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Hermite-Hadamard type inequalities for operator s-preinvex functions

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Abstract

In this paper, we introduce the concept of operator s-preinvex function, establish some new Hermite-Hadamard type inequalities for operator s-preinvex functions, and provide the estimates of both sides of Hermite-Hadamard type inequality in which some operator s-preinvex functions of positive selfadjoint operators in Hilbert spaces are involved.

Keywords: Hermite-Hadamard type inequality, operator s-convex function, operator preinvex function, operator s-preinvex function.
2010 MSC: 47A63, 26D07, 26D10, 26D15, 26D99.

1. Introduction and Preliminaries

Throughout this paper, let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_0 = [0, \infty)$.

The following inequality holds for any convex function f defined on $\mathbb{R}$ and $a, b \in \mathbb{R}$ with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave on $[a, b]$. The inequality (1.1) is well known in the literature as Hermite-Hadamard's inequality. We note that the Hermite-Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. The classical Hermite-Hadamard's inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$.

In [10], Hudzik and Maligranda considered s-convex function in the second sense. This class is defined in the following way.

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Definition 1.1 ([10])**.** For some fixed $s \in (0,1]$, a function $f : \mathbb{R}_0 \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.2}$$

holds for all $x, y \in \mathbb{R}_0$ and $\lambda \in [0,1]$. If the inequality (1.2) reverses, then f is said to be s-concave in the second sense on $\mathbb{R}_0$.

In [3], Dragomir and Fitzpatrick proved the following variant of Hadamard's inequality which holds for s-convex functions in the second sense.

Theorem 1.2 ([3])**.** *Suppose that $f : \mathbb{R}_0 \to \mathbb{R}_0$ is an s-convex function in the second sense, where $s \in (0,1]$ and let $a, b \in \mathbb{R}_0$ with $a < b$. If $f \in L([a,b])$, then the following inequality holds*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3).

In [2], the authors obtained the estimate of the left-hand side of Hermite-Hadamard's inequality for s-convex functions.

Theorem 1.3 ([2])**.** *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on I°, such that $f' \in L([a,b])$, where $a, b \in I$ with $a < b$. If $|f'|$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$, then the following inequality holds*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \right| \le \frac{b-a}{2(s+1)}\left[\frac{|f'(a)|+2(s+1)\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|}{2(s+2)}\right]. \tag{1.4}$$

In [12], Kirmaci et al. gave the estimate of the rift-hand side of Hermite-Hadamard's inequality for s-convex functions.

Theorem 1.4 ([12])**.** *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $f' \in L([a,b])$ and $|f'|$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \right| \le \frac{(b-a)(2^{s+1}+1)}{2^s(s+1)(s+2)}\left[\frac{|f'(a)|+|f'(b)|}{2}\right]. \tag{1.5}$$

Hermite-Hadamard's inequality has several applications in nonlinear analysis and the geometry of Banach spaces, see [11]. In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [9].

Let X be a vector space, $x, y \in X$, $x \ne y$. Define the segment

$$[x,y] := (1-t)x + ty, \quad t \in [0,1].$$

We consider the function $f : [x,y] \to \mathbb{R}$ and the associated function

$$\begin{aligned} &g(x,y) : [0,1] \to \mathbb{R}, \\ &g(x,y)(t) := f((1-t)x+ty), \quad t \in [0,1]. \end{aligned}$$

Note that f is convex on $[x,y]$ if and only if $g(x,y)$ is convex on $[0,1]$. For any convex function defined on a segment $[x,y] \in X$, we have the Hermite-Hadamard integral inequality (see [4], p.2 and [5], p.2)

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f((1-t)x+ty)\mathrm{d}t \le \frac{f(x)+f(y)}{2}, \tag{1.6}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x,y):[0;1]\to\mathbb{R}$.

Now we review the operator order in $B(H)$ which is the set of all bounded linear operators on a Hilbert space $(H;\langle .,.\rangle)$, and the continuous functional calculus for a bounded self-adjoint operator. For self-adjoint operators $A,B\in B(H)$, we write $A\le B$ if $\langle Ax,x\rangle\le\langle Bx,x\rangle$ for every vector $x\in H$, we call it the operator order.

Let A be a bounded self-adjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all continuous complex-valued functions defined on the spectrum of A, denoted $Sp(A)$, the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [6], p.3). For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$, we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;

(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi(f^*)=\Phi(f)^*$;

(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;

(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$ for $t\in Sp(A)$.

With this notation, we define

$$f(A):=\Phi(f)\quad\text{for all}\quad f\in C(Sp(A)) \tag{1.7}$$

and we call it the continuous functional calculus for a bounded self-adjoint operator A.

If A is a bounded self-adjoint operator and f is a real-valued continuous function on $Sp(A)$, then $f(t)\ge 0$ for any $t\in Sp(A)$ implies that $f(A)\ge 0$, i.e. $f(A)$ is a positive operator on H. Moreover, if both f and g are real-valued functions on $Sp(A)$ such that $f(t)\le g(t)$ for any $t\in Sp(A)$, then $f(A)\le g(A)$ in the operator order in $B(H)$.

