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Iterative algorithms with perturbations for Lipschitz pseudocontractive mappings in Banach spaces

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Abstract

In this paper, we present an iterative algorithm with perturbations for Lipschitz pseudocontractive mappings in Banach spaces. Consequently, we give the convergence analysis of the suggested algorithm. Our result improves the corresponding results in the literature.

Keywords: Strong convergence, pseudocontractive mapping, fixed point, Banach space.
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1. Introduction

Let E be a real Banach space and E^* be the dual space of E. Let J denote the normalized duality mapping from E into 2^{E^*} defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \|x\|\}, \quad x \in E,$$

where $\langle \cdot, \cdot\rangle$ denote the generalized duality pairing between E and E^*. It is well known that if E is smooth, then J is single-valued. In the sequel, we shall denote the single-valued normalized duality mapping by j.

Recall that a mapping T with domain $D(T)$ and range $R(T)$ in E is called *pseudocontractive* if the inequality

$$\|x - y\| \le \|x - y + r((I - T)x - (I - T)y)\| \tag{1.1}$$

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holds for each $x, y \in D(T)$ and for all $r > 0$. From a result of Kato [17], we know that (1.1) is equivalent to (1.2) below there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 \tag{1.2}$$

for all $x, y \in D(T)$.

The class of pseudocontractive mapping is one of the most important classes of mappings in nonlinear analysis. Interest in pseudocontractive mappings stems mainly from their firm connection with the class of accretive mappings, where a mapping A with domain $D(A)$ and range $R(A)$ in E is called *accretive* if the inequality

$$\|x-y\| \leq \|x-y+s(Ax-Ay)\|$$

holds for every $x, y \in D(A)$ and for all $s > 0$.

Within the past 30 years or so, many authors have been devoted to the existence of zeros of accretive mappings or fixed points of pseudocontractive mappings and iterative construction of zeros of accretive mappings, and of fixed points of pseudocontractive mappings (see [9, 13, 19, 21, 22]).

Especially, in 2000, Morales and Jung [20] studied existence of paths for pseudocontractive mappings in Banach spaces. They proved the following result.

Theorem 1.1. *Let E be a Banach space. Suppose that C is a nonempty closed convex subset of E and $T : C \to E$ is a continuous pseudocontractive mapping satisfying the weakly inward condition: $T(x) \in \overline{I_C(x)}$ ($\overline{I_C(x)}$ is the closure of $I_C(x)$) for each $x \in C$, where $I_C(x) = x + \{c(u-x) : u \in E \text{ and } c \geq 1\}$. Then for each $z \in C$, there exists a unique continuous path $t \longmapsto y_t \in C$, $t \in [0, 1)$, satisfying the following equation*

$$y_t = tTy_t + (1-t)z.$$

At the same time, several algorithms have been introduced and studied by various authors for approximating fixed points of pseudocontractive mappings in Hilbert spaces and Banach spaces, you may consult in [3, 4, 5, 23, 27, 29, 30, 32].

In 1953, Mann [18] introduced an iterative algorithm which is now referred to as the Mann iterative algorithm. Most of the literatures deal with the special case of the general Mann iterative algorithm which is defined by

$$x_0 \in C, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0, \tag{1.3}$$

where C is a convex subset of a Banach space E, $T : C \to C$ is a mapping and $\{\alpha_n\}$ is a sequence of positive numbers satisfying certain control conditions.

It is well known that the Mann iterative algorithm can be employed to approximate fixed points of nonexpansive mappings and zeros of strongly accretive mappings in Hilbert spaces or Banach spaces. Many convergence theorems have been announced and published by a good numbers of authors. For more details, see [2, 10, 11, 12, 14, 15, 25, 26, 28, 31]. A natural question rises:

Question 1.2. *Does the Mann iterative algorithm always converge for continuous pseudocontractive mappings or even Lipschitz pseudocontractive mappings?*

However in 2001, Chidume and Mutangadura [6] provided an example of a Lipschitz pseudocontractive mapping with a unique fixed point for which the Mann iterative algorithm failed to converge and they stated there "This resolves a long standing open problem". Therefore, it is an interesting topic to construct some new iterative algorithms for approximating the fixed points of pseudocontractive mappings. Now we recall some important results in the literature as follows.

