t) + \boldsymbol{e}_\lambda(t)) \\
=& \frac{1}{\sqrt{\epsilon(1 - (\frac{\kappa_g(t) - \lambda}{|1 - \lambda\kappa_g(t)|})^2)}} \Big(\frac{\kappa_g(t) - \lambda}{|1 - \lambda\kappa_g(t)|} \frac{\boldsymbol{\gamma}(t) + \lambda\boldsymbol{e}(t)}{\sqrt{\epsilon(\lambda^2 - 1)}} + \frac{1 - \lambda\kappa_g(t)}{|1 - \lambda\kappa_g(t)|} \frac{1}{\sqrt{\epsilon(\lambda^2 - 1)}} (\boldsymbol{e}(t) + \lambda\boldsymbol{\gamma}(t)) \Big) \\
=& \frac{1}{\sqrt{\epsilon(1 - \kappa_g^2(t))(1 - \lambda^2)}} \Big(\frac{1 - \lambda^2}{\sqrt{\epsilon(\lambda^2 - 1)}} \kappa_g(t)\boldsymbol{\gamma}(t) + \frac{1 - \lambda^2}{\sqrt{\epsilon(\lambda^2 - 1)}} \boldsymbol{e}(t) \Big) \\
=& \frac{1 - \lambda^2}{\sqrt{\epsilon(\lambda^2 - 1)}\sqrt{\epsilon(\lambda^2 - 1)}} \frac{\kappa_g(t)\boldsymbol{\gamma}(t) + \boldsymbol{e}(t)}{\sqrt{\epsilon(1 - \kappa_g^2(t))}} \\
=& - \epsilon E_{\boldsymbol{\gamma}}(t).
\end{aligned}$$

□

Remark 4.4. Let $(\boldsymbol{\gamma}, \nu)$ be a null torus framed immersion. If t_0 is a singular point of the front γ, then $\lim_{t \to t_0} |\kappa_g(t)| = \infty$. By the equality

$$\kappa_{g\lambda}(t) = \frac{\kappa_g(t) - \lambda}{|1 - \lambda\kappa_g(t)|},$$

we have $\lim_{t \to t_0} |\kappa_{g\lambda}(t)| \neq 0$.

We now define the evolute of a null torus front.

Definition 4.5. Let $(\boldsymbol{\gamma}, \nu) : I \to \mathrm{T}_2^4$ be a null torus framed immersion. We define an evolute $\mathcal{E}_{\boldsymbol{\gamma}} : I \to \mathrm{Q}_\epsilon^2$ of $\boldsymbol{\gamma}$ as follows. If t is a regular point,

$$\mathcal{E}_\gamma(t) = \frac{1}{\sqrt{\epsilon(1 - \kappa_g^2(t))}} (\kappa_g \boldsymbol{\gamma}(t) + \boldsymbol{e}(t)).$$

If t_0 is a singular point, for any $t \in (t_0 - \delta, t_0 + \delta)$

$$\mathcal{E}_\gamma(t) = \frac{-\epsilon}{\sqrt{\epsilon(1 - \kappa_{g\lambda}^2(t))}} (\kappa_{g\lambda} \boldsymbol{\gamma}_\lambda(t) + \boldsymbol{e}_\lambda(t)),$$

where δ is a sufficiently small positive real number and $\lambda \in \mathbb{R}$ is satisfied the condition $\lambda \neq 1/\kappa_g(t)$.

We give another representation of the evolute by using the moving frame and its geodesic curvature.

