



Fixed point results for multivalued contractive type maps

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Abstract

Using generalized distance in metric spaces, we prove some fixed point results for multivalued generalized contractive type maps. Consequently, several known fixed point results are either improved or generalized. An interesting example in support of the result is also presented. ©2016 All rights reserved.

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1. Introduction and Preliminaries

Using the concept of Hausdorff metric, Nadler [13] introduced a notion of multivalued contraction maps and proved a multivalued version of the well-known Banach contraction principle [1], which states that each closed bounded multi-valued contraction map on a complete metric space has a fixed point. Since then various fixed point results concerning multivalued contractions have appeared. Feng and Liu [5] extended Nadler's fixed point theorem without using the concept of Hausdorff metric. While in [8] Klim and Wardowski generalized their result. Ćirić [4] obtained some interesting fixed point results which extend and generalize many known results including these cited results.

In [7], Kada et al. introduced the concept of w -distance on a metric space and studied the properties, examples and some classical results with respect to w -distance. Using this generalized distance, Suzuki and Takahashi [15] have introduced notions of single-valued and multivalued weakly contractive maps and proved fixed point results for such maps. Consequently, they generalized the Banach contraction principle and Nadler's fixed point result. Some other fixed point results concerning w -distance can be found in [9, 10, 11, 16, 19].

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In [14], Susuki generalized the concept of w -distance by introducing the notion of τ -distance on metric space (X, d) . In [14], Suzuki improved several classical results including the Caristi's fixed point theorem for single-valued maps with respect to τ -distance. Recently, Ume [17] generalized the notion of τ -distance by introducing a concept of u -distance in metric spaces. Several interesting fixed point results have been studied and generalized with respect to u -distance; see [2, 3, 12, 17, 18].

Motivated by the work in [2, 3, 17], we prove some fixed point results for multivalued contractive type maps induced by the generalized distance in metric spaces. Our results unify and generalize many known fixed point results including the well-known corresponding fixed point results of Ciric [4], Klim and Wardowski [8], Latif and Abdou [10].

Let (X, d) be a metric space, 2^X a collection of nonempty subsets of X , and $CB(X)$ a collection of nonempty closed bounded subsets of X , $Cl(X)$ a collection of nonempty closed subsets of X , $K(X)$ a collection of nonempty compact subsets of X and H the Hausdorff metric induced by d . Then for any $A, B \in CB(X)$,

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

An element $x \in X$ is called a *fixed point* of a multivalued map $T : X \rightarrow 2^X$ if $x \in T(x)$. We denote $Fix(T) = \{x \in X : x \in T(x)\}$. A sequence $\{x_n\}$ in X is called an *orbit* of T at $x_0 \in X$ if $x_n \in T(x_{n-1})$ for all $n \geq 1$. A map $f : X \rightarrow \mathbb{R}$ is called *T -orbitally lower semicontinuous* if for any orbit $\{x_n\}$ of T and $x \in X$, $x_n \rightarrow x$ implies that $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$.

In [14], Susuki generalized the concept of w -distance by introducing the following notion of τ -distance on metric space (X, d) .

A function $p : X \times X \rightarrow \mathbb{R}_+$ is a τ -distance on X if it satisfies the following conditions for any $x, y, z \in X$:

- (τ_1) $p(x, z) \leq p(x, y) + p(y, z)$;
- (τ_2) $\eta(x, 0) = 0$ and $\eta(x, t) \geq t$ for all $x \in X$ and $t \geq 0$, and η is concave and continuous in its second variable;
- (τ_3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ imply $p(u, x) \leq \lim_n \inf p(u, x_n)$ for all $u \in X$;
- (τ_4) $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$;
- (τ_5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

Examples and properties of τ -distance are given in [14]. In [14], Suzuki improved several classical results including the Caristi's fixed point theorem for single valued maps with respect to τ -distance.