The first result was introduced in 1974 by Ishikawa [16] who proved the following theorem.

Theorem 1.3. *If C is a compact convex subset of a Hilbert space H, $T : C \to C$ is a Lipschitz pseudocontractive mappings and x_0 is any point in C, then the sequence $\{x_n\}$ converges strongly to a fixed point of T, where $\{x_n\}$ is defined iteratively for each positive integer $n \geq 0$ by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers satisfying the following conditions

(i) $0 \leq \alpha_n \leq \beta_n \leq 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty$.

Since its publication in 1974, the above theorem, as far as we know has never been extended to more general Banach spaces.

The second result was introduced by Bruck [1] in 1974. He proved the following theorem.

Theorem 1.4. *Let U be a maximal monotone operator on H with $0 \in R(U)$. Suppose that $\{\lambda_n\}$ and $\{\theta_n\}$ are acceptably paired, $z \in H$ and the sequence $\{x_n\} \subset D(U)$ satisfies*

$$x_{n+1} = x_n - \lambda_n(v_n + \theta_n(x_n - z)), v_n \in U(x_n) \tag{1.4}$$

for $n \geq 1$. If $\{x_n\}$ and $\{v_n\}$ are bounded, then $\{x_n\}$ converges strongly to x^*, the point of $U^{-1}(0)$ closest to z.

The recursion formula (1.4) has recently been modified by Chidume and Zegeye [8] and then applied to approximate fixed points of Lipschitz pseudocontractive mappings in real Banach spaces with uniformly Gâteaux differentiable norm.

The third result was introduced in 1993 by Schu [24] who proved the following theorem.

Theorem 1.5. *Let C be a nonempty closed convex and bounded subset of a Hilbert space H, $T : C \to C$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L \geq 0$, $\{\lambda_n\} \subset (0,1)$ with $\lim_{n\to\infty} \lambda_n = 1$, $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$ such that $(\{\alpha_n\}, \{\mu_n\})$ has property (A), $\{(1-\mu_n)(1-\lambda_n)^{-1}\}$ is bounded and $\lim_{n\to\infty} \frac{1-\mu_n}{\alpha_n} = 0$, where $k_n := (1+\alpha_n^2(1+L)^2)^{\frac{1}{2}}$ and $\mu_n := \frac{\lambda_n}{k_n}$, $\forall n \geq 1$. Fix an arbitrary point $w \in K$ and define*

$$z_{n+1} := \mu_{n+1}(\alpha_n T z_n + (1-\alpha_n)z_n) + (1-\mu_{n+1})w. \tag{1.5}$$

Then the sequence $\{z_n\}$ defined by (1.5) converges strongly to the unique fixed point of T closest to w.

Here the pair of sequences $(\{\alpha_n\}, \{\mu_n\}) \subset (0,\infty) \times (0,1)$ is said to have *property* (A) if and only if the following conditions hold:

(i) $\{\alpha_n\}$ is decreasing;
(ii) $\{\mu_n\}$ is strictly increasing;
(iii) there exists a strictly increasing sequence $\{\beta_n\} \subset \mathbb{N}$ such that
  (a) $\lim_n \frac{\alpha_n - \alpha_{n+\beta_n}}{1-\mu_n} = 0$;
  (b) $\lim_n (1-\mu_{n+\beta_n})(1-\mu_n)^{-1} = 1$;
  (c) $\lim_n \beta_n(1-\mu_n) = \infty$.