Theorem 4.6. *Under the condition of* $|\alpha(t)| \neq |\ell(t)|$*, the evolute of a null torus front* $\mathcal{E}_{\boldsymbol{\gamma}}(t) : I \to \mathrm{Q}^2_\epsilon$ *is represented by*

$$\mathcal{E}_{\boldsymbol{\gamma}}(t) = \frac{-\ell(t)\boldsymbol{\gamma}(t) + \alpha(t)\nu(t)}{\sqrt{\epsilon(\alpha^2(t) - \ell^2(t))}} \tag{4.1}$$

and $\mathcal{E}_{\boldsymbol{\gamma}}(t)$ *is a null torus front.*

Proof. First suppose that $\boldsymbol{\gamma}$ is a regular curve. Since $\dot{\boldsymbol{\gamma}}(t) = \alpha(t)\mu(t)$, we have $|\alpha(t)| \neq 0$ and

$$\boldsymbol{t}(t) = \frac{\alpha(t)}{|\alpha(t)|}\mu(t), \qquad \boldsymbol{e}(t) = \boldsymbol{\gamma}(t) \wedge \boldsymbol{t}(t) = \frac{\alpha(t)}{|\alpha(t)|}\nu(t).$$

By Proposition 3.4, $\kappa_g(t) = -\ell(t)/|\alpha(t)|$. Then

$$\mathcal{E}_\gamma(t) = \frac{-\ell(t)\boldsymbol{\gamma}(t) + \alpha(t)\nu(t)}{\sqrt{\epsilon(\alpha^2(t) - \ell^2(t))}}.$$

Second suppose that t_0 is a singular point of $\boldsymbol{\gamma}$. Consider $\boldsymbol{\gamma}_\lambda$ in hyperbolic plane, we know $\boldsymbol{\gamma}_\lambda$ is a regular curve around the neighbourhood of t_0 with $\lambda \neq 1/\kappa_g(t)$. Since $\dot{\boldsymbol{\gamma}}_\lambda(t) = (\alpha(t) + \lambda\ell(t))/\sqrt{1-\lambda^2}\mu(t)$, we have $|\alpha(t) + \lambda\ell(t)| \neq 0$ and

$$\begin{aligned}
\boldsymbol{t}_\lambda(t) &= \frac{\alpha(t) + \lambda\ell(t)}{|\alpha(t) + \lambda\ell(t)|}\mu(t),\\
\boldsymbol{e}_\lambda(t) &= \boldsymbol{\gamma}_\lambda(t) \wedge \boldsymbol{t}_\lambda(t) = \frac{\alpha(t) + \lambda\ell(t)}{|\alpha(t) + \lambda\ell(t)|}\frac{\nu(t) + \lambda\boldsymbol{\gamma}(t)}{\sqrt{1-\lambda^2}}.
\end{aligned}$$

By lemma 4.1, $\kappa_{g\lambda}(t) = -(\ell(t) + \lambda\alpha(t))/|\alpha(t) + \lambda\ell(t)|$.
Then,

$$\begin{aligned}
&\mathcal{E}_\gamma(t)\\
=&\mathcal{E}_{\gamma_\lambda}(t)\\
=&\frac{1}{\sqrt{-(1-\kappa^2_{g\lambda}(t))}}(\kappa_{g\lambda}(t)\boldsymbol{\gamma}_\lambda(t) + \boldsymbol{e}_\lambda(t))\\
=&\frac{|\alpha(t) + \lambda\ell(t)|}{\sqrt{(\ell(t) + \lambda\alpha(t))^2 - (\alpha(t) + \lambda\ell(t))^2}}\left(-\frac{\ell(t) + \lambda\alpha(t)}{|\alpha(t) + \lambda\ell(t)|}\frac{\boldsymbol{\gamma}(t) + \lambda\nu(t)}{\sqrt{1-\lambda^2}} + \frac{\alpha(t) + \lambda\ell(t)}{|\alpha(t) + \lambda\ell(t)|}\frac{\nu(t) + \lambda\boldsymbol{\gamma}(t)}{\sqrt{1-\lambda^2}}\right)\\
=&\frac{1}{\sqrt{(1-\lambda^2)(\ell^2(t) - \alpha^2(t))}}\frac{(-\ell(t)(1-\lambda^2)\boldsymbol{\gamma}(t) + \alpha(t)(1-\lambda^2)\nu(t))}{\sqrt{1-\lambda^2}}\\
=&\frac{-\ell(t)\boldsymbol{\gamma}(t) + \alpha(t)\nu(t)}{\sqrt{\ell^2(t) - \alpha^2(t)}}.
\end{aligned}$$