A real valued continuous function f on an interval $I\subseteq\mathbb{R}$ is said to be operator convex (operator concave) if

$$f((1-\lambda)A+\lambda B)\le(\ge)(1-\lambda)f(A)+\lambda f(B)$$

in the operator order in $B(H)$, for all $\lambda\in[0,1]$ and for every bounded self-adjoint operators A and B in $B(H)$ whose spectra are contained in I.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [6] and the references therein.

In [7], Ghazanfari et al. gave the concept of operator preinvex function and obtained Hermite-Hadamard type inequality for operator preinvex function.

Definition 1.5 ([7])**.** Let X be a real vector space, a set $S\subseteq X$ is said to be invex with respect to the map $\eta:S\times S\to X$, if for every $x,y\in S$ and $t\in[0,1]$,

$$x+t\eta(x,y)\in S. \tag{1.8}$$

It is obvious that every convex set is invex with respect to the map $\eta(x,y)=x-y$, but there exist invex sets which are not convex (see [1]).

Let $S\subseteq X$ be an invex set with respect to $\eta:S\times S\to X$. For every $x,y\in S$, the η-path P_{xv} joining the points x and $v:=x+\eta(y,x)$ is defined as follows

$$P_{xv}:=\{z:z=x+t\eta(y,x),t\in[0,1]\}.$$

The mapping η is said to satisfy the condition (C) if for every $x,y\in S$ and $t\in[0,1]$,

$$\eta(y,y+t\eta(x,y))=-t\eta(x,y),\quad \eta(x,y+t\eta(x,y))=(1-t)\eta(x,y). \tag{C}$$

Note that for every $x, y \in S$ and every $t_1, t_2 \in [0,1]$ from condition (C) we have

$$\eta(y + t_2\eta(x,y), y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y), \tag{1.9}$$

see [13], [16] for details.

Let A be a C^*-algebra, denote by A_{sa} the set of all self-adjoint elements in A.

Definition 1.6 ([7])**.** Let $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)_{sa}$. Then, the continuous function $f : \mathbb{R} \to \mathbb{R}$ is said to be operator preinvex with respect to η on S, if for every $A, B \in S$ and $t \in [0,1]$,

$$f(A + t\eta(B,A)) \leq (1-t)f(A) + tf(B) \tag{1.10}$$

in the operator order in $B(H)$.

Every operator convex function is operator preinvex with respect to the map $\eta(A,B) = A - B$, but the converse does not hold (see [7]).

Theorem 1.7 ([7])**.** *Let $S \subseteq B(H)_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)_{sa}$ and η satisfies condition (C). If for every $A, B \in S$ and $V = A + \eta(B,A)$ the function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is operator preinvex with respect to η on η-path P_{AV} with spectra of A and spectra of V in the interval I. Then we have the inequality*

$$f\left(\frac{A+V}{2}\right) \leq \int_0^1 f((A + t\eta(B,A))\mathrm{d}t \leq \frac{f(A)+f(B)}{2}. \tag{1.11}$$

In [8], Ghazanfari defined the operator s-convex function and proved Hermite-Hadamard type inequality for operator s-convex function as follows.

We denote by $B(H)^+$ the set of all positive operators in $B(H)$ and

$$C(H) := \{A \in B(H)^+ : AB + BA \geq 0 \quad \text{for all} \quad B \in B(H)^+\}. \tag{1.12}$$

It is obvious that $C(H)$ is a closed convex cone in $B(H)$.

Definition 1.8 ([8])**.** Let I be an interval in $\mathbb{R}_0$ and S be a convex subset of $B(H)^+$. A continuous function $f : I \to \mathbb{R}$ is said to be operator s-convex on I for operators in S if

$$f(\lambda A + (1-\lambda)B) \leq \lambda^s f(A) + (1-\lambda)^s f(B) \tag{1.13}$$

in the operator order in $B(H)$, for all $\lambda \in [0,1]$ and for every positive operators A and B in S whose spectra are contained in I and for some fixed $s \in (0,1]$.

Theorem 1.9 ([8])**.** *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be an operator s-convex function on the interval I for operators in $S \subseteq B(H)^+$ and for some fixed $s \in (0,1]$. Then for all positive operators A and B in S with spectra in I we have the inequality*

$$2^{s-1} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f((1-t)A + tB)\mathrm{d}t \leq \frac{f(A)+f(B)}{s+1}. \tag{1.14}$$

Motivated by the above results we investigate in this paper the operator version of the Hermite-Hadamard inequality for operator s-preinvex functions.

2. Main results

In order to verify our main results, the following preliminary definition and lemmas are necessary.

Definition 2.1. Let I be an interval in $\mathbb{R}_0$ and $S \subseteq B(H)^+_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)^+_{sa}$. Then, the continuous function $f : I \to \mathbb{R}$ is said to be operator s-preinvex with respect to η on I for operators in S, if

$$f(A + t\eta(B, A)) \leq (1 - t)^s f(A) + t^s f(B) \tag{2.1}$$

in the operator order in $B(H)$, for all $t \in [0, 1]$ and every positive operators A and B in S whose spectra are contained in I and for some fixed $s \in (0, 1]$.

It is obvious that every operator 1-preinvex function is operator preinvex, and every operator s-convex function is operator s-preinvex with respect to the map $\eta(A, B) = A - B$.

