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Existence and uniqueness of the weak solution for a contact problem

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Abstract

We study a mathematical model which describes the antiplane shear deformation of a cylinder in frictionless contact with a rigid foundation. The material is assumed to be electro-viscoelastic with long-term memory, the friction is modeled with Tresca's law and the foundation is assumed to be electrically conductive. First we derive the classical variational formulation of the model which is given by a system coupling an evolutionary variational equality for the displacement field, a time-dependent variational equation for the potential field and a differential equation for the bounding field. Then we prove the existence of a unique weak solution for the model. The proof is based on arguments of evolution equations and the Banach fixed point theorem.

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1. Introduction

Antiplane shear deformations are one the simplest examples of deformations that solids can undergo: in the antiplane shear of a cylindrical body, the displacement is parallel to the generators of the cylinder and is dependent of the axial coordinate. For this reason, considerable progress has been made in their modeling

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and analysis, and the engineering literature concerning this topic (see [2, 3, 6, 8, 9, 11]) is rather extensive. Mathematical and mechanical state of the art on contact mechanics can be found in [1, 10, 12, 13].

In the present paper we study an antiplane contact problem for electro-viscoelastic materials with long-term memory. We consider the case of antiplane shear deformation, i.e., the displacement is parallel to the generators of the cylinder and is dependent of the axial coordinate (see [4, 5, 7]). Our interest is to describe a physical process in which antiplane shear, contact, the state of material with long-term memory as well as piezoelectric effect are involved, leading to a well posedness mathematical problem. In the variational formulation, this kind of problem leads to an integro-differential inequality. The main result we provide concerns the existence of a unique weak solution to the model.

The rest of the paper is structured as follows. In Section 2 we describe the model of the frictional contact process between an electro-viscoelastic body and a conductive deformable foundation. In Section 3 we derive the variational formulation. It consists of a variational inequality for the displacement field coupled with a time-dependent variational equation for the electric potential. We then state our main result, the existence of a unique weak solution to the model in Theorem 4.2, which we proceed to prove in Section 4.

2. The mathematical model

We consider a piezoelectric body $\mathcal{B}$ identified with a region in $\mathbb{R}^3$ it occupies in a fixed and undistorted reference configuration. We assume that $\mathcal{B}$ is a cylinder with generators parallel to the x_3-axis with a cross-section which is a regular region Ω in the x_1, x_2-plane, $Ox_1x_2x_3$ being a Cartesian coordinate system. The cylinder is assumed to be sufficiently long so that the end effects in the axial direction are negligible (for more details see [5, 11]). Thus, $\mathcal{B} = \Omega \times (-\infty, +\infty)$. The cylinder is acted upon by body forces of density $\mathbf{f}_0$ and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe the boundary conditions, we denote by $\partial\Omega = \Gamma$ the boundary of Ω and we assume a partition of Γ into three open disjoint sets Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open sets Γ_a and Γ_b, on the other hand. We assume that the one-dimensional measures of Γ_1 and Γ_a, denoted $\operatorname{meas}\Gamma_1$ and $\operatorname{meas}\Gamma_a$, are positive. The cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and therefore the displacement field vanishes there. Surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (-\infty, +\infty)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (-\infty, +\infty)$ and a surface electrical charge of density q_2 is prescribed on $\Gamma_b \times (-\infty, +\infty)$. The cylinder is in contact over $\Gamma_3 \times (-\infty, +\infty)$ with a conductive obstacle, the so called foundation. The contact is frictional and is modeled with Tresca's law. We are interested in the deformation of the cylinder on the time interval $[0, T]$. We assume that

$$\mathbf{f}_0 = (0, 0, f_0) \quad \text{with} \quad f_0 = f_0(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \tag{2.1}$$

$$\mathbf{f}_2 = (0, 0, f_2) \quad \text{with} \quad f_2 = f_2(x_1, x_2, t) : \Gamma_2 \times [0, T] \to \mathbb{R}, \tag{2.2}$$

$$q_0 = q_0(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \tag{2.3}$$

$$q_2 = q_2(x_1, x_2, t) : \Gamma_b \times [0, T] \to \mathbb{R}. \tag{2.4}$$

The forces (2.1), (2.2) and the electric charges (2.3), (2.4) would be expected to give rise to deformations and to electric charges of the piezoelectric cylinder corresponding to a displacement $\mathbf{u}$ and to an electric potential field φ which are independent on x_3 and have the form

$$\mathbf{u} = (0, 0, u) \quad \text{with } u = u(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \tag{2.5}$$

$$\varphi = \varphi(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}. \tag{2.6}$$

Such kind of deformation, associated to a displacement field of the form (2.5), is called an *antiplane shear* (see for instance [5]-[7] for details).

In this paper the indices i and j will denote components of vectors and tensors and run from 1 to 3, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding spatial variable; also, a dot above represents the time derivative. We use $\mathcal{S}^3$ for the linear space of second order symmetric tensors on $\mathbb{R}^3$ or, equivalently, the space of symmetric matrices of order 3, and " $\cdot$ ", $\|\cdot\|$ will represent the inner products and the Euclidean norms on $\mathbb{R}^3$ and $\mathcal{S}^3$. We have

$$\mathbf{u}\cdot\mathbf{v}=u_iv_i, \quad \|\mathbf{v}\|=(\mathbf{v}\cdot\mathbf{v})^{1/2} \quad \text{for all } \mathbf{u}=(u_i),\ \mathbf{v}=(v_i)\in\mathbb{R}^3,$$

and

$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\|=(\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2} \quad \text{for all } \boldsymbol{\sigma}=(\sigma_{ij}),\ \boldsymbol{\tau}=(\tau_{ij})\in\mathcal{S}^3.$$