If we take $\tilde{\nu}(t) = \mu(t)$, then $(\mathcal{E}_\gamma(t), \tilde{\nu}(t))$ is a framed immersion. In fact,

$$\begin{aligned}
\dot{\mathcal{E}}_\gamma(t) =& \frac{\dot{\alpha}(t)\ell(t) - \alpha(t)\dot{\ell}(t)}{(\ell^2(t) - \alpha^2(t))^{3/2}}(\ell(t)\nu(t) - \alpha(t)\boldsymbol{\gamma}(t))\\
=& \frac{d(\alpha(t)/\ell(t))}{dt}\frac{\ell^2(t)}{(\ell^2(t) - \alpha^2(t))^{3/2}}(\ell(t)\nu(t) - \alpha(t)\boldsymbol{\gamma}(t)).
\end{aligned}$$

Accordingly, $\langle \mathcal{E}_\gamma(t), \tilde{\nu}(t)\rangle = \langle \dot{\mathcal{E}}_\gamma(t), \tilde{\nu}(t)\rangle = 0$. And

$$\dot{\tilde{\nu}}(t) = -\ell(t)\nu(t) + \alpha(t)\boldsymbol{\gamma}(t) = 0$$

equals to $\ell(t) = \alpha(t) = 0$. Since $(\boldsymbol{\gamma}, \nu)$ is a null torus framed immersion, we get $\dot{\tilde{\nu}}(t) \neq 0$. It follows that $\mathcal{E}_\gamma$ is a null torus front.

On the other hand, if $\boldsymbol{\gamma}_\lambda$ is in S^2_1, we have $\mathcal{E}_\gamma(t) = -\mathcal{E}_{\gamma_\lambda}(t)$. We can give the proof by the same way, that we omitted here. □

Remark 4.7. If $|\alpha| = |\ell|$, the evolute $\mathcal{E}_\gamma$ of the null torus front is exactly located on the null cone. We can characterize it by some null directions, for example $-\ell\boldsymbol{\gamma} + \alpha\nu$.

Remark 4.8. Let $(\boldsymbol{\gamma}, \nu)$ be a null torus framed immersion, then $(\boldsymbol{\gamma}, -\nu)$ is also a null torus framed immersion. However $\mathcal{E}_{\boldsymbol{\gamma}}$ does not change. It follows that we can define an evolute of a non co-orientable front by taking double covering of $\boldsymbol{\gamma}$.

Remark 4.9. By the representation (4.1), we may define the evolute of a null torus front even if $\boldsymbol{\gamma}$ have non-isolated singularities, under the condition $|\ell(t)| \neq |\alpha(t)|$.

Corollary 4.10. *Under the above notations, the evolute of an evolute of a null torus front $\mathcal{E}_{\boldsymbol{\gamma}}$ is given by*

$$\mathcal{E}_{\mathcal{E}_\gamma}(t) = \frac{(\ell^2(t) - \alpha^2(t))^{3/2}\mathcal{E}_{\boldsymbol{\gamma}}(t) + (\dot{\alpha}(t)\ell(t) - \alpha(t)\dot{\ell}(t))\mu(t)}{\sqrt{(\ell^2(t) - \alpha^2(t))^3 - (\dot{\alpha}(t)\ell(t) - \alpha(t)\dot{\ell}(t))^2}}.$$

If t_0 is a singular point of $\boldsymbol{\gamma}$, then $\alpha(t_0) = 0$. As a corollary of Theorem 4.6, we have following.

Corollary 4.11. *If t_0 is a singular point of $\boldsymbol{\gamma}$, then*

$$\mathcal{E}_{\boldsymbol{\gamma}}(t_0) = -\frac{\ell(t_0)}{|\ell(t_0)|}\boldsymbol{\gamma}(t_0).$$

Remark 4.12. The evolute of a null torus front lies in Q^2_ϵ. However, it only lies in H^2_0 at a singular point t_0, which can be taken as the limit position of the evolute of its parallel curves in Q^2_ϵ. In other words, the part in S^2_1 of the evolute must be regular, singularities only occur in H^2_0.

