Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 8 (2015), 1134–1140
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

# Best proximity points for multiplicative proximal contraction mapping on multiplicative metric spaces

Chirasak Mongkolkeha[a], Wutiphol Sintunavarat[b,*]

[a] *Department of Mathematics, Statistics and Computers, Faculty of Liberal Arts and Science, Kasetsart University, Kamphaeng-Saen Campus, Nakhonpathom 73140, Thailand.*

[b] *Department of Mathematics and Statistics, Faculty of Science and Technology, Thammasat University Rangsit Center, Pathumthani 12121, Thailand.*

Communicated by Yeol Je Cho

**Abstract**

In this paper, we introduce the concept of multiplicative proximal contraction mapping which is a Banach's contraction for non-self mapping in the framework of multiplicative metric spaces and we also prove best proximity point theorems for such mappings. Some illustrative example is furnished which demonstrate the validity of the hypotheses and degree of utility of our results. 

*Keywords:* Best proximity points, fixed points, multiplicative metric spaces, multiplicative proximal contraction mappings.
*2010 MSC:* 47H10, 47H10.

## 1. Introduction

In 2008, Bashirov et al. [1] studied the usefulness of a new calculus, called multiplicative calculus due to Michael Grossman and Robert Katz in the the period from 1967 till 1970. By using the concepts of multiplicative absolute values, Bashirov et al. [1] defined a new distance so called multiplicative distance. Afterward, Özavşar and Çevikel [2] introduced the concept of multiplicative metric spaces by using the idea of multiplicative distance, and gave some topological properties in such space. They also introduced the concept of multiplicative Banach's contraction mapping and proved fixed point results for such mapping in multiplicative metric spaces.

*Corresponding author
*Email addresses:* `faascsm@ku.ac.th` (Chirasak Mongkolkeha), `wutiphol@mathstat.sci.tu.ac.th`; `poom_teun@hotmail.com` (Wutiphol Sintunavarat)

*Received 2015-02-03*

The aim of this paper is to introduce the new classes of proximal contractions which are more general than class of multiplicative Banach's contraction for non-self mapping. We also give the necessary condition to have best proximity points and give some illustrative example of our main results. Our main results generalize, extend and improve the corresponding results on the topics given in the literature.

## 2. Preliminaries

In this section, we give some definitions and basic concept of multiplicative metric space for our consideration. Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{R}$ the sets of positive integers, positive real numbers and real numbers, respectively.

**Definition 2.1** ([1]). Let $X$ be a nonempty set. A mapping $d : X \times X \to \mathbb{R}$ is said to be a multiplicative metric if it satisfying the following conditions:

(m1) $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and only if $x = y$,
(m2) $d(x, y) = d(y, x)$ for all $x, y \in X$
(m3) $d(x, z) \leq d(x, y) \cdot d(y, z)$ for all $x, y, z \in X$ (multiplicative triangle inequality).

Also, the ordered pair $(X, d)$ is called multiplicative metric space.

**Example 2.2** ([2]). Let $d^* : (\mathbb{R}^+)^n \times (\mathbb{R}^+)^n \to \mathbb{R}$ be defined as follows

$$d^*(x, y) = \left|\frac{x_1}{y_1}\right|^* \cdot \left|\frac{x_2}{y_2}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*,$$

where $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in (\mathbb{R}^+)^n$ and $|\cdot|^* : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as follows

$$|a|^* = \begin{cases} a & \text{if } a \geq 1, \\ \frac{1}{a} & \text{if } a < 1. \end{cases}$$

Then $((\mathbb{R}^+)^n, d^*)$ is a multiplicative metric space.

The following notations and results given by Özavşar and Çevikel [2].

**Definition 2.3** ([2]). Let $(X, d)$ be a multiplicative metric space, $x \in X$ and $\varepsilon > 1$. Define the following set:

$$B_\varepsilon(x) := \{y \in X : d(x, y) < \varepsilon\},$$

which is called the multiplicative open ball of radius $\varepsilon$ with center $x$.

Similarly, one can describe the multiplicative closed ball as follows:

$$\overline{B}_\varepsilon(x) := \{y \in X : d(x, y) \leq \varepsilon\}.$$

**Definition 2.4** ([2]). Let $(X, d)$ be a multiplicative metric space, $\{x_n\}$ be a sequence in $X$ and $x \in X$. If, for any multiplicative open ball $B_\varepsilon(x)$, there exists a natural number $N$ such that, for all $n \geq N$, $x_n \in B_\varepsilon(x)$, then the sequence $\{x_n\}$ is said to be multiplicative convergent to the point $x$, which is denoted by $x_n \to_* x$ as $n \to \infty$.