The infinitesimal strain tensor is denoted by $\boldsymbol{\varepsilon}(\mathbf{u})=(\varepsilon_{ij}(\mathbf{u}))$ and the stress field by $\boldsymbol{\sigma}=(\sigma_{ij})$. We also denote by $\mathbf{E}(\varphi)=(E_i(\varphi))$ the electric field and by $\mathbf{D}=(D_i)$ the electric displacement field. Here and in the following, in order to simplify the notation, we do not indicate the dependence of various functions on x_1, x_2, x_3 or t and we recall that

$$\varepsilon_{ij}(\mathbf{u})=\frac{1}{2}(u_{i,j}+u_{j,i}), \quad E_i(\varphi)=-\varphi_{,i}\,.$$

The material is modeled by the following electro-viscoelastic constitutive law with long-term memory:

$$\boldsymbol{\sigma}=\lambda(\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}))\,\mathbf{I}+2\mu\boldsymbol{\varepsilon}(\mathbf{u})+2\int_0^t\theta(t-s)\,\boldsymbol{\varepsilon}(\mathbf{u}(\mathbf{s}))ds-\mathcal{E}^*\mathbf{E}(\varphi), \tag{2.7}$$

$$\mathbf{D}=\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u})+\beta\mathbf{E}(\varphi), \tag{2.8}$$

where λ and μ are the Lamé coefficients, $\theta:[0,T]\longrightarrow\mathbb{R}$ is the viscosity coefficient, tr $\boldsymbol{\varepsilon}(\mathbf{u})=\varepsilon_{ii}(\mathbf{u})$, $\mathbf{I}$ is the unit tensor in $\mathbb{R}^3$, β is the electric permittivity constant, $\mathcal{E}$ represents the third-order piezoelectric tensor and $\mathcal{E}^*$ is its transpose. In the antiplane context (2.5), (2.6), using the constitutive equations (2.7), (2.8) it follows that the stress field and the electric displacement field are given by

$$\boldsymbol{\sigma}=\begin{pmatrix}0&0&\boldsymbol{\sigma}_{13}\\0&0&\boldsymbol{\sigma}_{23}\\\boldsymbol{\sigma}_{31}&\boldsymbol{\sigma}_{32}&0\end{pmatrix}, \tag{2.9}$$

$$\mathbf{D}=\begin{pmatrix}eu_{,1}-\beta\varphi_{,1}\\eu_{,2}-\beta\varphi_{,2}\\0\end{pmatrix}, \tag{2.10}$$

where

$$\boldsymbol{\sigma}_{13}=\boldsymbol{\sigma}_{31}=\mu\partial_{x_1}u+\int_0^t\theta(t-s)\ \partial_{x_1}u(s)ds$$

and

$$\boldsymbol{\sigma}_{23}=\boldsymbol{\sigma}_{32}=\mu\partial_{x_2}u+\int_0^t\theta(t-s)\ \partial_{x_2}u(s)ds.$$

We assume that

$$\mathcal{E}\boldsymbol{\varepsilon}=\begin{pmatrix}e(\varepsilon_{13}+\varepsilon_{31})\\e(\varepsilon_{23}+\varepsilon_{32})\\e\varepsilon_{33}\end{pmatrix} \quad \forall\boldsymbol{\varepsilon}=(\varepsilon_{ij})\in\mathcal{S}^3, \tag{2.11}$$

where e is a piezoelectric coefficient. We also assume that the coefficients θ, μ, β and e depend on the spatial variables x_1, x_2, but are independent on the spatial variable x_3. Since $\mathcal{E}\boldsymbol{\varepsilon}\cdot\mathbf{v}=\boldsymbol{\varepsilon}\cdot\mathcal{E}^*\mathbf{v}$ for all $\boldsymbol{\varepsilon}\in\mathcal{S}^3$, $\mathbf{v}\in\mathbb{R}^3$, it follows from (2.11) that

$$\mathcal{E}^*\mathbf{v}=\begin{pmatrix}0&0&ev_1\\0&0&ev_2\\ev_1&ev_2&ev_3\end{pmatrix} \quad \forall\mathbf{v}=(v_i)\in\mathbb{R}^3. \tag{2.12}$$

We further assume that the process is mechanically quasistatic and electrically static and therefore is governed by the equilibrium equations

$$\mathrm{Div}\,\boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0}, \quad D_{i,i} - q_0 = 0 \quad \text{in } \mathcal{B} \times (0,T), \tag{2.13}$$

where $\mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j})$ represents the divergence of the tensor field $\boldsymbol{\sigma}$.

Taking into account (2.1), (2.3), (2.5), (2.6), (2.9) and (2.10), the equilibrium equations above reduce to the scalar equations

$$\mathrm{div}(\mu\nabla u) + \int_0^t \theta(t-s)\,\mathrm{div}(\nabla u(s))ds + \mathrm{div}(e\nabla\varphi) + f_0 = 0, \text{in } \Omega \times (0,T), \tag{2.14}$$

$$\mathrm{div}(e\nabla u - \beta\nabla\varphi) = q_0. \tag{2.15}$$

Here and below we use the notation

$$\mathrm{div}\,\boldsymbol{\tau} = \tau_{1,1} + \tau_{1,2} \quad \text{for } \boldsymbol{\tau} = (\tau_1(x_1,x_2,t), \tau_2(x_1,x_2,t)),$$

$$\nabla v = (v_{,1}, v_{,2}), \quad \partial_\nu v = v_{,1}\,\nu_1 + v_{,2}\,\nu_2 \quad \text{for } v = v(x_1,x_2,t).$$

We now describe the boundary conditions. During the process the cylinder is clamped on $\Gamma_1 \times (-\infty,+\infty)$ and the electric potential vanishes on $\Gamma_1 \times (-\infty,+\infty)$. Thus, (2.5) and (2.6) imply that

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{2.16}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T). \tag{2.17}$$

Let $\boldsymbol{\nu}$ denote the unit normal on $\Gamma \times (-\infty,+\infty)$. We have

$$\boldsymbol{\nu} = (\nu_1, \nu_2, 0) \quad \text{with} \quad \nu_i = \nu_i(x_1,x_2) : \Gamma \to \mathbb{R}, \quad i = 1,2. \tag{2.18}$$