Lemma 2.2 ([14])**.** *Let $A, B \in B(H)^+$. Then $AB + BA$ is positive if and only if $f(A + B) \leq f(A) + f(B)$ for all non-negative operator monotone functions f on $\mathbb{R}_0$.*

Now, we give an example of operator s-preinvex function.

Example 2.3. Suppose that 1_H is the identity operator on a Hilbert space H, and

$$S := (1_H, 5 \cdot 1_H) = \{A \in B(H)^+_{sa} : 1_H < A < 5 \cdot 1_H\}.$$

The map $\eta : S \times S \to B(H)^+_{sa}$ is defined by $\eta(A, B) = A - B$ for all $A > B \geq 0$ in the operator order in $B(H)$. Clearly η satisfies condition (C) and S is an invex set with respect to η. From Lemma 2.2 and (1.12), the continuous function $f(t) = t^s (0 < s \leq 1)$ is operator s-preinvex with respect to η on S for operators in $C(H)$.

The following lemma is a generalization of Proposition 1 in [7].

Lemma 2.4. *Let $S \subseteq B(H)^+_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)^+_{sa}$ and $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a continuous function on the interval I. Suppose that η satisfies condition (C) on S. Then for every $A, B \in S$ and $V = A + \eta(B, A)$ and for some fixed $s \in (0, 1]$, the function f is operator s-preinvex with respect to η on η-path P_{AV} with spectra of A and with spectra of V in the interval I if and only if the function $\varphi_{x,A,B} : [0, 1] \to \mathbb{R}$ defined by*

$$\varphi_{x,A,B}(t) := \langle f(A + t\eta(B, A))x, x\rangle \tag{2.2}$$

is s-convex on $[0, 1]$ for every $x \in H$ with $\|x\| = 1$.

Proof. Suppose that $x \in H$ with $\|x\| = 1$ and $\varphi_{x,A,B} : [0, 1] \to \mathbb{R}$ is s-convex on $[0, 1]$ for some fixed $s \in (0, 1]$. For every $C_1 := A + t_1\eta(B, A) \in P_{AV}$, $C_2 := A + t_2\eta(B, A) \in P_{AV}$, fix $\lambda \in [0, 1]$, by (2.2) we have

$$\begin{aligned}\langle f(C_1 + \lambda\eta(C_2, C_1))x, x\rangle &= \langle f(A + ((1 - \lambda)t_1 + \lambda t_2)\eta(B, A))x, x\rangle \\ &= \varphi_{x,A,B}((1 - \lambda)t_1 + \lambda t_2) \\ &\leq (1 - \lambda)^s \varphi_{x,A,B}(t_1) + \lambda^s \varphi_{x,A,B}(t_2) \\ &= (1 - \lambda)^s \langle f(C_1)x, x\rangle + \lambda^s \langle f(C_2)x, x\rangle.\end{aligned} \tag{2.3}$$

Hence, f is operator s-preinvex with respect to η on η-path P_{AV}.

Conversely, let $A, B \in S$ and f be operator s-preinvex with respect to η on η-path P_{AV} for some fixed $s \in (0, 1]$. Suppose that $t_1, t_2 \in [0, 1]$. Then for every $\lambda \in [0, 1]$ and $x \in H$ with $\|x\| = 1$, we have

$$\begin{aligned}\varphi_{x,A,B}((1 - \lambda)t_1 + \lambda t_2) &= \langle f(A + t_1\eta(B, A) + \lambda\eta(A + t_2\eta(B, A), A + t_1\eta(B, A)))x, x\rangle \\ &\leq (1 - \lambda)^s \langle f(A + t_1\eta(B, A))x, x\rangle + \lambda^s \langle f(A + t_2\eta(B, A))x, x\rangle \\ &= (1 - \lambda)^s \varphi_{x,A,B}(t_1) + \lambda^s \varphi_{x,A,B}(t_2).\end{aligned} \tag{2.4}$$

Therefore, $\varphi_{x,A,B}$ is s-convex on $[0, 1]$. The proof of Lemma 2.4 is complete. □

The following theorem is the generalization of Hermite-Hadamard's inequality for operator s-preinvex functions.

Theorem 2.5. *Let $S \subseteq B(H)^+_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)^+_{sa}$ and η satisfy condition (C) on S. If for every $A, B \in S$ and $V = A + \eta(B, A)$ and for some fixed $s \in (0, 1]$, the continuous function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is operator s-preinvex with respect to η on η-path P_{AV} with spectra of A and with spectra of V in the interval I. Then we have the inequality*

$$2^{s-1} f\left(\frac{A+V}{2}\right) \leq \int_0^1 f(A + t\eta(B, A))\mathrm{d}t \leq \frac{f(A) + f(B)}{s+1}. \tag{2.5}$$

Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0, 1]$, we have

$$\langle (A + t\eta(B, A))x, x\rangle = \langle Ax, x\rangle + t\langle \eta(B, A)x, x\rangle \in I, \tag{2.6}$$

since $\langle Ax, x\rangle \in Sp(A) \subseteq I$ and $\langle Vx, x\rangle \in Sp(V) \subseteq I$.

Continuity of f and (2.6) imply that the operator valued integral $\int_0^1 f(A + t\eta(B, A))\mathrm{d}t$ exists.