Subsequently, Chidume and Udomene [7] extended Theorem 1.5 to real Banach spaces with the following assumptions on iterative parameters which are simper than the above iterative parameters:

(i)$'$ $\{\alpha_n\}$ is decreasing and $\lim_{n\to\infty} \alpha_n = 0$;
(ii)$'$ $\lim_{n\to\infty} \mu_n = 1$ and $\sum_{n=1}^{\infty}(1-\mu_n) = \infty$;
(iii) $(a)'$ $\lim_{n\to\infty} \frac{1-\mu_n}{\alpha_n} = 0$; $(b)'$ $\lim_{n\to\infty} \frac{\alpha_n^2}{1-\mu_n} = 0$;

$(c)'$ $\lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{(1-\mu_n)^2} = 0$; $(d)'$ $\lim_{n\to\infty} \frac{\alpha_{n-1}-\alpha_n}{\alpha_{n-1}(1-\mu_n)}$.

On the other hand, there are perturbations always occurring in the iterative processes because the manipulations are inaccurate. It is no doubt that researching the convergent problems of iterative methods with perturbations members is a significant job.

In this paper, we present an iterative algorithm with perturbations for Lipschitz pseudocontractive mappings in Banach spaces. Consequently, we give the convergence analysis of the suggested algorithm. Our result improves the corresponding results in the literature.

2. Preliminaries

Let $S := \{x \in E : \|x\| = 1\}$ denote the unit sphere of a Banach space E. The space E is said to have a *Gâteaux differentiable norm* (or E is said to be *smooth*) if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S$, and E is said to have a *uniformly Gâteaux differentiable norm* if for each $y \in S$ the limit (2.1) is attained uniformly for $x \in S$.

We need the following lemmas for proof of our main results.

Lemma 2.1 ([20]). *Let E be a Banach space. Suppose K is a nonempty closed convex subset of E and $T : K \to E$ is a continuous pseudocontractive mapping satisfying the weakly inward condition. Then for $y_0 \in K$, there exists a unique path $t \to y_t \in K$, $t \in [0, 1)$, satisfying the following condition:*

$$y_t = tTy_t + (1-t)y_0.$$

Furthermore, if E is assumed to be a reflexive Banach space possessing a uniformly Gâteaux differentiable norm and is such that every closed convex and bounded subset of K has the fixed point property for non-expansive self-mappings, then as $t \to 1$, the path $\{y_t : t \in [0, 1)\}$ converges strongly to a fixed point Qu of T.

Lemma 2.2 ([25]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that
(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Theorem 3.1. *Let K be a nonempty closed convex subset of a real reflexive Banach space E with a uniformly Gâteaux differentiable norm. Let $T : K \to K$ be a Lipschitz pseudocontractive mapping with Lipschitz constant $L > 0$ and $F(T) \ne \emptyset$, where $F(T)$ is fixed point sets of T. Suppose that every closed convex and bounded subset of K has the fixed point property for nonexpansive self-mappings. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0, 1)$ which satisfy the following conditions:*
$(C1)$ $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;
$(C2)$ $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$ *and* $\lim_{n\to\infty} \frac{\alpha_n^2}{\beta_n} = 0$;
$(C3)$ $\lim_{n\to\infty} \frac{1}{\beta_n} \left| \frac{1-\beta_{n-1}}{1-\beta_n} - \frac{\alpha_n \beta_{n-1}}{\alpha_{n-1}\beta_n} \right| = 0$.
For any $u \in K$, let $\{x_n\}$ be a sequence generated from arbitrary $x_1 \in K$ by

$$x_{n+1} = \beta_n u_n + (1-\beta_n)(\alpha_n T x_n + (1-\alpha_n)x_n), \quad \forall n \ge 0, \tag{3.1}$$

where $\{u_n\} \subset K$ is a perturbation satisfying $u_n \to u \in K$ as $n \to \infty$. Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a fixed point Qu of T, where Q is the unique sunny nonexpansive retract from K onto $F(T)$.

Proof. First we prove that the sequence $\{x_n\}$ is bounded. We will show this fact by induction. According to conditions $(C1)$ and $(C2)$, there exists a sufficiently large positive integer m such that

$$1 - 2(L+1)(L+2)\left(\beta_n + 2\alpha_n + \frac{\alpha_n^2}{\beta_n}\right) > 0, \quad n \geq m. \tag{3.2}$$

Fix a $p \in F(T)$ and take a constant $M_1 > 0$ such that

$$\max\{\|x_0 - p\|, \|x_1 - p\|, \cdots, \|x_m - p\|, 2\|u_m - p\|\} \leq M_1. \tag{3.3}$$

Next, we show that $\|x_{m+1} - p\| \leq M_1$.