For a vector $\mathbf{v}$ we denote by v_ν and $\mathbf{v}_\tau$ its normal and tangential components on the boundary, given by

$$v_\nu = \mathbf{v}\cdot\boldsymbol{\nu}, \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu\boldsymbol{\nu}. \tag{2.19}$$

For a given stress field σ we denote by σ_ν and σ_τ the normal and the tangential components on the boundary, that is

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}. \tag{2.20}$$

From (2.9), (2.10) and (2.18) we deduce that the Cauchy stress vector and the normal component of the electric displacement field are given by

$$\boldsymbol{\sigma}\boldsymbol{\nu} = (0,0,\mu\partial_\nu u + \int_0^t \theta(t-s)\,\partial_\nu u(s)ds + e\partial_\nu\varphi), \quad \mathbf{D}\cdot\boldsymbol{\nu} = e\partial_\nu u - \beta\partial_\nu\varphi. \tag{2.21}$$

Taking into account relations (2.2), (2.4) and (2.21), the traction condition on $\Gamma_2 \times (-\infty,\infty)$ and the electric conditions on $\Gamma_b \times (-\infty,\infty)$ are given by

$$\mu\partial_\nu u + \int_0^t \theta(t-s)\,\partial_\nu u(s)ds + e\partial_\nu\varphi = f_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{2.22}$$

$$e\partial_\nu u - \beta\partial_\nu\varphi = q_2 \quad \text{on } \Gamma_b \times (0,T). \tag{2.23}$$

We now describe the frictional contact condition and the electric conditions on $\Gamma_3 \times (-\infty,+\infty)$. First, from (2.5) and (2.18) we infer that the normal displacement vanishes ($u_\nu = 0$), which shows that the contact is bilateral, that is, the contact is kept during all the process. Using now (2.5) and (2.18)–(2.20) we conclude that

$$\mathbf{u}_\tau = (0,0,u), \quad \boldsymbol{\sigma}_\tau = (0,0,\sigma_\tau), \tag{2.24}$$

where

$$\sigma_\tau = (0, 0, \mu\partial_\nu u + \int_0^t \theta(t-s)\,\partial_\nu u(s)ds + e\partial_\nu\varphi). \tag{2.25}$$

We assume that the friction is invariant with respect to the x_3 axis and is modeled with Tresca's friction law, that is

$$\begin{cases} |\boldsymbol{\sigma}_\tau(t)| \le g, \\ |\boldsymbol{\sigma}_\tau(t)| < g \Rightarrow \dot{\boldsymbol{u}}_\tau(t) = \mathbf{0}, \qquad \text{on } \Gamma_3 \times (0,T), \\ |\boldsymbol{\sigma}_\tau(t)| = g \Rightarrow \exists\, \beta \ge 0 \text{ such that } \boldsymbol{\sigma}_\tau = -\beta \dot{\boldsymbol{u}}_\tau. \end{cases} \tag{2.26}$$

Here $g : \Gamma_3 \to \mathbb{R}_+$ is a given function, the friction bound, and $\dot{\mathbf{u}}_\tau$ represents the tangential velocity on the contact boundary (see [4], [9] and [12] for details). Using now (2.24) it is straightforward to see that the friction law (2.26) implies

$$\begin{cases} |\mu\partial_\nu u + \int_0^t \theta(t-s)\,\partial_\nu u(s)ds + e\partial_\nu\varphi| \le g, \\ |\mu\partial_\nu u + \int_0^t \theta(t-s)\,\partial_\nu u(s)ds + e\partial_\nu\varphi| < g \Rightarrow \dot{u}(t) = 0, \ \text{on } \Gamma_3 \times (0,T), \\ |\mu\partial_\nu u + \int_0^t \theta(t-s)\,\partial_\nu u(s)ds + e\partial_\nu\varphi| = g \Rightarrow \exists\, \beta \ge 0 \quad \text{such that} \\ \mu\partial_\nu u + \int_0^t \theta(t-s)\,\partial_\nu u(s)ds + e\partial_\nu\varphi = -\beta\dot{u}. \end{cases} \tag{2.27}$$

Next, since the foundation is electrically conductive and the contact is bilateral, we assume that the normal component of the electric displacement field or the free charge is proportional to the difference between the potential on the foundation and the body's surface. Thus,

$$\mathbf{D}\cdot\boldsymbol{\nu} = k\,(\varphi - \varphi_F) \quad \text{on} \quad \Gamma_3 \times (0,T),$$

where φ_F represents the electric potential of the foundation and k is the electric conductivity coefficient. We use (2.21) and the previous equality to obtain

$$e\partial_\nu u - \beta\partial_\nu\varphi = k\,(\varphi - \varphi_F) \quad \text{on } \Gamma_3 \times (0,T). \tag{2.28}$$

Finally, we prescribe the initial displacement,

$$u(0) = u_0 \quad \text{in } \Omega, \tag{2.29}$$

where u_0 is a given function on Ω.

We collect the above equations and conditions to obtain the following mathematical model which describes the antiplane shear of an electro-viscoelastic cylinder in frictional contact with a conductive foundation.

2.1. Problem $\mathcal{P}$

Find the displacement field $u : \Omega \times [0,T] \to \mathbb{R}$ and the electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\operatorname{div}(\mu\nabla u) + \int_0^t \theta(t-s)\operatorname{div}(\nabla u(s))ds + \operatorname{div}(e\nabla\varphi) + f_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{2.30}$$

$$\operatorname{div}(e\nabla u - \alpha\nabla\varphi) = q_0 \quad \text{in } \Omega \times (0,T), \tag{2.31}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{2.32}$$

$$\mu\partial_\nu u + \int_0^t \theta(t-s)\,\partial_\nu u(s)ds + e\partial_\nu\varphi = f_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{2.33}$$