Since η satisfies condition (C) and f is s-preinvex with respect to η, for every $t \in [0, 1]$, we have

$$\begin{aligned} f\left(A + \frac{1}{2}\eta(B, A)\right) &\leq \frac{1}{2^s} f(A + t\eta(B, A)) + \frac{1}{2^s} f(A + (1-t)\eta(B, A)) \\ &\leq \frac{1}{2^s}[(1-t)^s + t^s][f(A) + f(B)]. \end{aligned} \tag{2.7}$$

Integrating the inequality (2.7) over $t \in [0, 1]$ and taking into account that

$$\int_0^1 f(A + t\eta(B, A))\mathrm{d}t = \int_0^1 f(A + (1-t)\eta(B, A))\mathrm{d}t, \tag{2.8}$$

we obtain the inequality (2.5), which completes the proof of Theorem 2.5. □

Remark 2.6. Choosing $s = 1$ and $\eta(B, A) = B - A$ respectively, we obtain Theorem 1.7 and Theorem 1.9.

Now we establish the estimates of both sides of Hermite-Hadamard type inequality in which some operator s-preinvex functions of selfadjoint operators in Hilbert spaces are involved.

Theorem 2.7. *Let the function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}_0$ be continuous, $S \subseteq B(H)^+_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)^+_{sa}$, and η satisfy condition (C) on S. If for every $A, B \in S$ and $V = A + \eta(B, A)$ and for some fixed $s \in (0, 1]$, the function f is operator s-preinvex with respect to η on η-path P_{AV} with spectra of A and with spectra of V in the interval I. Then for every $a, b \in (0, 1)$ with $a < b$ and every $x \in H$ with $\|x\| = 1$, the following inequality holds,*

$$\begin{aligned} &\left|\left\langle \int_0^{(a+b)/2} f(A + u\eta(B, A))\mathrm{d}u \quad x, x\right\rangle - \frac{1}{b-a}\int_a^b \left\langle \int_0^t f(A + u\eta(B, A))\mathrm{d}u \quad x, x\right\rangle \mathrm{d}t\right| \\ &\qquad \leq \frac{b-a}{4(s+1)(s+2)}\Big[\langle f(A + a\eta(B, A))x, x\rangle \\ &\qquad\quad + 2(s+1)\left\langle f\left(A + \frac{a+b}{2}\eta(B, A)\right)x, x\right\rangle + \langle f(A + b\eta(B, A))x, x\rangle\Big]. \end{aligned} \tag{2.9}$$

Moreover, we have

$$\begin{aligned} &\left\| \int_0^{(a+b)/2} f(A + u\eta(B, A))\mathrm{d}u - \frac{1}{b-a}\int_a^b \int_0^t f(A + u\eta(B, A))\mathrm{d}u\mathrm{d}t \right\| \\ &\qquad \leq \frac{b-a}{2(s+1)}\left[\frac{\|f(A + a\eta(B, A))\| + 2(s+1)\left\|f\left(A + \frac{a+b}{2}\eta(B, A)\right)\right\| + \|f(A + b\eta(B, A))\|}{2(s+2)}\right]. \end{aligned} \tag{2.10}$$

Proof. Let $A, B \in S$ and $a, b \in (0,1)$ with $a < b$. For $x \in H$ with $\|x\| = 1$, we define the function $\varphi : [a,b] \subseteq [0,1] \to \mathbb{R}_0$ by

$$\varphi(t) := \left\langle \int_0^t f(A + u\eta(B,A))\mathrm{d}u \quad x, x \right\rangle.$$

Utilizing the continuity of f, the continuity property of the inner product, and the properties of the integral of operator-valued functions, we have

$$\left\langle \int_0^t f(A + u\eta(B,A))\mathrm{d}u \quad x, x \right\rangle = \int_0^t \langle f(A + u\eta(B,A))x, x\rangle \mathrm{d}u.$$

Since $f(A + u\eta(B,A)) \geq 0$, $\varphi(t) \geq 0$ for all $t \in [a,b]$. Obviously for every $t \in [a,b]$, we have

$$\varphi'(t) = \langle f(A + t\eta(B,A))x, x\rangle \geq 0,$$

hence, $|\varphi'(t)| = \varphi'(t)$.

Since f is operator s-preinvex with respect to η on η-path P_{AV} for some fixed $s \in (0,1]$, by Lemma 2.4 φ' is s-convex. Applying Theorem 1.3 to the function φ implies that

$$\left|\varphi\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b \varphi(t)\mathrm{d}t\right| \leq \frac{b-a}{4(s+1)(s+2)}\left[\varphi'(a) + 2(s+1)\varphi'\left(\frac{a+b}{2}\right) + \varphi'(b)\right], \tag{2.11}$$

and we know that the inequality (2.9) holds. Taking supremum over both sides of inequality (2.9) for all x with $\|x\| = 1$, we deduce that the inequality (2.10) holds. Theorem 2.7 is thus proved. □

Corollary 2.8. *Under the assumptions of Theorem 2.7, it turns out that*

$$\begin{aligned}\left|\left\langle \int_0^{(a+b)/2} f(A + u\eta(B,A))\mathrm{d}u \quad x, x\right\rangle - \frac{1}{b-a}\int_a^b \left\langle \int_0^t f(A + u\eta(B,A))du \quad x, x\right\rangle \mathrm{d}t\right| \\ \leq \frac{(2^{2-s}+1)(b-a)}{2(s+1)(s+2)}\left[\frac{\langle f(A + a\eta(B,A))x, x\rangle + \langle f(A + b\eta(B,A))x, x\rangle}{2}\right].\end{aligned} \tag{2.12}$$