In [17], Ume generalized the notion of τ -distance by introducing u -distance as follows:

A function $p : X \times X \rightarrow \mathbb{R}_+$ is called u -distance on X if there exists a function $\theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that the followings hold for each $x, y, z \in X$:

- (u_1) $p(x, z) \leq p(x, y) + p(y, z)$.
- (u_2) $\theta(x, y, 0, 0) = 0$ and $\theta(x, y, s, t) \geq \min\{s, t\}$ for each $s, t \in \mathbb{R}_+$, and for every $\epsilon > 0$, there exists $\delta > 0$ such that $|s - s_0| < \delta$, $|t - t_0| < \delta$, $s, s_0, t, t_0 \in \mathbb{R}_+$ and $y \in X$ imply

$$|\theta(x, y, s, t) - \theta(x, y, s_0, t_0)| < \epsilon.$$

- (u_3)

$$\lim_{n \rightarrow \infty} x_n = x, \lim_{n \rightarrow \infty} \sup\{\theta(w_n, z_n, p(w_n, x_m), p(z_n, x_m)) : m \geq n\} = 0$$

imply

$$p(y, x) \leq \liminf_{n \rightarrow \infty} p(y, x_n)$$

(u₄)

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup\{p(x_n, w_m) : m \geq n\} &= 0, \quad \lim_{n \rightarrow \infty} \sup\{p(y_n, z_m) : m \geq n\} = 0, \\ \lim_{n \rightarrow \infty} \theta(x_n, w_n, s_n, t_n) &= 0, \quad \lim_{n \rightarrow \infty} \theta(y_n, z_n, s_n, t_n) = 0 \end{aligned}$$

imply

$$\lim_{n \rightarrow \infty} \theta(w_n, z_n, s_n, t_n) = 0$$

or

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup\{p(w_n, x_m) : m \geq n\} &= 0, \quad \lim_{n \rightarrow \infty} \sup\{p(z_m, y_n) : m \geq n\} = 0, \\ \lim_{n \rightarrow \infty} \theta(x_n, w_n, s_n, t_n) &= 0, \quad \lim_{n \rightarrow \infty} \theta(y_n, z_n, s_n, t_n) = 0 \end{aligned}$$

imply

$$\lim_{n \rightarrow \infty} \theta(w_n, z_n, s_n, t_n) = 0;$$

(u₅)

$$\lim_{n \rightarrow \infty} \theta(w_n, z_n, p(w_n, x_n), p(z_n, x_n)) = 0, \quad \lim_{n \rightarrow \infty} \theta(w_n, z_n, p(w_n, y_n), p(z_n, y_n)) = 0$$

imply

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$$

or

$$\lim_{n \rightarrow \infty} \theta(a_n, b_n, p(x_n, a_n), p(x_n, b_n)) = 0, \quad \lim_{n \rightarrow \infty} \theta(a_n, b_n, p(y_n, a_n), p(y_n, b_n)) = 0$$

imply

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0.$$

Remark 1.1 ([17]).

- (a) Suppose that θ from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R}_+ is a mapping satisfying (u2) to (u5). Then there exists a mapping η from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R}_+ such that η is nondecreasing in its third and fourth variables, respectively, satisfying (u2) η to (u5) η , where (u2) η to (u5) η stand for substituting η for θ in (u2) to (u5), respectively.
- (b) In the light of (a), we may assume that θ is nondecreasing in its third and fourth variables, respectively, for a function θ from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R}_+ satisfying (u2) to (u5).
- (c) Each τ -distance p on a metric space (X, d) is also a u -distance on X .

Here we present some examples of u -distance which are not τ -distance. (For the detail, see [17]).

Example 1.2. Let $X = \mathbb{R}_+$ with the usual metric. Define $p : X \times X \rightarrow \mathbb{R}_+$ by $p(x, y) = (\frac{1}{4})x^2$. And define $\theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by $\theta(x, y, s, t) = s$ for all $x, y \in X$ and $s, t \in \mathbb{R}_+$. Then p is a u -distance on X but not a τ distance on X .

Example 1.3. Let X be a normed space with norm $\|\cdot\|$. and let θ be the same as in Example 1.2. Then a function $p : X \times X \rightarrow \mathbb{R}_+$ defined by $p(x, y) = \|x\|$ for every $x, y \in X$, is a u -distance on X but not a

τ -distance.

It follows from the above examples and Remark 1.1(c) that u -distance is a proper extension of τ -distance. Other useful examples on u -distance and some related fixed point results are also given in [17]. Most recently, using u -distance Ume [18] also proved some fixed point result for Kannan type maps.

Definition 1.4 ([17]). Let (X, d) be a metric space and let p be a u -distance on X . A sequence $\{x_n\}$ in X is called p -Cauchy [17] if there exists a function θ from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R}_+ satisfying (u2)–(u5) and a sequence $\{z_n\}$ of X such that

$$\lim_{n \rightarrow \infty} \sup \{\theta(z_n, z_n, p(z_n, x_m), p(z_n, x_m)) : m \geq n\} = 0,$$

or

$$\lim_{n \rightarrow \infty} \sup \{\theta(z_n, z_n, p(x_m, z_n), p(x_m, z_n)) : m \geq n\} = 0.$$

The following lemmas concerning u -distance are crucial for the proofs of our results.