$$\begin{cases} |\mu\partial_\nu u + \int_0^t \theta(t-s)\,\partial_\nu u(s)ds + e\partial_\nu\varphi| \le g, \\ |\mu\partial_\nu u + \int_0^t \theta(t-s)\,\partial_\nu u(s)ds + e\partial_\nu\varphi| < g \Rightarrow \dot{u}(t) = 0, \text{ on } \Gamma_3 \times (0,T), \\ |\mu\partial_\nu u + \int_0^t \theta(t-s)\,\partial_\nu u(s)ds + e\partial_\nu\varphi| = g \Rightarrow \exists\, \beta \ge 0 \quad \text{such that} \\ \mu\partial_\nu u + \int_0^t \theta(t-s)\,\partial_\nu u(s)ds + e\partial_\nu\varphi = -\beta\dot{u}, \end{cases} \tag{2.34}$$

$$e\partial_\nu u - \alpha\partial_\nu\varphi = q_2 \quad \text{on } \Gamma_b \times (0,T), \tag{2.35}$$

$$e\partial_\nu u - \alpha\partial_\nu\varphi = k\,(\varphi - \varphi_F) \quad \text{on } \Gamma_3 \times (0,T), \tag{2.36}$$

$$u(0) = u_0 \quad \text{in } \Omega. \tag{2.37}$$

Note that once the displacement field u and the electric potential φ which solve Problem $\mathcal{P}$ are known, the stress tensor $\boldsymbol{\sigma}$ and the electric displacement field $\mathbf{D}$ can be obtained by using the constitutive laws (2.9) and (2.10), respectively.

3. Variational formulation and main result

We now derive the variational formulation of Problem $\mathcal{P}$. To this end we introduce the function spaces

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1\}, \quad W = \{\,\psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_a\},$$

where we write w for the trace γw of a function $w \in H^1(\Omega)$ on Γ. Since $\operatorname{meas}\Gamma_1 > 0$ and $\operatorname{meas}\Gamma_a > 0$, it is well known that V and W are real Hilbert spaces with the inner products

$$(u,v)_V = \int_\Omega \nabla u \cdot \nabla v\, dx \quad \forall u, v \in V, \quad (\varphi,\psi)_W = \int_\Omega \nabla\varphi \cdot \nabla\psi\, dx \quad \forall \varphi,\, \psi \in W.$$

Moreover, the associated norms

$$\|v\|_V = \|\nabla v\|_{L^2(\Omega)^2} \quad \forall v \in V, \quad \|\psi\|_W = \|\nabla\psi\|_{L^2(\Omega)^2} \quad \forall \psi \in W \tag{3.1}$$

are equivalent on V and W, respectively, with the usual norm $\|\cdot\|_{H^1(\Omega)}$. By Sobolev's trace theorem we deduce that there exist two positive constants $c_V > 0$ and $c_W > 0$ such that

$$\|v\|_{L^2(\Gamma_3)} \le c_V \|v\|_V \quad \forall v \in V, \quad \|\psi\|_{L^2(\Gamma_3)} \le c_W \|\psi\|_W \quad \forall \psi \in W. \tag{3.2}$$

Given a real Banach space $(X, \|\cdot\|_X)$ we use the usual notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ where $1 \le p \le \infty,\ k = 1, 2, \ldots$. We also denote by $C([0,T];X)$ the space of continuous and continuously differentiable functions on $[0,T]$ with values in X, with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X$$

and we use the standard notations for the Lebesgue space $L^2(0,T;X)$ as well as the Sobolev space $W^{1,2}(0,T;X)$. In particular, recall that the norm on the space $L^2(0,T;X)$ is given by the formula

$$\|u\|^2_{L^2(0,T;X)} = \int_0^T \|u(t)\|_X^2\, dt$$

and the norm on the space $W^2(0,T;X)$ is given by the formula

$$\|u\|^2_{W^{1,2}(0,T;X)} = \int_0^T \|u(t)\|_X^2\, dt + \int_0^T \|\dot{u}(t)\|_X^2\, dt.$$

We suppress the argument X when $X = \mathbb{R}$, thus, for example, we use the notation $W^2(0,T)$ for the space $W^2(0,T;\mathbb{R})$ and the notation $\|\cdot\|_{W^2(0,T)}$ for the norm $\|\cdot\|_{W^2(0,T;\mathbb{R})}$.

In the study of Problem $\mathcal{P}$ we assume that the viscosity coefficient satisfies

$$\theta \in W^{1,2}(0,T) \tag{3.3}$$

and the electric permittivity coefficient satisfies

$$\alpha \in L^\infty(\Omega) \text{ and there exists } \alpha^* > 0 \text{ such that } \ \alpha(\mathbf{x}) \geq \alpha^* \text{ a.e. } \mathbf{x} \in \Omega. \tag{3.4}$$

We also assume that the Lamé coefficient and the piezoelectric coefficient satisfy

$$\mu \in L^\infty(\Omega) \quad \text{and} \quad \mu(\mathbf{x}) > 0 \ \text{ a.e. } \mathbf{x} \in \Omega, \tag{3.5}$$

$$e \in L^\infty(\Omega). \tag{3.6}$$

The forces, tractions, volume and surface free charge densities have the regularity

$$f_0 \in W^{1,2}(0,T;L^2(\Omega)), \quad f_2 \in W^{1,2}(0,T;L^2(\Gamma_2)), \tag{3.7}$$

$$q_0 \in W^{1,2}(0,T;L^2(\Omega)), \quad q_2 \in W^{1,2}(0,T;L^2(\Gamma_b)). \tag{3.8}$$

The electric conductivity coefficient and the friction bound function g satisfy the following properties:

$$k \in L^\infty(\Gamma_3) \quad \text{and} \quad k(\mathbf{x}) \geq 0 \ \text{ a.e. } \mathbf{x} \in \Gamma_3, \tag{3.9}$$

$$g \in L^\infty(\Gamma_3) \quad \text{and} \quad g(\mathbf{x}) \geq 0 \ \text{ a.e. } \mathbf{x} \in \Gamma_3. \tag{3.10}$$