Furthermore, we have

$$\begin{aligned}\left\|\int_0^{(a+b)/2} f(A + u\eta(B,A))\mathrm{d}u - \frac{1}{b-a}\int_a^b \int_0^t f(A + u\eta(B,A))\mathrm{d}u\mathrm{d}t\right\| \\ \leq \frac{(2^{2-s}+1)(b-a)}{2(s+1)(s+2)}\left[\frac{\|f(A + a\eta(B,A))\| + \|f(A + b\eta(B,A))\|}{2}\right].\end{aligned} \tag{2.13}$$

Proof. As the proof of Theorem 2.7, employing s-convexity of φ and (2.11) yield the results of Corollary 2.8. □

Corollary 2.9. *With the conditions of Theorem 2.7, if $s = 1$, then*

$$\begin{aligned}\left|\left\langle \int_0^{(a+b)/2} f(A + u\eta(B,A))\mathrm{d}u \quad x, x\right\rangle - \frac{1}{b-a}\int_a^b \left\langle \int_0^t f(A + u\eta(B,A))\mathrm{d}u \quad x, x\right\rangle \mathrm{d}t\right| \\ \leq \frac{b-a}{4}\left[\frac{\langle f(A + a\eta(B,A))x, x\rangle + 4\big\langle f\big(A + \frac{a+b}{2}\eta(B,A)\big)x, x\big\rangle + \langle f(A + b\eta(B,A))x, x\rangle}{6}\right].\end{aligned} \tag{2.14}$$

In addition, we have

$$\begin{aligned}\left\|\int_0^{(a+b)/2} f(A + u\eta(B,A))\mathrm{d}u - \frac{1}{b-a}\int_a^b \int_0^t f(A + u\eta(B,A))\mathrm{d}u\mathrm{d}t\right\| \\ \leq \frac{b-a}{4}\left[\frac{\|f(A + a\eta(B,A))\| + 4\big\|f\big(A + \frac{a+b}{2}\eta(B,A)\big)\big\| + \|f(A + b\eta(B,A))\|}{6}\right].\end{aligned} \tag{2.15}$$

Corollary 2.10. *Under the assumptions of Theorem 2.7, if $\eta(B,A)=B-A$, then*

$$\left|\left\langle \int_0^{(a+b)/2} f((1-u)A+uB)\mathrm{d}u \quad x,x\right\rangle - \frac{1}{b-a}\int_a^b\left\langle \int_0^t f((1-u)A+uB)\mathrm{d}u \quad x,x\right\rangle \mathrm{d}t\right|$$
$$\leq \frac{b-a}{4(s+1)(s+2)}\Big[\langle f((1-a)A+aB)x,x\rangle$$
$$+2(s+1)\left\langle f\left(\frac{2-a-b}{2}A+\frac{a+b}{2}B\right)x,x\right\rangle+\langle f((1-b)A+bB)x,x\rangle\Big]. \tag{2.16}$$

Moreover, we have

$$\left\|\int_0^{(a+b)/2} f((1-u)A+uB)\mathrm{d}u-\frac{1}{b-a}\int_a^b\int_0^t f((1-u)A+uB)\mathrm{d}u\mathrm{d}t\right\|$$
$$\leq \frac{b-a}{2(s+1)}\left[\frac{\|f((1-a)A+aB)\|+2(s+1)\left\|f\left(\frac{2-a-b}{2}A+\frac{a+b}{2}B\right)\right\|+\|f((1-b)A+bB)\|}{2(s+2)}\right]. \tag{2.17}$$

Remark 2.11. Corollaries 2.8, 2.9 and 2.10 are generalizations of Theorem 5 in [2] and Theorem 2.2 in [15], respectively.

Theorem 2.12. *Let the function $f: I\subseteq \mathbb{R}_0\to\mathbb{R}_0$ be continuous, $S\subseteq B(H)^+_{sa}$ be an invex set with respect to $\eta: S\times S\to B(H)^+_{sa}$, and η satisfy condition (C) on S. If for every $A,B\in S$ and $V=A+\eta(B,A)$ and for some fixed $s\in(0,1]$, the function f is operator s-preinvex with respect to η on η-path P_{AV} with spectra of A and with spectra of V in the interval I. Then for every $a,b\in(0,1)$ with $a<b$ and every $x\in H$ with $\|x\|=1$, the following inequality holds,*

$$\left|\frac{1}{2}\left\langle\int_0^a f(A+u\eta(B,A))\mathrm{d}u \quad x,x\right\rangle+\frac{1}{2}\left\langle\int_0^b f(A+u\eta(B,A))\mathrm{d}u \quad x,x\right\rangle\right.$$
$$\left.-\frac{1}{b-a}\int_a^b\left\langle\int_0^t f(A+u\eta(B,A))\mathrm{d}u \quad x,x\right\rangle\mathrm{d}t\right|$$
$$\leq\frac{(b-a)(2^{s+1}+1)}{2^s(s+1)(s+2)}\left[\frac{\langle f(A+a\eta(B,A))x,x\rangle+\langle f(A+b\eta(B,A))x,x\rangle}{2}\right]. \tag{2.18}$$