Lemma 1.5 ([17]). Let (X, d) be a metric space and let p be a u -distance on X . If $\{x_n\}$ is a p -Cauchy sequence in X , then $\{x_n\}$ is a Cauchy sequence.

Lemma 1.6 ([6]). Let (X, d) be a metric space and let p be a u -distance on X . If $\{x_n\}$ is a p -Cauchy sequence and $\{y_n\}$ is a sequence satisfying

$$\lim_{n \rightarrow \infty} \sup \{p(x_n, y_m) : m \geq n\} = 0,$$

then $\{y_n\}$ is also a p -Cauchy sequence and $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.

Lemma 1.7 ([17]). Let (X, d) be a metric space and let p be a u -distance on X . Suppose that a sequence $\{x_n\}$ of X satisfies

$$\lim_{n \rightarrow \infty} \sup \{p(x_n, x_m) : m > n\} = 0,$$

or

$$\lim_{n \rightarrow \infty} \sup \{p(x_m, x_n) : m > n\} = 0.$$

Then $\{x_n\}$ is a p -Cauchy sequence.

2. The Results

Using the u -distance, we prove some general results on the existence of fixed points for multivalued contractive type maps. In the sequel, we consider X as a complete metric space with metric d and p a u -distance on X .

Theorem 2.1. Let $T : X \rightarrow Cl(X)$ be a multivalued map. Assume that the following conditions hold:

(I) there exist functions $\varphi : [0, \infty) \rightarrow (0, 1)$ and $\mu : [0, \infty) \rightarrow [b, 1)$, with $b > 0$, μ nondecreasing such that

$$\mu(t) > \varphi(t) \quad \text{and} \quad \limsup_{r \rightarrow t^+} \mu(r) > \limsup_{r \rightarrow t^+} \varphi(r)$$

(II) for any $x \in X$, there exists $y \in T(x)$ satisfying

$$\mu(p(x, y))p(x, y) \leq p(x, T(x)) = \inf_{y \in T(x)} p(x, y),$$

and

$$p(y, T(y)) \leq \varphi(p(x, y))p(x, y),$$

(III) a real valued function f on X defined by $f(x) = p(x, T(x))$ is T -orbitally lower semicontinuous.

Then there exists $v_0 \in X$ such that $f(v_0) = 0$. Further if $p(v_0, v_0) = 0$, then $v_0 \in T(v_0)$.

Proof. Let x_0 be an arbitrary element in X , then there exists $x_1 \in T(x_0)$ such that

$$\mu(p(x_0, x_1))p(x_0, x_1) \leq p(x_0, T(x_0)), \quad (2.1)$$

and

$$p(x_1, T(x_1)) \leq \varphi(p(x_0, x_1))p(x_0, x_1). \quad (2.2)$$

From (2.1) and (2.2), we have

$$p(x_1, T(x_1)) \leq \frac{\varphi(p(x_0, x_1))}{\mu(p(x_0, x_1))} p(x_0, T(x_0)). \quad (2.3)$$

Define a function $\psi : [0, \infty) \rightarrow [0, \infty)$ by

$$\psi(t) = \frac{\varphi(t)}{\mu(t)} \quad \text{for all } t \in [0, \infty).$$

Since $\varphi(t) < \mu(t)$, we have

$$\psi(t) < 1, \quad (2.4)$$

and

$$\limsup_{r \rightarrow t^+} \psi(r) < 1 \quad \text{for all } t \in [0, \infty). \quad (2.5)$$

Now, using (2.3), we get

$$p(x_1, T(x_1)) \leq \psi(p(x_0, x_1))p(x_0, T(x_0)).$$

Similarly, there exists $x_2 \in T(x_1)$ such that

$$\mu(p(x_1, x_2))p(x_1, x_2) \leq p(x_1, T(x_1)),$$

and

$$p(x_2, T(x_2)) \leq \varphi(p(x_1, x_2))p(x_1, x_2).$$

Then by definition of ψ , we get

$$p(x_2, T(x_2)) \leq \psi(p(x_1, x_2))p(x_1, T(x_1)).$$

Continuing this process, we get an orbit $\{x_n\}$ of T at x_0 such that

$$\mu(p(x_n, x_{n+1}))p(x_n, x_{n+1}) \leq p(x_n, T(x_n)), \quad (2.6)$$

and

$$p(x_{n+1}, T(x_{n+1})) \leq \varphi(p(x_n, x_{n+1}))p(x_n, x_{n+1}). \quad (2.7)$$