**Lemma 2.5** ([2]). *Let $(X, d)$ be a multiplicative metric space, $\{x_n\}$ be a sequence in $X$ and $x \in X$. Then $x_n \to_* x$ as $n \to \infty$ if and only if $d(x_n, x) \to_* 1$ as $n \to \infty$.*

**Lemma 2.6** ([2]). *Let $(X, d)$ be a multiplicative metric space and $\{x_n\}$ be a sequence in $X$. If the sequence $\{x_n\}$ is multiplicative convergent, then the multiplicative limit point is unique.*

**Definition 2.7** ([2]). Let $(X, d)$ be a multiplicative metric space and $\{x_n\}$ be a sequence in $X$. The sequence $\{x_n\}$ is called a *multiplicative Cauchy sequence* if, for all $\varepsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq N$.

**Lemma 2.8** ([2])**.** *Let $(X, d)$ be a multiplicative metric space and $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_n, x_m) \to_* 1$ as $m, n \to \infty$.*

**Theorem 2.9** ([2])**.** *Let $(X, d)$ be a multiplicative metric space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $X$ such that $x_n \to_* x \in X$ and $y_n \to_* y \in X$ as $n \to \infty$. Then $d(x_n, y_n) \to_* (x, y)$ as $n \to \infty$.*

**Definition 2.10** ([2])**.** Let $(X, d)$ be a multiplicative metric space and $A \subseteq X$. Then we call $x \in A$, a multiplicative interior point of $A$ if there exists an $\varepsilon >1$ such that $B_\varepsilon(x) \subseteq A$. The collection of all interior points of $A$ is called multiplicative interior of $A$ and denoted by $int(A)$.

**Definition 2.11** ([2])**.** Let $(X, d)$ be a multiplicative metric space and $A \subseteq X$. If every point of $A$ is a multiplicative interior point of $A$, i.e., $A = int(A)$, then $A$ is called a multiplicative open set.

**Definition 2.12** ([2])**.** Let $(X, d)$ be a multiplicative metric space. A subset $S \subseteq X$ is called multiplicative closed in $(X, d)$ if $S$ contains all of its multiplicative limit points.

**Theorem 2.13** ([2])**.** *Let $(X, d)$ be a multiplicative metric space. A subset $S \subseteq X$ is multiplicative closed if and only if $X \setminus S$, the complement of $S$, is multiplicative open.*

**Theorem 2.14** ([2])**.** *Let $(X, d)$ be a multiplicative metric space and $S \subseteq X$. Then the set $S$ is multiplicative closed if and only if every multiplicative convergent sequence in $S$ has a multiplicative limit point that belongs to $S$.*

**Theorem 2.15** ([2])**.** *Let $(X, d)$ be a multiplicative metric space and $S \subseteq X$. Then $(S, d)$ is complete if and only if $S$ is multiplicative closed.*

**Theorem 2.16** ([2])**.** *Let $(X, d_X)$ and $(Y, d_Y)$ be two multiplicative metric spaces, $f : X \to Y$ be a mapping and $\{x_n\}$ be any sequence in $X$. Then $f$ is multiplicative continuous at the point $x \in X$ if and only if $f(x_n) \to_* f(x)$ as $n \to \infty$ for every sequence $\{x_n\}$ with $x_n \to_* x$ as $n \to \infty$.*

Next, we give the notations $A_0$, $B_0$ and $d(A, B)$ for nonempty subsets $A$ and $B$ of a multiplicative metric space $(X, d)$ in the same sense in metric spaces.

Let $A$ and $B$ be nonempty subsets of a multiplicative metric space $(X, d)$, we recall the following notations and notions that will be used in what follows.

$$\begin{aligned} d(A, B) &:= \inf\{d(x, y) : x \in A \text{ and } y \in B\}, \\ A_0 &:= \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}, \\ B_0 &:= \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}. \end{aligned}$$

**Definition 2.17.** Let $A$ be nonempty subset of a multiplicative metric space $(X, d)$. A mapping $g : A \to A$ is said to be isometry if $d(gx, gy) = d(x, y)$ for all $x, y \in A$.

**Definition 2.18.** Let $A$ and $B$ be nonempty subsets of a multiplicative metric space $(X, d)$. A point $x \in A$ is called a best proximity point of a mapping $T : A \to B$ if it satisfies the condition that $d(x, Tx) = d(A, B)$.

It can be observed that a best proximity reduces to a fixed point if the underlying mapping is a self-mapping.

**Definition 2.19.** A subset $A$ of a multiplicative metric space $(X, d)$ is said to be approximatively compact with respect to $B$ if every sequence $\{x_n\}$ in $A$ satisfies the condition that $d(y, x_n) \to_* d(y, A)$ as $n \to \infty$ for some $y \in B$ has a convergent subsequence.

We observe that each set is approximatively compact with respect to itself.

## 3. Main Result

In this section, we introduce the new class of proximal contractions in the framework of multiplicative metric spaces so called multiplicative proximal contraction mappings and prove best proximity theorems for mappings in such class on multiplicative metric spaces.