Since T is pseudocontractive, we have

$$\langle (I-T)x_{m+1} - (I-T)p, j(x_{m+1} - p)\rangle \geq 0. \tag{3.4}$$

From (3.1) and (3.4), we obtain

$$\begin{aligned}
\|x_{m+1} - p\|^2 &= \langle x_{m+1} - p, j(x_{m+1} - p)\rangle \\
&= \beta_m \langle u_m - p, j(x_{m+1} - p)\rangle + (1-\beta_m)\alpha_m \langle Tx_m - p, j(x_{m+1} - p)\rangle \\
&\quad + (1-\beta_m)(1-\alpha_m)\langle x_m - p, j(x_{m+1} - p)\rangle \\
&= \beta_m \langle u_m - p, j(x_{m+1} - p)\rangle + (1-\beta_m)\alpha_m \langle Tx_m - Tx_{m+1}, j(x_{m+1} - p)\rangle \\
&\quad + (1-\beta_m)\alpha_m \langle Tx_{m+1} - x_{m+1}, j(x_{m+1} - p)\rangle \\
&\quad + (1-\beta_m)\alpha_m \langle x_{m+1} - x_m, j(x_{m+1} - p)\rangle + \langle x_m - p, j(x_{m+1} - p)\rangle \\
&\quad - \beta_m \langle x_{m+1} - p, j(x_{m+1} - p)\rangle - \beta_m \langle x_m - x_{m+1}, j(x_{m+1} - p)\rangle \\
&\leq \beta_m \|u_m - p\|\|x_{m+1} - p\| + (1-\beta_m)\alpha_m \|Tx_m - Tx_{m+1}\|\|x_{m+1} - p\| \\
&\quad + (1-\beta_m)\alpha_m \|x_{m+1} - x_m\|\|x_{m+1} - p\| + \|x_m - p\|\|x_{m+1} - p\| \\
&\quad - \beta_m \|x_{m+1} - p\|^2 + \beta_m \|x_m - x_{m+1}\|\|x_{m+1} - p\| \\
&\leq \beta_m \|u_m - p\|\|x_{m+1} - p\| + (\alpha_m + \beta_m)(L+1)\|x_{m+1} - x_m\|\|x_{m+1} - p\| \\
&\quad + \|x_m - p\|\|x_{m+1} - p\| - \beta_m \|x_{m+1} - p\|^2.
\end{aligned}$$

It follows that

$$(1+\beta_m)\|x_{m+1} - p\| \leq \|x_m - p\| + \beta_m \|u_m - p\| + (L+1)(\alpha_m + \beta_m)\|x_{m+1} - x_m\|. \tag{3.5}$$

By (3.1) and (3.3), we have

$$\begin{aligned}
\|x_{m+1} - x_m\| &= \|\beta_m(u_m - p) + (1-\beta_m)\alpha_m(Tx_m - p) \\
&\quad + \alpha_m(\beta_m - 1)(x_m - p) - \beta_m(x_m - p)\| \\
&\leq \beta_m \|u_m - p\| + (1-\beta_m)\alpha_m L\|x_m - p\| + [\alpha_m(1-\beta_m) + \beta_m]\|x_m - p\| \\
&\leq (L+2)(\alpha_m + \beta_m)M_1.
\end{aligned} \tag{3.6}$$

Substitute (3.6) into (3.5) to obtain

$$\begin{aligned}
(1+\beta_m)\|x_{m+1} - p\| &\leq \|x_m - p\| + \beta_m \|u_m - p\| + (L+1)(L+2)(\alpha_m + \beta_m)^2 M_1 \\
&\leq \left(1 + \frac{1}{2}\beta_m\right)M_1 + (L+1)(L+2)(\alpha_m + \beta_m)^2 M_1,
\end{aligned}$$

that is,

$$\begin{aligned}\|x_{m+1}-p\| &\le \left[1-\frac{(\beta_m/2)-(L+1)(L+2)(\alpha_m+\beta_m)^2}{1+\beta_m}\right]M_1\\ &= \left\{1-\frac{(\beta_m/2)[1-2(L+1)(L+2)\left(\beta_m+2\alpha_m+(\alpha_m^2/\beta_m)\right)]}{1+\beta_m}\right\}M_1\\ &\le M_1.\end{aligned}$$