Furthermore, we have

$$\left\|\frac{1}{2}\int_0^a f(A+u\eta(B,A))\mathrm{d}u+\frac{1}{2}\int_0^b f(A+u\eta(B,A))\mathrm{d}u-\frac{1}{b-a}\int_a^b\int_0^t f(A+u\eta(B,A))\mathrm{d}u\mathrm{d}t\right\|$$
$$\leq\frac{(b-a)(2^{s+1}+1)}{2^s(s+1)(s+2)}\left[\frac{\|f(A+a\eta(B,A))\|+\|f(A+b\eta(B,A))\|}{2}\right]. \tag{2.19}$$

Proof. With the inequality (1.5) and the similar approach of the proof of Theorem 2.7, it is a simple verification. We omit the routine details. □

Corollary 2.13. *With the conditions of Theorem 2.12, if $s=1$, then*

$$\left|\frac{1}{2}\left\langle\int_0^a f(A+u\eta(B,A))\mathrm{d}u \quad x,x\right\rangle+\frac{1}{2}\left\langle\int_0^b f(A+u\eta(B,A))\mathrm{d}u \quad x,x\right\rangle\right.$$
$$\left.-\frac{1}{b-a}\int_a^b\left\langle\int_0^t f(A+u\eta(B,A))\mathrm{d}u \quad x,x\right\rangle\mathrm{d}t\right|$$
$$\leq\frac{5(b-a)}{12}\left[\frac{\langle f(A+a\eta(B,A))x,x\rangle+\langle f(A+b\eta(B,A))x,x\rangle}{2}\right]. \tag{2.20}$$

Moreover, we have

$$\left\|\frac{1}{2}\int_0^a f(A+u\eta(B,A))\mathrm{d}u+\frac{1}{2}\int_0^b f(A+u\eta(B,A))\mathrm{d}u-\frac{1}{b-a}\int_a^b\int_0^t f(A+u\eta(B,A))\mathrm{d}u\mathrm{d}t\right\| \\ \leq \frac{5(b-a)}{12}\left[\frac{\|f(A+a\eta(B,A))\|+\|f(A+b\eta(B,A))\|}{2}\right]. \tag{2.21}$$

Corollary 2.14. *Under the assumptions of Theorem 2.12, if $\eta(B,A)=B-A$, then*

$$\Bigg|\frac{1}{2}\left\langle\int_0^a f((1-u)A+uB)\mathrm{d}u\quad x,x\right\rangle+\frac{1}{2}\left\langle\int_0^b f((1-u)A+uB)\mathrm{d}u\quad x,x\right\rangle \\ -\frac{1}{b-a}\int_a^b\left\langle\int_0^t f((1-u)A+uB)\mathrm{d}u\quad x,x\right\rangle\mathrm{d}t\Bigg| \\ \leq \frac{(b-a)(2^{s+1}+1)}{2^s(s+1)(s+2)}\left[\frac{\langle f((1-a)A+aB)x,x\rangle+\langle f((1-b)A+bB)x,x\rangle}{2}\right]. \tag{2.22}$$

In addition, we have

$$\left\|\frac{1}{2}\int_0^a f((1-u)A+uB)\mathrm{d}u+\frac{1}{2}\int_0^b f((1-u)A+uB)\mathrm{d}u-\frac{1}{b-a}\int_a^b\int_0^t f((1-u)A+uB)\mathrm{d}u\mathrm{d}t\right\| \\ \leq \frac{(b-a)(2^{s+1}+1)}{2^s(s+1)(s+2)}\left[\frac{\|f((1-a)A+aB)\|+\|f((1-b)A+bB)\|}{2}\right]. \tag{2.23}$$

Remark 2.15. Corollaries 2.13 and 2.14 are generalizations of Theorem 1.4 and Theorem 4 in [12], respectively.

In what follows, Hermite-Hadamard type inequalities for the product of two operator s-preinvex functions are established.

For some fixed $s_1,s_2\in(0,1]$, let $f:I\subseteq\mathbb{R}_0\to\mathbb{R}$ be an operator s_1-preinvex function and $g:I\to\mathbb{R}$ be an operator s_2-preinvex function on the interval I. Then for all positive operators A and B on a Hilbert space H with spectra in I, we define real functions $M(A,B)$ and $N(A,B)$ on H by

$$\begin{aligned} M(A,B)(x)&=\langle f(A)x,x\rangle\langle g(A)x,x\rangle+\langle f(B)x,x\rangle\langle g(B)x,x\rangle, \quad x\in H,\\ N(A,B)(x)&=\langle f(A)x,x\rangle\langle g(B)x,x\rangle+\langle f(B)x,x\rangle\langle g(A)x,x\rangle, \quad x\in H. \end{aligned} \tag{2.24}$$

We note that, the Beta function is defined as follows:

$$\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad x>0,y>0. \tag{2.25}$$

The following two theorems are the generalization of Theorem 3.1 and Theorem 3.2 in [8] respectively for operator s-preinvex functions.