Then we give a natural generalization for a null torus frontal in form. If there exists a unique smooth function $\Omega(t) : I \to \mathbb{R}$ such that $\alpha(t) = \Omega(t)\ell(t)$, let

$$\mathcal{E}_{\boldsymbol{\gamma}}(t) = \frac{-\boldsymbol{\gamma}(t) + \Omega(t)\nu(t)}{\sqrt{\epsilon(-1 + \Omega^2(t))}}$$

be the evolute of the null torus frontal.

It is well known that a singular point of the evolute of a regular plane curve is corresponding to a vertex of this curve, namely $\dot{\kappa}(t_0) = 0$. Here, we extend the notion to hyperbolic plane curves.

For a null torus framed immersion (or a null torus framed curve) $(\boldsymbol{\gamma}, \nu)$ with the geodesic curvature (ℓ, α), t_0 is a *geodesic vertex* of $\boldsymbol{\gamma}$ if

$$(d/dt)(\alpha/\ell)(t_0) = 0 \text{ (or } (d/dt)\Omega(t_0) = 0),$$

namely, $(d/dt)\mathcal{E}_{\boldsymbol{\gamma}}(t_0) = 0$. Note that if t_0 is a regular point of $\boldsymbol{\gamma}$, the definition of the geodesic vertex coincides with usual geodesic vertex for regular curves.

In 1945, Jackson gave the four-vertex theorem for surfaces of constant curvature [9]. Moreover, in [8] the authors also declared that four vertices theorem holds for curves in H^2_+. As a further and intensive exploration, we discuss the correlations between the singularites and geodesic vertices for a closed null torus framed curve. First, we give a result for a closed null torus framed immersion without inflection points. And then we consider the closed null torus framed curve with inflection points.

Theorem 4.13. *Let $(\boldsymbol{\gamma},\nu):[0,2\pi)\to \mathrm{T}^4_2$ be a closed null torus framed immersion without inflection points.*

(1) *If $\boldsymbol{\gamma}$ has at least $2p$ singular points which degenerate more than $3/2$ cusp, then $\boldsymbol{\gamma}$ has at least $4p$ geodesic vertices.*

(2) *If $\boldsymbol{\gamma}$ has at least p singular points, then $\boldsymbol{\gamma}$ has at least p geodesic vextices where $p\geq 2$.*

Proof. (1) We declare that if t_0 is a singular point of $\boldsymbol{\gamma}$ which degenerate more than 3/2 cusp, then t_0 is a geodesic vertex of the front $\boldsymbol{\gamma}$. In fact, because of $\alpha(t_0)=\dot{\alpha}(t_0)=0$, we have $(d/dt)(\alpha/\ell)(t_0)=0$.

Let t_i be a singular point of $\boldsymbol{\gamma}$ for each $i\in\{1,\dots,n\}$. Suppose that at least two of them are degenerate more than 3/2 cusp. So there is at least one geodesic vertex between the two adjacent singular points. Because there is no inflection points of $\boldsymbol{\gamma}$, the sign of the geodesic curvature of $\boldsymbol{\gamma}$ on regular part will not change. Accordingly, either $\lim_{t\to t_i}\kappa_g(t)=\infty$ or $\lim_{t\to t_i}\kappa_g(t)=-\infty$ for all $i\in\{1,\dots,n\}$. This concludes there exist $t\in(t_i,t_{i+1})$ such that $\dot{\kappa_g}(t)=0$ for all $i\in\{1,\dots,n\}$. Besides, as $\boldsymbol{\gamma}$ is closed, there exists $t\in[0,t_1)\cup(t_n,2\pi)$ such that $\dot{\kappa_g}(t)=0$. Then, $\boldsymbol{\gamma}$ has at least four vertices. It is easy to see that if $\boldsymbol{\gamma}$ has at least $2p$ singular points which degenerate more than 3/2 cusp, then $\boldsymbol{\gamma}$ has at least $4p$ geodesic vertices.