By induction, we get

$$\|x_n-p\|\le M_1, \quad \forall n\ge 0, \tag{3.7}$$

which implies that $\{x_n\}$ is bounded and so is $\{Tx_n\}$.

Set $\gamma_n=\frac{\beta_n}{\alpha_n+\beta_n-\alpha_n\beta_n}=\frac{\frac{\beta_n}{\alpha_n}}{1+\frac{\beta_n}{\alpha_n}-\beta_n}$ for all $n\ge 0$. Noting that $\lim_{n\to\infty}\frac{\beta_n}{\alpha_n}=\lim_{n\to\infty}\beta_n=0$, thus we deduce $\gamma_n\to 0$ as $n\to\infty$. It follows from Lemma 2.1 that there exists a unique sequence $z_n\in K$ satisfying

$$z_n=\gamma_n u+(1-\gamma_n)Tz_n. \tag{3.8}$$

We note that (3.8) can be rewritten as the follows

$$z_n=\beta_n u+(1-\beta_n)(\alpha_n Tz_n+(1-\alpha_n)z_n).$$

From (3.1) and (3.2), we have

$$\begin{aligned}\|x_{n+1}-z_n\|^2 &= \beta_n\langle u_n-u, j(x_{n+1}-z_n)\rangle+(1-\beta_n)(1-\alpha_n)\langle x_n-z_n, j(x_{n+1}-z_n)\rangle\\ &\quad+(1-\beta_n)\alpha_n\langle Tx_n-Tz_n, j(x_{n+1}-z_n)\rangle\\ &= \beta_n\langle u_n-u, j(x_{n+1}-z_n)\rangle+(1-\beta_n)(1-\alpha_n)\langle x_n-z_n, j(x_{n+1}-z_n)\rangle\\ &\quad+(1-\beta_n)\alpha_n\langle Tx_{n+1}-Tz_n, j(x_{n+1}-z_n)\rangle\\ &\quad+(1-\beta_n)\alpha_n\langle Tx_n-Tx_{n+1}, j(x_{n+1}-z_n)\rangle\\ &\le \beta_n\|u_n-u\|\|x_{n+1}-z_n\|+(1-\beta_n)(1-\alpha_n)\|x_n-z_n\|\|x_{n+1}-z_n\|\\ &\quad+(1-\beta_n)\alpha_n\|x_{n+1}-z_n\|^2+(1-\beta_n)\alpha_n L\|x_{n+1}-x_n\|\|x_{n+1}-z_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-z_n\| &\le \frac{\beta_n}{1-(1-\beta_n)\alpha_n}\|u_n-u\|+\frac{(1-\beta_n)(1-\alpha_n)}{1-(1-\beta_n)\alpha_n}\|x_n-z_n\|\\ &\quad+\frac{(1-\beta_n)\alpha_n L}{1-(1-\beta_n)\alpha_n}\|x_{n+1}-x_n\|\\ &\le \frac{\beta_n}{1-(1-\beta_n)\alpha_n}\|u_n-u\|+\frac{(1-\beta_n)(1-\alpha_n)}{1-(1-\beta_n)\alpha_n}\|x_n-z_{n-1}\|\\ &\quad+\frac{(1-\beta_n)(1-\alpha_n)}{1-(1-\beta_n)\alpha_n}\|z_n-z_{n-1}\|+\frac{(1-\beta_n)\alpha_n L}{1-(1-\beta_n)\alpha_n}\|x_{n+1}-x_n\|.\end{aligned} \tag{3.9}$$

Next, we will estimate $\|x_{n+1}-x_n\|$ and $\|z_n-z_{n-1}\|$.