Thus

$$p(x_{n+1}, T(x_{n+1})) \leq \psi(p(x_n, x_{n+1}))p(x_n, T(x_n)). \quad (2.8)$$

Since $\psi(t) < 1$ for all $t \in [0, \infty)$, we get

$$p(x_{n+1}, T(x_{n+1})) < p(x_n, T(x_n)). \quad (2.9)$$

Thus the sequence of non-negative real numbers $\{p(x_n, T(x_n))\}$ is decreasing and bounded below, thus convergent. Now, we want to show that the sequence $\{p(x_n, x_{n+1})\}$ is also decreasing. Suppose to the contrary, that $p(x_n, x_{n+1}) \leq p(x_{n+1}, x_{n+2})$, then as $\mu(t)$ is nondecreasing, we have

$$\mu(p(x_n, x_{n+1})) \leq \mu(p(x_{n+1}, x_{n+2})). \quad (2.10)$$

Now using (2.6), (2.7) and (2.10) with substituting $n + 1$ by n in (2.6), we get

$$\begin{aligned} p(x_{n+1}, x_{n+2}) &\leq \frac{\varphi(p(x_n, x_{n+1}))}{\mu(p(x_{n+1}, x_{n+2}))} p(x_n, x_{n+1}) \\ &\leq \frac{\varphi(p(x_n, x_{n+1}))}{\mu(p(x_n, x_{n+1}))} p(x_n, x_{n+1}) \\ &= \psi(p(x_n, x_{n+1})) p(x_n, x_{n+1}) \\ &< p(x_n, x_{n+1}), \end{aligned} \quad (2.11)$$

a contradiction. Thus the sequence $\{p(x_n, x_{n+1})\}$ is decreasing. Now let

$$\limsup_{n \rightarrow \infty} \psi(p(x_n, x_{n+1})) = \alpha.$$

Thus by (2.5), $\alpha < 1$. Then for any $q \in (\alpha, 1)$, there exists $n_0 \in \mathbb{N}$ such that

$$\psi(p(x_n, x_{n+1})) < q \quad \text{for all } n \geq n_0,$$

so from (2.8), for all $n \geq n_0$, we get

$$p(x_{n+1}, T(x_{n+1})) < qp(x_n, T(x_n)).$$

Thus by induction we get for all $n \geq n_0$

$$p(x_{n+1}, T(x_{n+1})) \leq q^{n+1-n_0} p(x_{n_0}, T(x_{n_0})). \quad (2.12)$$

Since $\mu(t) \geq b$, from (2.6) and (2.12), we have

$$p(x_n, x_{n+1}) \leq \frac{1}{b} p(x_n, T(x_n)) \leq \frac{1}{b} q^{n-n_0} p(x_{n_0}, T(x_{n_0})) \quad (2.13)$$

for all $n \geq n_0$. Note that $p(x_n, T(x_n)) \rightarrow 0$. Now, we show that $\{x_n\}$ is a Cauchy sequence. For all $m > n \geq n_0$, we have

$$\begin{aligned} p(x_n, x_m) &\leq \sum_{k=n}^{m-1} p(x_k, x_{k+1}) \\ &\leq \frac{1}{b} \sum_{k=n}^{m-1} q^{k-n_0} p(x_{n_0}, T(x_{n_0})) \\ &\leq \frac{1}{b} \left(\frac{q^{n-n_0}}{1-q} \right) p(x_{n_0}, T(x_{n_0})) \end{aligned} \quad (2.14)$$

and hence

$$\lim_{n \rightarrow \infty} \sup \{p(x_n, x_m) : m > n\} = 0.$$

Thus, by Lemma 1.7, $\{x_n\}$ is a p -Cauchy sequence and hence by Lemma 1.5, $\{x_n\}$ is a Cauchy sequence. Due to the completeness of X , there exists some $v_0 \in X$ such that $\lim_{n \rightarrow \infty} x_n = v_0$. Since f is T -orbitally lower semicontinuous and from (2.13), we have

$$0 \leq f(v_0) \leq \liminf_{n \rightarrow \infty} f(x_n) = \liminf_{n \rightarrow \infty} p(x_n, T(x_n)) = 0,$$

and thus, $f(v_0) = p(v_0, T(v_0)) = 0$. Thus there exists a sequence $\{v_n\} \subset T(v_0)$ such that $\lim_{n \rightarrow \infty} p(v_0, v_n) = 0$. It follows that

$$0 \leq \lim_{n \rightarrow \infty} \sup \{p(x_n, v_m) : m > n\}$$

$$\leq \lim_{n \rightarrow \infty} \sup \{p(x_n, v_0) + p(v_0, v_m) : m > n\} = 0. \quad (2.15)$$