Finally, we assume that the electric potential of the foundation and the initial displacement are such that

$$\varphi_F \in W^{1,2}(0,T;L^2(\Gamma_3)). \tag{3.11}$$

The initial data are chosen such that

$$u_0 \in V \tag{3.12}$$

and

$$a_\mu(u_0,v)_V + j(v) \geq (f(0),v)_V \quad \forall v \in V. \tag{3.13}$$

We now define the functional $j : [0,T] \longrightarrow \mathbb{R}_+$ by

$$j(v) = \int_{\Gamma_3} g|v|\,da \quad \forall v \in V, \tag{3.14}$$

and the mappings $f : [0,T] \to V$ and $q : [0,T] \to W$ by

$$(f(t),v)_V = \int_\Omega f_0(t)v\,dx + \int_{\Gamma_2} f_2(t)v\,da, \tag{3.15}$$

$$(q(t),\psi)_W = \int_\Omega q_0(t)\psi\,dx - \int_{\Gamma_b} q_2(t)\psi\,da + \int_{\Gamma_3} k\,\varphi_F(t)\psi\,da, \tag{3.16}$$

for all $v \in V$, $\psi \in W$ and $t \in [0,T]$. The definitions of f and q are based on Riesz's representation theorem. Moreover, it follows from assumptions (3.7)–(3.8) that the integrals above are well-defined and

$$f \in W^{1,2}(0,T;V), \tag{3.17}$$

$$q \in W^{1,2}(0,T;W). \tag{3.18}$$

Next, we consider the bilinear forms $a_\mu : V \times V \to \mathbb{R}$, $a_e : V \times W \to \mathbb{R}$, $a_e^* : W \times V \to \mathbb{R}$, and $a_\alpha : W \times W \to \mathbb{R}$, given by equalities

$$a_\mu(u,v) = \int_\Omega \mu\, \nabla u \cdot \nabla v\, dx, \tag{3.19}$$

$$a_e(u,\varphi) = \int_\Omega e\, \nabla u \cdot \nabla \varphi\, dx = a_e^*(\varphi,u), \tag{3.20}$$

$$a_\alpha(\varphi,\psi) = \int_\Omega \beta\, \nabla \varphi \cdot \nabla \psi\, dx + \int_{\Gamma_3} k\, \varphi\psi\, dx, \tag{3.21}$$

for all u, $v \in V$, $\varphi, \psi \in W$. Assumptions (3.14)–(3.16) imply that the integrals above are well defined and, using (3.1) and (3.2), it follows that the forms a_μ, a_e and a_e^* are continuous. Moreover, the forms a_μ and a_α are symmetric and, in addition, the form a_α is W-elliptic since

$$a_\alpha(\psi,\psi) \geq \alpha^* \|\psi\|_W^2 \quad \forall \psi \in W. \tag{3.22}$$

The variational formulation of Problem $\mathcal{P}$ is based on the following result.

Lemma 3.1. *If (u,φ) is a smooth solution to Problem $\mathcal{P}$, then $(u(t),\varphi(t)) \in X$ and*

$$\begin{aligned} a_\mu(u(t), v - \dot{u}(t)) + \left(\int_0^t \theta(t-s)u(s)\, ds, v - \dot{u}(t)\right)_V + a_e^*(\varphi(t), v - \dot{u}(t)) + j(v) - j(\dot{u}(t)) \\ \geq (f(t), v - \dot{u}(t))_V \quad \forall v \in V,\ t \in [0,T], \end{aligned} \tag{3.23}$$

$$a_\alpha(\varphi(t),\psi) - a_e(u(t),\psi) = (q(t),\psi)_W \quad \forall \psi \in W,\ t \in [0,T], \tag{3.24}$$

$$u(0) = u_0. \tag{3.25}$$

Proof. Let (u,φ) denote a smooth solution to Problem $\mathcal{P}$. We have $u(t) \in V$, $\dot{u}(t) \in V$ and $\varphi(t) \in W$ a.e. $t \in [0,T]$ and, from (2.30), (2.32) and (2.33), we obtain

$$\begin{aligned} &\int_\Omega \mu\, \nabla u(t) \cdot \nabla (v - \dot{u}(t))\, dx + \left(\int_0^t \theta(t-s)u(s)\, ds, v - \dot{u}(t)\right)_V + \int_\Omega e\, \nabla \varphi(t) \cdot \nabla (v - \dot{u}(t))\, dx \\ &\quad = \int_\Omega f_0(t)\, (v - \dot{u}(t))\, dx + \int_{\Gamma_2} f_2(t)\, (v - \dot{u}(t))\, da \\ &\qquad + \int_{\Gamma_3} (|\mu \partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)\, ds + e\partial_\nu \varphi(t)|)(v - \dot{u}(t))\, da, \quad \forall v \in V \ \ t \in (0,T), \end{aligned}$$

and from (2.31) and (2.35)–(2.36) we have

$$\begin{aligned} \int_\Omega \alpha\, \nabla \varphi(t) \cdot \nabla \psi\, dx - \int_\Omega e\, \nabla u(t) \cdot \nabla \psi\, dx = \int_\Omega q_0(t)\psi\, dx - \int_{\Gamma_b} q_2(t)\psi\, da \\ + \int_{\Gamma_3} k\, \varphi_F(t)\psi\, da, \quad \forall \psi \in W \ \ t \in (0,T). \end{aligned} \tag{3.26}$$

Using (3.14) and (2.34) we infer that

$$\begin{aligned} a_\mu(u(t), v - \dot{u}(t)) + \left(\int_0^t \theta(t-s)u(s)\, ds, v - \dot{u}(t)\right)_V + a_e^*(\varphi(t), v - \dot{u}(t)) \\ - \int_{\Gamma_3} (|\mu \partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)\, ds + e\partial_\nu \varphi(t)|)(v - \dot{u}(t))\, da \\ = (f(t), v - \dot{u}(t))_V, \quad \forall v \in V,\ t \in [0,T]. \end{aligned} \tag{3.27}$$

Taking into account relations (3.16) and (3.20)–(3.21), we find the second equality in Lemma 3.1, i.e. (3.24). Using the frictional contact condition (2.34) and (3.14) on $\Gamma_3 \times (0,T)$, we deduce that for all $t \in [0,T]$

$$j(\dot{u}(t)) = -\int_{\Gamma_3} (|\mu\partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)\,ds + e\partial_\nu\varphi(t)|)\dot{u}(t)\,da. \tag{3.28}$$

It is easy to see that

$$j(v) \geq -\int_{\Gamma_3} (|\mu\partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)\,ds + e\partial_\nu\varphi(t)|)v\,da, \quad \forall v \in V. \tag{3.29}$$

The first inequality in Lemma 3.1 follows now from (3.27) and (3.28)–(3.29).