First, from (3.1), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\beta_n(u_n-x_n)+(1-\beta_n)\alpha_n(Tx_n-x_n)\|\\ &\le \beta_n\|u_n-x_n\|+(1-\beta_n)\alpha_n\|Tx_n-x_n\|\\ &\le (\alpha_n+\beta_n)M,\end{aligned} \tag{3.10}$$

where $M>0$ is some constant such that $\sup_{n\ge 0}\{\|u_n-x_n\|, \|Tx_n-x_n\|\}$.

From (3.2), we have the following estimation

$$\begin{aligned}\|z_n - z_{n-1}\|^2 &= \gamma_n\langle u - z_{n-1}, j(z_n - z_{n-1})\rangle \\ &\quad + (1-\gamma_n)\langle Tz_n - Tz_{n-1} + Tz_{n-1} - z_{n-1}, j(z_n - z_{n-1})\rangle \\ &= \gamma_n\langle u - z_{n-1}, j(z_n - z_{n-1})\rangle + (1-\gamma_n)\langle Tz_n - Tz_{n-1}, j(z_n - z_{n-1})\rangle \\ &\quad + (1-\gamma_n)\langle Tz_{n-1} - z_{n-1}, j(z_n - z_{n-1})\rangle \\ &= \gamma_n\langle u - z_{n-1}, j(z_n - z_{n-1})\rangle + (1-\gamma_n)\langle Tz_n - Tz_{n-1}, j(z_n - z_{n-1})\rangle \\ &\quad - (1-\gamma_n)\frac{\gamma_{n-1}}{1-\gamma_{n-1}}\langle u - z_{n-1}, j(z_n - z_{n-1})\rangle \\ &\le \left|\gamma_n - (1-\gamma_n)\frac{\gamma_{n-1}}{1-\gamma_{n-1}}\right|\|u - z_{n-1}\|\|z_n - z_{n-1}\| \\ &\quad + (1-\gamma_n)\|z_n - z_{n-1}\|^2,\end{aligned}$$

which implies that

$$\|z_n - z_{n-1}\| \le \frac{\left|\gamma_n - (1-\gamma_n)\frac{\gamma_{n-1}}{1-\gamma_{n-1}}\right|}{\gamma_n}\|u - z_{n-1}\|. \tag{3.11}$$

Hence, from (3.9)-(3.11), we have

$$\begin{aligned}\|x_{n+1} - z_n\| &\le \frac{\beta_n}{1-(1-\beta_n)\alpha_n}\|u_n - u\| + \frac{(1-\beta_n)(1-\alpha_n)}{1-(1-\beta_n)\alpha_n}\|x_n - z_{n-1}\| \\ &\quad + \frac{(1-\beta_n)(1-\alpha_n)}{1-(1-\beta_n)\alpha_n} \times \frac{\left|\gamma_n - (1-\gamma_n)\frac{\gamma_{n-1}}{1-\gamma_{n-1}}\right|}{\gamma_n}\|u - z_{n-1}\| \\ &\quad + \frac{(1-\beta_n)\alpha_n L}{1-(1-\beta_n)\alpha_n}(\alpha_n + \beta_n)M \\ &\le \left[1 - \frac{\beta_n}{1-(1-\beta_n)\alpha_n}\right]\|x_n - z_{n-1}\| \\ &\quad + \frac{\beta_n}{1-(1-\beta_n)\alpha_n}\left\{\frac{\left|\gamma_n - (1-\gamma_n)\frac{\gamma_{n-1}}{1-\gamma_{n-1}}\right|}{\beta_n\gamma_n}\|u - z_{n-1}\|\right. \\ &\quad \left. + \frac{(1-\beta_n)\alpha_n L}{\beta_n}(\alpha_n + \beta_n)M + \|u_n - u\|\right\}..\end{aligned}$$