(2) It can be proved by the same method of (1). □

Theorem 4.14. *Let $(\boldsymbol{\gamma},\nu):[0,2\pi)\to \mathrm{T}^4_2$ be a closed null torus framed curve. If $\boldsymbol{\gamma}$ has at least $2p$ inflection points, then $\boldsymbol{\gamma}$ has at least $2p$ geodesic vertices.*

Proof. Let t_0 be an inflection point of $\boldsymbol{\gamma}$. We have $\kappa_g(t_0)=0$ and the sign of $\kappa_g(t)$ changes at the opposite site of t_0. Moreover, since $\boldsymbol{\gamma}$ is closed, the number of inflection points are even. Suppose that there are two inflection points t_1 and t_2. This concludes that there exists a point $t\in(t_1,t_2)$ such that $\dot{\kappa}(t)=0$. Because of the closed curve, there also exists a point $t\in[0,t_1)\cup(t_2,2\pi)$ such that $\dot{\kappa}(t)=0$. Therefore, $\boldsymbol{\gamma}$ has 2 vertices. It is obvious that if $\boldsymbol{\gamma}$ has at least $2n$ inflection points, then $\boldsymbol{\gamma}$ has at least $2n$ vertices. □

Then we give an example of a framed curve.

Example 4.15. By Example 3.5, we get the geodesic curvature of the null torus framed curve $(\boldsymbol{\gamma},\nu)$ as follows,

$$\begin{aligned}\alpha(t)=&t\sqrt{4+9t^2\sinh^2(t^2)},\\ \ell(t)=&\frac{3(9t^4\cosh^3(t^2)-9t^4\cosh(t^2)+8t^2\cosh(t^2)+2\sinh(t^2))}{9t^2\cosh^2(t^2)-9t^2+4}.\end{aligned}$$

We have $(\alpha(0),\ell(0))=(0,0)$, thus, $\boldsymbol{\gamma}$ is a null torus frontal. Take

$$\Omega(t)=\frac{t\left(3t^2\cosh^2(t^2)-3t^2+4/3\right)^{3/2}}{9t^4\cosh^3(t^2)-9t^4\cosh(t^2)+8t^2\cosh(t^2)+2\sinh(t^2)}.$$

Through a direct calculation, $\alpha(t)=\Omega(t)\ell(t)$.
Thus, $\mathcal{E}_{\boldsymbol{\gamma}}(t)=(\mathcal{E}_{\boldsymbol{\gamma}1},\mathcal{E}_{\boldsymbol{\gamma}1},\mathcal{E}_{\boldsymbol{\gamma}3})$, where

$$\begin{aligned}\mathcal{E}_{\boldsymbol{\gamma}1}&=-\frac{\cosh(t^2)}{\sqrt{1-P}}+\frac{tTS_1}{Q\sqrt{1-P}},\\ \mathcal{E}_{\boldsymbol{\gamma}2}&=-\frac{\sinh(t^2)\cos(t^3)}{\sqrt{1-P}}+\frac{tTS_2}{Q\sqrt{1-P}},\\ \mathcal{E}_{\boldsymbol{\gamma}2}&=-\frac{\sinh(t^2)\sin(t^3)}{\sqrt{1-P}}+\frac{tTS_3}{Q\sqrt{1-P}}.\end{aligned}$$

Here,

$$P=\frac{t^2(4+9t^2\sinh^2(t^2))T^2}{Q^2},$$

$$
\begin{aligned}
Q &= \cosh(t^2)(27t^4\cosh^2(t^2) - 27t^4 + 24t^2) + 6\sinh^2(t^2),\\
T &= 9t^2\cosh^2(t^2) - 9t^2 + 4,\\
S_1 &= 3t\sinh^2(t^2),\\
S_2 &= 3\sin(t^3) + 3t\sinh(t^2)\cosh(t^2)\cos(t^3),\\
S_3 &= -2\cos(t^3) + 3t\sinh(t^2)\cosh(t^2)\sin(t^3).
\end{aligned}
$$

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