Theorem 2.16. *Let $S\subseteq B(H)^+_{sa}$ be an invex set with respect to $\eta:S\times S\to B(H)^+_{sa}$ and η satisfy condition (C) on S. If for every $A,B\in S$ and $V=A+\eta(B,A)$ and for some fixed $s_1,s_2\in(0,1]$, the continuous function $f:I\subseteq\mathbb{R}_0\to\mathbb{R}$ is an operator s_1-preinvex function and $g:I\to\mathbb{R}$ is an operator s_2-preinvex function on the interval I with respect to η on η-path P_{AV} with spectra of A and with spectra of V in the interval I. Then we have the inequality*

$$\int_0^1\langle f(A+t\eta(B,A))x,x\rangle\langle g(A+t\eta(B,A))x,x\rangle\mathrm{d}t \\ \leq \frac{1}{s_1+s_2+1}\big[M(A,B)(x)+s_1\beta(s_1,s_2+1)N(A,B)(x)\big] \tag{2.26}$$

holds for any $x\in H$ with $\|x\|=1$, where $M(A,B)$ and $N(A,B)$ are defined in (2.24), and the Beta function is defined in (2.25).

Proof. For $x \in H$ with $\|x\| = 1$ and $t \in [0,1]$, we have

$$\langle (A + t\eta(B,A))x, x\rangle = \langle Ax, x\rangle + t\langle \eta(B,A)x, x\rangle \in I,$$

since $\langle Ax, x\rangle \in Sp(A) \subseteq I$ and $\langle Vx, x\rangle \in Sp(V) \subseteq I$.

From the continuity of f, g, it shows that the operator valued integral $\int_0^1 f(A + t\eta(B,A))\mathrm{d}t$, $\int_0^1 g(A + t\eta(B,A))\mathrm{d}t$, and $\int_0^1 (fg)(A + t\eta(B,A))\mathrm{d}t$ exist.

Since $f : I \to \mathbb{R}$ is operator s_1-preinvex and $g : I \to \mathbb{R}$ is operator s_2-preinvex for some fixed $s_1, s_2 \in (0,1]$, therefore for every $t \in [0,1]$ we derive

$$\begin{aligned}\langle f(A + t\eta(B,A))x, x\rangle &\langle g(A + t\eta(B,A))x, x\rangle \\ &\le (1-t)^{s_1+s_2}\langle f(A)x, x\rangle\langle g(A)x, x\rangle + (1-t)^{s_1}t^{s_2}\langle f(A)x, x\rangle\langle g(B))x, x\rangle \\ &\quad + t^{s_1}(1-t)^{s_2}\langle f(B)x, x\rangle\langle g(A)x, x\rangle + t^{s_1+s_2}\langle f(B)x, x\rangle\langle g(B))x, x\rangle. \end{aligned} \tag{2.27}$$

Integrating both sides of (2.27) over $t \in [0,1]$, we get the required inequality (2.26). The proof of Theorem 2.16 is complete. □

Corollary 2.17. *Under the assumptions of Theorem 2.16, if $s_1 = s_2 = s$, then*

$$\begin{aligned}\int_0^1 \langle f(A + t\eta(B,A))x, x\rangle&\langle g(A + t\eta(B,A))x, x\rangle \mathrm{d}t \\ &\le \frac{1}{2s+1}\big[M(A,B)(x) + s\beta(s, s+1)N(A,B)(x)\big]. \end{aligned} \tag{2.28}$$

Specially, if $s_1 = s_2 = 1$, then

$$\int_0^1 \langle f(A + t\eta(B,A))x, x\rangle\langle g(A + t\eta(B,A))x, x\rangle \mathrm{d}t \le \frac{2M(A,B)(x) + N(A,B)(x)}{6}. \tag{2.29}$$

Corollary 2.18. *With the conditions of Theorem 2.16, if $\eta(B,A) = B - A$, then*

$$\begin{aligned}\int_0^1 \langle f((1-t)A + tB)x,x\rangle&\langle g((1-t)A + tB)x, x\rangle \mathrm{d}t \\ &\le \frac{1}{s_1+s_2+1}\big[M(A,B)(x) + s_1\beta(s_1, s_2+1)N(A,B)(x)\big]. \end{aligned} \tag{2.30}$$

Theorem 2.19. *Let $S \subseteq B(H)^+_{sa}$ be an invex set with respect to $\eta : S \times S \to B(H)^+_{sa}$ and η satisfy condition (C) on S. If for every $A, B \in S$ and $V = A + \eta(B,A)$ and for some fixed $s_1, s_2 \in (0,1]$, the continuous function $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ is an operator s_1-preinvex function and $g : I \to \mathbb{R}$ is an operator s_2-preinvex function on the interval I with respect to η on η-path P_{AV} with spectra of A and with spectra of V in the interval I. Then we have that the inequality*

$$2^{s_1+s_2-1}\left\langle f\left(\frac{A+V}{2}\right)x, x\right\rangle\left\langle g\left(\frac{A+V}{2}\right)x, x\right\rangle \tag{2.31}$$

$$\begin{aligned}&\le \int_0^1 \langle f(A + t\eta(B,A))x, x\rangle\langle g(A + t\eta(B,A))x, x\rangle \mathrm{d}t \\ &\quad + \frac{1}{s_1+s_2+1}\big[N(A,B)(x) + s_1\beta(s_1, s_2+1)M(A,B)(x)\big]\end{aligned} \tag{2.32}$$

holds for any $x \in H$ with $\|x\| = 1$, where $M(A,B)$ and $N(A,B)$ are defined on H in (2.24) and the Beta function is defined in (2.25).