We note that

$$\begin{aligned}\frac{|\gamma_n - (1-\gamma_n)\frac{\gamma_{n-1}}{1-\gamma_{n-1}}|}{\beta_n\gamma_n} &= \frac{1-\beta_n}{1-\beta_{n-1}}\frac{1}{\beta_n}\left|\frac{1-\beta_{n-1}}{1-\beta_n} - \frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}\beta_n}\right| \\ &\to 0,\end{aligned}$$

and $\frac{(1-\beta_n)\alpha_n L}{\beta_n}(\alpha_n + \beta_n) = (1-\beta_n)L\frac{\alpha_n^2}{\beta_n} + (1-\beta_n)\alpha_n L \to 0$. Hence, by Lemma 2.2, we have $\|x_{n+1} - z_n\| \to 0$. By Lemma 2.1, the sequence $\{z_n\}$ given by (3.8) converges strongly to Qu. Hence, $\{x_n\}$ strongly converges to some fixed point Qu of T. This completes the proof. □

Remark 3.2. We can choose $\alpha_n = \frac{1}{(n+1)^{\frac{1}{3}}}$ and $\beta_n = \frac{1}{(n+1)^{\frac{1}{2}}}$. It is clear that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy conditions $(C1)$ and $(C2)$. Now, we validate that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy condition $(C3)$. As a matter of fact, from $(C3)$, we get

$$\begin{aligned}\frac{1}{\beta_n}\left|\frac{1-\beta_{n-1}}{1-\beta_n} - \frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}\beta_n}\right| &\le \frac{1}{\beta_n}\left|\frac{1-\beta_{n-1}}{1-\beta_n} - 1\right| + \frac{1}{\beta_n}\left|1 - \frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}\beta_n}\right| \\ &= \frac{1}{1-\beta_n}\left|\frac{\beta_n - \beta_{n-1}}{\beta_n}\right| + \frac{1}{\beta_n}\left|1 - \frac{\alpha_n\beta_{n-1}}{\alpha_{n-1}\beta_n}\right|.\end{aligned}$$

Note that

$$\frac{\beta_n - \beta_{n-1}}{\beta_n} = 1 - \frac{\beta_{n-1}}{\beta_n} = 1 - \left(\frac{n+1}{n}\right)^{\frac{1}{2}} \to 0,$$

and

$$\begin{aligned}\frac{1}{\beta_n}\left|1 - \frac{\alpha_n \beta_{n-1}}{\alpha_{n-1}\beta_n}\right| &= (n+1)^{\frac{1}{2}}\left|\left(1+\frac{1}{n}\right)^{\frac{1}{6}} - 1\right| \\ &\le (n+1)^{\frac{1}{2}}\frac{1}{n} \\ &\to 0.\end{aligned}$$

Therefore, $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy all conditions.

Remark 3.3. The assumptions in Theorem 3.1 imposed on iterative parameters are simper than the corresponding assumptions imposed on iterative parameters in [7].