Now, from Lemma 3.1 and condition (3.25) we obtain the following variational problem:

Problem $\mathcal{PV}$

Find a displacement field $u : [0,T] \to V$ and an electric potential field $\varphi : [0,T] \to W$ such that

$$\begin{aligned} a_\mu(u(t), v - \dot{u}(t)) + \left(\int_0^t \theta(t-s)u(s)\,ds, v - \dot{u}(t)\right)_V + a_e^*(\varphi(t), v - \dot{u}(t)) + j(v) - j(\dot{u}(t)) \\ \geq (f(t), v - \dot{u}(t))_V, \quad \forall v \in V,\ t \in [0,T], \end{aligned} \tag{3.30}$$

$$a_\alpha(\varphi(t), \psi) - a_e(u(t), \psi) = (q(t), \psi)_W, \quad \forall \psi \in W,\ t \in [0,T], \tag{3.31}$$

$$u(0) = u_0. \tag{3.32}$$

$\square$

Our main existence and uniqueness result, which we state now and prove in the next section, is the following:

Theorem 3.2. *Assume that* (3.3)–(3.18) *hold. Then the variational problem $\mathcal{PV}$ possesses a unique solution (u, φ) which satisfies*

$$u \in W^{1,2}(0,T;V), \quad \varphi \in W^{1,2}(0,T;W). \tag{3.33}$$

We note that an element (u, φ) which solves Problem $\mathcal{PV}$ is called a weak solution of the antiplane contact Problem $\mathcal{P}$. We conclude by Theorem 3.2 that the antiplane contact Problem $\mathcal{P}$ has a unique weak solution, provided that (3.3)–(3.18) hold.

4. Proof of Theorem 3.2

The proof of the theorem will be carried out in several steps. In the rest of this section we assume that (3.3)–(3.18) hold.

Step 1: First, we introduce the set

$$\mathcal{W} = \{\eta \in W^{1,2}(0,T;X) \quad \text{such that} \quad \eta(0) = 0_X\} \tag{4.1}$$

and we recall the following existence and uniquness result.

Lemma 4.1. *For all $\eta \in \mathcal{W}$ there exists a unique element $\eta \in W^{1,2}(0,T;X)$ satisfying the inequality and the data condition defined by the problem $\mathcal{PV}_\eta^1$, where problem $\mathcal{PV}_\eta^1$ is*

$$a(u_\eta(t), v - \dot{u_\eta}(t)) + (\eta(t), v - \dot{u}_\eta(t))_X + j(v) - j(\dot{u}_\eta(t)) \geq (f(t), v - \dot{u_\eta}(t))_X, \quad \forall v \in X,\ t \in [0,T], \tag{4.2}$$

$$u_\eta(0) = u_0. \tag{4.3}$$

The next theorem will be used in the proof of the above lemma:

Theorem 4.2 ([2], p. 117). *Let $(X, (\cdot,\cdot)_X)$ be a real Hilbert space and let $j : X \longrightarrow (-\infty, +\infty)$ be a convex lower semicontinuous functional. Assume that $j \neq +\infty$, that is*

$$D(j) = \{v \in X \quad j(v) < \infty\} \neq \emptyset.$$

Let $f \in W^{1,2}(0, T; X)$ and $u_0 \in X$ be such that

$$\sup_{v \in D(j)} = \{(f(0), v)_X - (u_0, v)_X - j(v)\} < +\infty.$$

Then the variational problem $\mathcal{PV}$ possesses a unique solution (u, φ) satisfying $u(0) = u_0$ and

$$(u(t), v - \dot{u}(t))_X + j(v) - j(\dot{u}(t)) \leq (f(t), v - \dot{u}(t))_X \quad \forall v \in X \quad a.e.\, t \in (0, T).$$

Proof of lemma 4.1. Let $a(\cdot,\cdot)$ be defined by

$$a(u, v) = (u, v)_a \quad \forall u, v \in X. \tag{4.4}$$

Note that $(\cdot,\cdot)_a$ is an inner product on the space X and $\|\cdot\|_a$ is the associated norm which is equivalent to the norm $\|\cdot\|_X$ on the space X. Then $(X, (\cdot,\cdot)_a)$ is a real Hilbert space.

We now define the function $f_\eta : [0, T] \longrightarrow X$ by

$$(f_\eta(t), v)_a = (f(t), v)_X - (\eta(t), v)_X \quad \forall v \in V, \quad t \in [0, T]. \tag{4.5}$$

It follows from (3.17) and (4.1) that

$$f_\eta(t) \in W^{1,2}(0, T; X). \tag{4.6}$$

Using now (4.5) at $t = 0$, we obtain

$$(f_\eta(0), v)_a = (f(0), v)_X - (\eta(0), v)_X \quad \forall v \in V, \quad t \in [0, T]. \tag{4.7}$$

Moreover, rewriting (4.4) at $t = 0$, we have

$$a(u_0, v) = (u_0, v)_a \quad \forall v \in X. \tag{4.8}$$

On the other hand, taking into account (4.1), (4.7) and (4.8), we obtain the equality

$$(f_\eta(0), v)_a - (u_0, v)_a - j(v) = (f(0), v)_X - a(u_0, v) - j(v) \quad \forall v \in V. \tag{4.9}$$

From assumption (3.13), we find

$$\sup_{v \in D(j)} = \{(f_\eta(0), v)_a - (u_0, v)_a - j(v)\} < +\infty. \tag{4.10}$$