Proof. Since $f : I \to \mathbb{R}$ is operator s_1-preinvex and $g : I \to \mathbb{R}$ be operator s_2-preinvex for some fixed $s_1, s_2 \in (0,1]$, therefore for every $t \in [0,1]$ we have

$$\begin{aligned}
&\left\langle f\left(\frac{A+V}{2}\right)x,x\right\rangle\left\langle g\left(\frac{A+V}{2}\right)x,x\right\rangle\\
&\qquad\le \frac{1}{2^{s_1}}\langle[f(A+t\eta(B,A))+f(A+(1-t)\eta(B,A))]x,x\rangle\\
&\qquad\quad\times \frac{1}{2^{s_2}}\langle[g(A+t\eta(B,A))+g(A+(1-t)\eta(B,A))]x,x\rangle\\
&\qquad\le \frac{1}{2^{s_1+s_2}}\big[\langle f(A+t\eta(B,A))x,x\rangle\langle g(A+t\eta(B,A))x,x\rangle\\
&\qquad\quad+\langle f(A+(1-t)\eta(B,A))x,x\rangle\langle g(A+(1-t)\eta(B,A))x,x\rangle\big]\\
&\qquad\quad+\frac{1}{2^{s_1+s_2}}\big\{[t^{s_1+s_2}+(1-t)^{s_1+s_2}][\langle f(A)x,x\rangle\langle g(B)x,x\rangle+\langle f(B)x,x\rangle\langle g(A)x,x\rangle]\\
&\qquad\quad+[t^{s_1}(1-t)^{s_2}+t^{s_2}(1-t)^{s_1}][\langle f(A)x,x\rangle\langle g(A)x,x\rangle+\langle f(B)x,x\rangle\langle g(B)x,x\rangle]\big\}.
\end{aligned}\tag{2.33}$$

By integrating over $t \in [0,1]$ and taking into account that

$$\begin{aligned}
&\int_0^1\langle f(A+t\eta(B,A))x,x\rangle\langle g(A+t\eta(B,A))x,x\rangle \mathrm{d}t\\
&\qquad\qquad=\int_0^1\langle f(A+(1-t)\eta(B,A))x,x\rangle\langle g(A+(1-t)\eta(B,A))x,x\rangle \mathrm{d}t,
\end{aligned}$$

we obtain the required inequality (2.31). Theorem 2.19 is thus proved. □

Corollary 2.20. *Under the assumptions of Theorem 2.19, if $s_1 = s_2 = s$, then*

$$\begin{aligned}
&2^{2s-1}\left\langle f\left(\frac{A+V}{2}\right)x,x\right\rangle\left\langle g\left(\frac{A+V}{2}\right)x,x\right\rangle\\
&\qquad\le\int_0^1\langle f(A+t\eta(B,A))x,x\rangle\langle g(A+t\eta(B,A))x,x\rangle \mathrm{d}t\\
&\qquad\quad+\frac{1}{2s+1}\big[N(A,B)(x)+s\beta(s,s+1)M(A,B)(x)\big].
\end{aligned}\tag{2.34}$$

In particular, if $s_1 = s_2 = 1$, then

$$\begin{aligned}
&2\left\langle f\left(\frac{A+V}{2}\right)x,x\right\rangle\left\langle g\left(\frac{A+V}{2}\right)x,x\right\rangle\\
&\qquad\le\int_0^1\langle f(A+t\eta(B,A))x,x\rangle\langle g(A+t\eta(B,A))x,x\rangle \mathrm{d}t\\
&\qquad\quad+\frac{2N(A,B)(x)+M(A,B)(x)}{6}.
\end{aligned}\tag{2.35}$$

Corollary 2.21. *With the conditions of Theorem 2.19, if $\eta(B,A) = B - A$, then*

$$\begin{aligned}
&2^{s_1+s_2-1}\left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle\left\langle g\left(\frac{A+B}{2}\right)x,x\right\rangle\\
&\qquad\le\int_0^1\langle f((1-t)A+tB)x,x\rangle\langle g((1-t)A+tB)x,x\rangle \mathrm{d}t\\
&\qquad\quad+\frac{1}{s_1+s_2+1}\big[N(A,B)(x)+s_1\beta(s_1,s_2+1)M(A,B)(x)\big].
\end{aligned}\tag{2.36}$$

Corollary 2.22. *With the assumptions of Theorem 2.16 and Theorem 2.19, we get*

$$\begin{aligned}2^{s_1+s_2-1}\left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle\left\langle g\left(\frac{A+B}{2}\right)x,x\right\rangle - \frac{1}{s_1+s_2+1}\big[N(A,B)(x)+s_1\beta(s_1,s_2+1)M(A,B)(x)\big]\\ \le \int_0^1 \langle f((1-t)A+tB)x,x\rangle\langle g((1-t)A+tB)x,x\rangle \mathrm{d}t\\ \le \frac{1}{s_1+s_2+1}\big[M(A,B)(x)+s_1\beta(s_1,s_2+1)N(A,B)(x)\big]. \qquad (2.37)\end{aligned}$$

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