Since $\{x_n\}$ is a p -Cauchy sequence, thus it follows from (2.15) and Lemma 1.6 that $\{v_n\}$ is also a p -Cauchy sequence and $\lim_{n \rightarrow \infty} d(x_n, v_n) = 0$. Thus, by Lemma 1.5, $\{v_n\}$ is a Cauchy sequence in the complete space. Due to closedness of $T(v_0)$, there exists $z_0 \in X$ such that $\lim_{n \rightarrow \infty} v_n = z_0 \in T(v_0)$. Consequently, using (u_3) we get

$$p(v_0, z_0) \leq \liminf_{n \rightarrow \infty} p(v_0, v_n) = 0,$$

and thus $p(v_0, z_0) = 0$. But, since $\lim_{n \rightarrow \infty} x_n = v_0$, $\lim_{n \rightarrow \infty} v_n = z_0$ and $\lim_{n \rightarrow \infty} d(x_n, v_n) = 0$, we have $v_0 = z_0$. Hence $v_0 \in T(v_0)$ and $p(v_0, v_0) = 0$. \square

Remark 2.2. Theorem 2.1 is an improved version of Ciric's result [4, Theorem 6] and generalizes many fixed point results including the results of Latif and Abdou [10, Theorem 2.3] and Klim and Wardowski [8, Theorem 2.1].

We also have the following interesting result by replacing the hypothesis (III) of Theorem 2.1 with another suitable condition.

Theorem 2.3. *Suppose that all the hypotheses of Theorem 2.1 except (III) hold. Assume that*

$$\inf \{p(x, v) + p(x, T(x))\} : x \in X > 0,$$

for every $v \in X$ with $v \notin T(v)$. Then $Fix(T) \neq \emptyset$.

Proof. Following the proof of Theorem 2.1, there exists an orbit $\{x_n\}$ of T , which is Cauchy sequence in a complete metric space X . Thus, there exists $v_0 \in X$ such that $\lim_{n \rightarrow \infty} x_n = v_0$. Thus, using (2.14) we have for all $n \geq n_0$

$$p(x_n, v_0) \leq \lim_{m \rightarrow \infty} \inf p(x_n, x_m) \leq \left(\frac{1}{b} \left(\frac{q^{n-n_0}}{1-q}\right)\right) p(x_{n_0}, T(x_{n_0})).$$

Since $x_{n+1} \in T(x_n)$, we note that $p(x_n, T(x_n)) \leq p(x_n, x_{n+1})$ and from (2.13) we have

$$p(x_n, x_{n+1}) \leq \frac{1}{b} q^{n-n_0} p(x_{n_0}, T(x_{n_0})),$$

thus, we get

$$p(x_n, T(x_n)) \leq \frac{1}{b} q^{n-n_0} p(x_{n_0}, T(x_{n_0})).$$

Assume that $v_0 \notin T(v_0)$. Then, we have

$$\begin{aligned} 0 &< \inf \{p(x, v_0) + p(x, T(x)) : x \in X\} \\ &\leq \inf \{p(x_n, v_0) + p(x_n, T(x_n)) : n \geq n_0\} \\ &\leq \inf \left\{ \left(\frac{1}{b} \left(\frac{q^{n-n_0}}{1-q}\right)\right) p(x_{n_0}, T(x_{n_0})) + \frac{1}{b} q^{n-n_0} p(x_{n_0}, T(x_{n_0})) : n \geq n_0 \right\} \\ &= \frac{(2-q)}{b(1-q)q^{n_0}} p(x_{n_0}, T(x_{n_0})) \inf \{q^n : n \geq n_0\} = 0, \end{aligned}$$

which is impossible and hence $v_0 \in T(v_0)$. \square

Now, we present a result which is a generalization of the fixed point results of Klim and Wardowski [8, Theorem 2.2] and Ciric [4, Theorem 7].