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References

[1] R. E. Bruck, *A strongly convergent iterative method for the solution of* $0 \in Ux$ *for a maximal monotone operator* U *in Hilbert space*, J. Math. Anal. Appl., **48** (1974), 114–126. 1
[2] S. S. Chang, *On Chidume's open questions and approximate solutions of multivalued strongly accretive mapping equations in Banach spaces*, J. Math. Anal. Appl., **216** (1997), 94–111. 1
[3] C. E. Chidume, *Approximation of fixed points of strongly pseudocontractive mappings*, Proc. Amer. Math. Soc., **120** (1994), 545–551. 1
[4] C. E. Chidume, *Global iteration schemes for strongly pseudocontractive maps*, Proc. Amer. Math. Soc., **126** (1998), 2641–2649. 1
[5] C. E. Chidume, C. Moore, *Fixed point iteration for pseudocontractive maps*, Proc. Amer. Math. Soc., **127** (1999), 1163–1170. 1
[6] C. E. Chidume, S. A. Mutangadura, *An example on the Mann iteration method for Lipschitz pseudocontractions*, Proc. Amer. Math. Soc., **129** (2001), 2359–2363. 1
[7] C. E. Chidume, A. Udomene, *Strong convergence theorems for uniformly continuous pseudocontractive maps*, J. Math. Anal. Appl., **323** (2006), 88–99. 1, 3.3
[8] C. E. Chidume, H. Zegeye, *Approximate fixed point sequences and convergence theorems for Lipschitz pseudocontractive maps*, Proc. Amer. Math. Soc., **132** (2004), 831–840. 1
[9] K. Deimling, *Zeros of accretive operators*, Manuscripta Math., **13** (1974), 365–374. 1
[10] L. Deng, *Iteration processes for nonlinear Lipschitzian strongly accretive mappings in* L_p *spaces*, J. Math. Anal. Appl., **188** (1994), 128–140. 1
[11] L. Deng, X. P. Ding, *Iterative approximation of Lipschitz strictly pseudocontractive mappings in uniformly smooth Banach spaces*, Nonlinear Anal., **24** (1995), 981–987. 1
[12] X. P. Ding, *Iterative process with errors to locally strictly pseudocontractive maps in Banach spaces*, Comput. Math. Appl., **32** (1996), 91–97. 1
[13] K. Goebel, W. A. Wirk, *Topics in metric fixed point theory*, Cambridge University Press, Cambridge, (1990). 1
[14] N. J. Huang, C. J. Gao, X. P. Huang, *New iteration procedures with errors for multivalued* ϕ*-strongly pseudocontractive and* ϕ*-strongly accretive mappings*, Comput. Math. Appl., **43** (2002), 1381–1390. 1
[15] N. J. Huang, Y. J. Cho, B. S. Lee, J. S. Jung, *Convergence of iterative processes with errors for set-valued pseudocontractive and accretive type mappings in Banach spaces*, Comput. Math. Appl., **40** (2000), 1127–1139. 1
[16] S. Ishikawa, *Fixed point by a new iteration method*, Proc. Amer. Math. Soc., **4** (1974), 147–150. 1
[17] T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan, **19** (1967), 508–520. 1
[18] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506–510. 1
[19] R. H. Martin, *Differential equations on closed subsets of Banach space*, Trans. Amer. Math. Soc., **179** (1973), 399–414. 1
[20] C. H. Morales, J. S. Jung, *Convergence of paths for pseudocontractive mappings in Banach spaces*, Proc. Amer. Math. Soc., **128** (2000), 3411–3419. 1, 2.1
[21] S. Reich, *An iterative procedure for constructing zero of accretive sets in Banach spaces*, Nonlinear Anal., **2** (1978), 85–92. 1

[22] S. Reich, *Constructive techniques for accretive and monotone operators*, Applied Nonlinear Analysis, Academic Press, New York, (1979). 1

[23] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl., **75** (1980), 287–292. 1

[24] J. Schu, *Approximating fixed points of Lipschitzian pseudocontractive mappings*, Houston J. Math., **19** (1993), 107–115. 1

[25] H. K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl., **298** (2004), 279–291. 1, 2.2

[26] Y. Xu, *Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl., **224** (1998), 91–101. 1

[27] Y. Yao, Y. C. Liou, R. Chen, *Strong convergence of an iterative algorithm for pseudocontractive mapping in Banach spaces*, Nonlinear Anal., **67** (2007), 3311–3317. 1

[28] Y. Yao, Y. C. Liou, S. M. Kang, *Strong convergence of an iterative algorithm on an infinite countable family of nonexpansive mappings*, Appl. Math. Comput., **208** (2009), 211–218. 1

[29] H. Zegeye, E. Prempeh, *Strong convergence of approximants to fixed points of Lipschitzian pseudocontractive maps*, Comput. Math. Appl., **44** (2002), 339–346. 1

[30] H. Zegeye, N. Shahzad, T. Mekonen, *Viscosity approximation methods for pseudocontractive mappings in Banach spaces*, Appl. Math. Comput., **185** (2007), 538–546. 1

[31] L. C. Zeng, *Iterative approximation of solutions to nonlinear equations of strongly accretive operators in Banach spaces*, Nonlinear Anal., **31** (1998), 589–598. 1

[32] H. Zhou, *Convergence theorems of common fixed points for a finite family of Lipschitz pseudocontractions in Banach spaces*, Nonlinear Anal., **68** (2008), 2977–2983. 1