Given that (3.13), (3.14), (4.6) and (4.10) are satisfied, we can use Theorem 4.2 on the space $(X, (\cdot,\cdot)_a)$, therefore there exists a unique element u_η satisfying

$$u_\eta \in W^{1,2}(0, T; X) \text{ such that } u_\eta = u_0 \tag{4.11}$$

and

$$(u_\eta(t), v - \dot{u}_\eta(t))_a + j(v) - j(\dot{u}_\eta(t)) \geq (f_\eta(t), v - \dot{u}_\eta(t))_a, \quad \forall v \in X \quad a.e. \quad t \in (0, T). \tag{4.12}$$

Using (4.4) and (4.7), we obtain the relations (4.2) and (4.3) in Lemma 4.1. This concludes the existence and uniqueness part of the proof of Lemma 4.1. □

Step 2: We use the displacement field u_η obtained in Lemma 4.1 to define the following variational problem for the electric potential field:

Problem $\mathcal{PV}_\eta^2$

Find an electrical potential $\varphi_\eta : [0,T] \to W$ such that

$$a_\alpha(\varphi_\eta(t), \psi) - a_e(u_\eta(t), \psi) = (q(t), \psi)_W \quad \forall \psi \in W,\, t \in [0,T]. \tag{4.13}$$

The well posedness of problem $\mathcal{PV}_\eta^2$ follows.

Lemma 4.3. *There exists a unique solution $\varphi_\eta \in W^{1,2}(0,T;W)$ which satisfies (4.13). Moreover, if φ_{η_1} and φ_{η_2} are the solutions of (4.13) corresponding to η_1, $\eta_2 \in C([0,T];V)$ then, there exists $c > 0$, such that*

$$\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\|_W \le c\, \|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V \quad \forall\, t \in [0,T]. \tag{4.14}$$

Proof. Let $t \in [0,T]$. We use the properties of the bilinear form a_β and the Lax-Milgram lemma to see that there exists a unique element $\varphi_\eta(t) \in W$ which solves (4.13) at any moment $t \in [0,T]$. Consider now t_1, $t_2 \in [0,T]$. From (4.13), we get

$$a_\alpha(\varphi_\eta(t_1), \psi) - a_e(u_\eta(t_1), \psi) = (q(t_1), \psi)_W \quad \forall \psi \in W,\, t_1 \in [0,T] \tag{4.15}$$

and

$$a_\alpha(\varphi_\eta(t_2), \psi) - a_e(u_\eta(t_2), \psi) = (q(t_2), \psi)_W \quad \forall \psi \in W,\, t_2 \in [0,T]. \tag{4.16}$$

Using (4.15), (4.16) and (3.22) we find that

$$\alpha^* \|\varphi(t_1) - \varphi(t_2)\|_W^2 \le (\|e\|_{L^\infty(\Omega)} \|u(t_1) - u(t_2)\|_V + \|q(t_1) - q(t_2)\|_W) \|\varphi(t_1) - \varphi(t_2)\|_W$$

and it follows from the previous inequality that

$$\|\varphi(t_1) - \varphi(t_2)\|_W \le c\, (\|u(t_1) - u(t_2)\|_V + \|q(t_1) - q(t_2)\|_W). \tag{4.17}$$

Then, the regularity $u_\eta \in W^{1,2}(0,T;V)$ combined with (3.18) and (4.17) imply that $\varphi_\eta \in W^{1,2}(0,T;W)$ which concludes the proof. □

Now, for all $\eta \in \mathcal{W}$ we denote by u_η the solution of Problem $\mathcal{PV}_\eta^1$ obtained in Lemma 4.1 and by φ_η the solution of Problem $\mathcal{PV}_\eta^2$ obtained in Lemma 4.3.

Step 3: We consider the operator $\Lambda : \mathcal{W} \longrightarrow \mathcal{W}$. We use Riesz's representation theorem to define the element $\Lambda\eta(t) \in \mathcal{W}$ by equality

$$(\Lambda\eta(t), w)_{\mathcal{W}} = \int_0^t \theta(t-s) u_\eta(s)\, ds + a_e^*(\varphi_\eta(t), w), \quad \forall \eta \in \mathcal{W}, \quad \forall w \in W \quad t \in [0,T]. \tag{4.18}$$

Clearly, for a given $\eta \in \mathcal{W}$ the function $t \mapsto \Lambda\eta(t)$ belongs to $\mathcal{W}$. In this step we show that the operator $\Lambda : \mathcal{W} \to \mathcal{W}$ has unique fixed point.

Lemma 4.4. *There exists a unique $\eta^* \in \mathcal{W}$ such that $\Lambda\eta^* = \eta^*$.*

Proof. Let η_1, $\eta_2 \in \mathcal{W}$ and $t \in [0,T]$. We denote by u_i and φ_i the functions u_{η_i} and φ_{η_i} obtained in Lemmas 4.1 and 4.3, for $i = 1, 2$. Using (4.18) and (3.20) we obtain

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X^2 \le C \left(\int_0^t \|u_1(s) - u_2(s)\|_X^2\, ds + \|\varphi_1(t) - \varphi_2(t)\|_W^2 \right), \quad \forall t \in [0,T]. \tag{4.19}$$

The constant C represents a generic positive number which may depend on $\|\theta\|_{W^{1,2}(0,T)}$, T and e, and whose value may change from place to place.