Theorem 2.4. *Let $T : X \rightarrow Cl(X)$ be a multivalued map. Assume that the following conditions hold:*

(I) *there exists a function $\varphi : [0, \infty) \rightarrow [0, 1)$ such that for each $t \in [0, \infty)$*

$$\limsup_{r \rightarrow t^+} \varphi(r) < 1;$$

(II) for any $x \in X$, there exists $y \in T(x)$ satisfying

$$p(x, y) = p(x, T(x)),$$

and

$$p(y, T(y)) \leq \varphi(p(x, y))p(x, y);$$

(III) a real valued function f on X defined by $f(x) = p(x, T(x))$ is T -orbitally lower semicontinuous. Then there exists $v_0 \in X$ such that $f(v_0) = 0$. Further if $p(v_0, v_0) = 0$, then $v_0 \in T(v_0)$.

Proof. Let $x_0 \in X$ be any initial point. Then from (II) we can choose $x_1 \in T(x_0)$ such that

$$p(x_0, x_1) = p(x_0, T(x_0)) \tag{2.16}$$

and

$$p(x_1, T(x_1)) \leq \varphi(p(x_0, x_1))p(x_0, x_1), \quad \varphi(p(x_0, x_1)) < 1. \tag{2.17}$$

Continuing this process, we get an orbit $\{x_n\}$ of T in X such that

$$p(x_n, x_{n+1}) = p(x_n, T(x_n)), \tag{2.18}$$

and

$$p(x_{n+1}, T(x_{n+1})) \leq \varphi(p(x_n, x_{n+1}))p(x_n, x_{n+1}), \quad \varphi(p(x_n, x_{n+1})) < 1. \tag{2.19}$$

Using (2.18) and (2.19), we get

$$\begin{aligned} p(x_n, T(x_n)) - p(x_{n+1}, T(x_{n+1})) &\geq p(x_n, x_{n+1}) - \varphi(p(x_n, x_{n+1}))p(x_n, x_{n+1}) \\ &= [1 - \varphi(p(x_n, x_{n+1}))]p(x_n, x_{n+1}) > 0, \end{aligned} \tag{2.20}$$

and hence the sequences $\{p(x_n, T(x_n))\}$ and so $\{p(x_n, x_{n+1})\}$ are decreasing and bounded below thus convergent. Now, using the similar arguments as in the proof of Theorem 2.1, we can get that for all $m > n \geq n_0$,

$$\lim_{n \rightarrow \infty} \sup\{p(x_n, x_m) : m > n\} = 0.$$

Thus, by Lemma 1.7, $\{x_n\}$ is a p -Cauchy sequence and hence by Lemma 1.5, $\{x_n\}$ is a Cauchy sequence. Due to the completeness of X , there exists some $v_0 \in X$ such that $\lim_{n \rightarrow \infty} x_n = v_0$. The rest of the proof is the same as in Theorem 2.1. \square

Finally, we present an example which shows that Theorem 2.4 is a genuine generalization of Theorem 2.2 of Klim-Wardowski [8].

Example 2.5. Let $X = [0, \infty)$ with the usual metric d . Define a function $p : X \times X \rightarrow [0, \infty)$, by $p(x, y) = x + y$, for all $x, y \in X$. Then p is a u -distance on X . Note that $p \neq d$. Now, for any real number $k > 1$, define $T : X \rightarrow Cl(X)$ by

$$T(x) = \left\{ \frac{x}{k} \right\} \cup [1 + x, \infty), \quad \text{for all } x \in [0, \infty),$$

and define a constant function $\varphi : [0, \infty) \rightarrow [0, 1)$ by $\varphi(t) = \frac{1}{k}$, for all $t \in [0, \infty)$. Note that $\varphi(t) < 1$ for all $t \in [0, \infty)$. And for each $x \in X$ we have

$$f(x) = p(x, T(x)) = x + \frac{x}{k} = \left(\frac{k+1}{k}\right)x.$$

Thus, f is continuous. Now for each $x \in [0, \infty)$ there exists $y = \frac{x}{k} \in T(x)$ satisfying

$$p(x, y) = p(x, \frac{x}{k}) = p(x, T(x))$$

and

$$p(y, T(y)) = \frac{x}{k} + \frac{x}{k^2} = \frac{1}{k} \left(\frac{k+1}{k} \right) x = \varphi(p(x, y))p(x, y).$$

Therefore, all assumptions of Theorem 2.4 are satisfied and note that $Fix(T) = \{0\}$. Note that T do not satisfy the hypotheses of [8, Theorem 2.2] because $T(x)$ is not compact for all $x \in X$ and the u -distance p is not a metric d .

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