Since $u_\eta \in W^{1,2}(0,T;V)$ and $\varphi_\eta \in W^{1,2}(0,T;W)$ we deduce from inequality (4.19) that $\Lambda\eta \in W^{1,2}(0,T;X)$. On the other hand, (4.13) and arguments similar as those used in the proof of (4.17) yield

$$\|\varphi_1(t) - \varphi_2(t)\|_W \;\le\; C\, \|u_1(t) - u_2(t)\|_V. \tag{4.20}$$

Using now (4.20) in (4.19), we get

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X^2 \le C\,(\int_0^t \|u_1(s) - u_2(s)\|_X^2\, ds + \|u_1(t) - u_2(t)\|_V), \quad a.e.\, t \in [0,T]. \tag{4.21}$$

Taking into account (4.2) in Lemma 4.1, we have the inequalities

$$a(u_1(t), v - \dot{u_1}(t)) + (\eta_1(t), v - \dot{u}_1(t))_X + j(v) - j(\dot{u}_1(t)) \ge (f(t), v - \dot{u_1}(t))_X, \quad \forall v \in X,\ t \in [0,T]$$

and

$$a(u_2(t), v - \dot{u_2}(t)) + (\eta_2(t), v - \dot{u}_2(t))_X + j(v) - j(\dot{u}_2(t)) \ge (f(t), v - \dot{u_2}(t))_X, \quad \forall v \in X,\ t \in [0,T],$$

for all $v \in X$, $a.e.\, s \in (0,T)$. We choose $v = \dot{u}_2(s)$ in the first inequality, $v = \dot{u}_1(s)$ in the second inequality, and add the results to obtain

$$\frac{1}{2}\|u_1(s) - u_2(s)\|_X^2 \ \le\ -(\eta_1(s) - \eta_2(s), \dot{u}_1(s) - \dot{u}_2(s))_X \quad a.e.\, s \in (0,T).$$

Let $t \in [0,T]$. Integrating the previous inequality from 0 to t and using (4.3), we get

$$\frac{1}{2}\|u_1(t) - u_2(t)\|_X^2 \le -(\eta_1(t) - \eta_2(t), u_1(t) - u_2(t))_X + \int_0^t (\dot{\eta}_1(s) - \dot{\eta}_2(s), u_1(s) - u_2(s))_X\, ds.$$

We deduce that

$$C\|u_1(t) - u_2(t)\|_X^2 \le \|\eta_1(t) - \eta_2(t)\|_X \|u_1(t) - u_2(t)\|_X + \int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X \|u_1(s) - u_2(s)\|_X\, ds.$$

Using Young's inequality, we get

$$\|u_1(t) - u_2(t)\|_X^2 \le C(\|\eta_1(t) - \eta_2(t)\|_X^2 + \int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X^2\, ds + \int_0^t \|u_1(s) - u_2(s)\|_X^2\, ds. \tag{4.22}$$

On the other hand, as

$$\eta_1(t) - \eta_2(t) = \int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X^2\, ds,$$

we can obtain

$$\|\eta_1(t) - \eta_2(t)\|_X^2 \ \le\ C\int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X^2\, ds. \tag{4.23}$$

Now, using (4.23) in (4.22), we have

$$\|u_1(t) - u_2(t)\|_X^2 \ \le\ C\left(\int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X^2\, ds + \int_0^t \|u_1(s) - u_2(s)\|_X^2\, ds\right).$$

Taking into account Gronwall's inequality we deduce

$$\|u_1(t) - u_2(t)\|_X^2 \ \le\ C\int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X^2\, ds \tag{4.24}$$

which yields

$$\int_0^t \|u_1(s) - u_2(s)\|_X^2\, ds \ \le\ C\int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X^2\, ds. \tag{4.25}$$

From (4.21), (4.24) and (4.25) we obtain

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X^2 \ \le\ C\int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_X^2\, ds.$$

Iterating the last inequality m times we infer that

$$\|\Lambda^m\eta_1(t)-\Lambda^m\eta_2(t)\|_X^2 \le C^m\int_0^t\int_0^{s_1}\cdots\int_0^{s_{m-1}}\|\dot\eta_1(s_m)-\dot\eta_2(s_m)\|_X^2\,ds_m\cdots ds_1,$$

where Λ^m denotes the power of the operator Λ. The last inequality implies

$$\|\Lambda^m\eta_1(t)-\Lambda^m\eta_2(t)\|_{W^{1,2}(0,T;X)}^2 \le \frac{C^mT^m}{m!}\,\|\eta_1-\eta_2\|_{W^{1,2}(0,T;X)}^2,$$

whence it follows that, for a sufficiently large m, the power Λ^m of Λ is a contraction, since

$$\lim_{m\longrightarrow+\infty}\frac{C^mT^m}{m!}=0.$$

By Banach's fixed point theorem, there exists a unique element $\eta^*\in\mathcal{W}$ such that $\Lambda^m\eta^*=\eta^*$. Moreover, since

$$\Lambda^m(\Lambda\eta^*)=\Lambda(\Lambda^m\eta^*)=\Lambda\eta^*,$$

we deduce that $\Lambda\eta^*$ is also a fixed point of the operator Λ^m. By the uniqueness of the fixed point, we conclude that $\Lambda\eta^*=\eta^*$. □

Step 4: We now have all the ingredients to provide the proof of Theorem 3.2.

For the existence part, let $\eta^*\in W^{1,2}(0,T;V)$ be the fixed point of the operator Λ, and let u_{η^*}, φ_{η^*} be the solutions of problems $\mathcal{PV}^1_\eta$ and $\mathcal{PV}^2_\eta$, respectively, for $\eta=\eta^*$. It follows from (4.18) that

$$(\eta^*(t),v)_V=\int_0^t\theta(t-s)u_\eta*(s)\,ds+a_e^*(\varphi_\eta*(t),w)\quad\forall v\in V,\ t\in[0,T]$$

and, therefore, (4.2), (4.3) and (4.13) imply that $(u_{\eta^*},\varphi_{\eta^*})$ is a solution of problem $\mathcal{PV}$. The regularity (3.33) of the solution follows from Lemmas 4.1 and 4.3.

The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ. It can also be obtained by using arguments similar as those used in [5] and [11].

5. Conclusion

We presented a model for an antiplane contact problem for electro-viscoelastic materials with long-term memory. The problem was set as a variational inequality for the displacements and a variational equality for the electric potential. The existence of the unique weak solution for the problem was established by using arguments from the theory of evolutionary variational inequalities and a fixed point theorem. This work opens the way to study further problems with other conditions.

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