**Definition 3.1.** Let $A$ and $B$ be nonempty subsets of a multiplicative metric space $(X, d)$. A mapping $T : A \to B$ is called a multiplicative proximal contraction if there exists $\alpha \in [0, 1)$ satisfying the following condition:

$$\left.\begin{array}{l} \text{for } u, v, x, y \in A, \\ d(u, Tx) = d(A, B), \\ d(v, Ty) = d(A, B) \end{array}\right\} \implies d(u, v) \leq d(x, y)^{\alpha}.$$

It is easy to see that a self-mapping that is a multiplicative proximal contraction is precisely a Banach's contraction which is due to Özavşar and Çevikel [2]. However, a nonself-proximal contraction mapping is not necessarily a Banach's contraction mapping.

**Theorem 3.2.** *Let $(X, d)$ be a complete multiplicative metric space and $A$, $B$ be nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty and $B$ is approximatively compact with respect to $A$. Suppose that $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

*(a) $T$ is a multiplicative proximal contraction;*

*(b) $T(A_0) \subseteq B_0$;*

*(c) $g$ is an isometry;*

*(d) $A_0 \subseteq g(A_0)$.*

*Then there exists a unique point $x^* \in A$ such that*

$$d(gx^*, Tx^*) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by*

$$d(gx_n, Tx_{n-1}) = d(A, B)$$

*converges to the element $x^*$.*

*Proof.* Let $x_0$ be a fixed element in $A_0$. In view of the fact that $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element $x_1 \in A_0$ such that

$$d(gx_1, Tx_0) = d(A, B).$$

Since $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element $x_2 \in A_0$ such that

$$d(gx_2, Tx_1) = d(A, B).$$

Since $T$ is a multiplicative proximal contraction and $g$ is isometry, we get

$$\begin{array}{rcl} d(x_2, x_1) & = & d(gx_2, gx_1) \\ & \leq & d(x_1, x_0)^{\alpha}. \end{array} \tag{3.1}$$

Again, since $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists an element $x_3 \in A_0$ such that

$$d(gx_3, Tx_2) = d(A, B).$$

It follows from $T$ is a multiplicative proximal contraction, $g$ is an isometry and (3.1) that

$$\begin{aligned} d(x_3, x_2) &= d(gx_3, gx_2) \\ &\leq d(x_2, x_1)^{\alpha} \\ &\leq d(x_1, x_0)^{\alpha^2}. \end{aligned}$$

By the same method, for each $n \in \mathbb{N}$, we can find $x_n, x_{n+1} \in A_0$ such that

$$d(gx_n, Tx_{n-1}) = d(A, B)$$

and

$$d(gx_{n+1}, Tx_n) = d(A, B). \tag{3.2}$$

This implies that

$$\begin{aligned} d(x_{n+1}, x_n) &= d(gx_{n+1}, gx_n) \\ &\leq d(x_n, x_{n-1})^{\alpha} \\ &\leq d(x_{n-1}, x_{n-2})^{\alpha^2} \\ &\vdots \\ &\leq d(x_1, x_0)^{\alpha^n} \end{aligned}$$

for all $n \in \mathbb{N}$. Next, we will show that $\{x_n\}$ is a Cauchy sequence. Let $m, n \in \mathbb{N}$ with $m > n$, then we get

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) \cdot d(x_{m-1}, x_{m-2}) \cdots d(x_{n+1}, x_n) \\ &\leq d(x_1, x_0)^{\alpha^{m-1}} \cdot d(x_1, x_0)^{\alpha^{m-2}} \cdots d(x_1, x_0)^{\alpha^n} \\ &= d(x_1, x_0)^{\alpha^{m-1}+\alpha^{m-2}+\cdots+\alpha^n} \\ &\leq d(x_1, x_0)^{\frac{\alpha^n}{1-\alpha}}. \end{aligned}$$

Taking $m, n \to \infty$ in the above inequality, we obtain that $d(x_m, x_n) \to_* 1$. Hence $\{x_n\}$ is a Cauchy sequence. Since $A$ is a closed subsets of complete multiplicative metric space $X$, then the sequence $\{x_n\}$ converges to some element $x \in A$. Notice that,

$$\begin{aligned} d(gx, B) &\leq d(gx, Tx_n) \\ &\leq d(gx, gx_{n+1}) \cdot d(gx_{n+1}, Tx_n) \\ &= d(gx, gx_{n+1}) \cdot d(A, B) \\ &\leq d(gx, gx_{n+1}) \cdot d(gx, B) \end{aligned}$$

for all $n \in \mathbb{N}$. Since, $g$ is continuous and the sequence $\{x_n\}$ converges to $x$, then the sequence $\{gx_n\}$ converges to $gx$, that is $d(gx, gx_n) \to_* 1$ as $n \to \infty$. Therefore, $d(gx, Tx_n) \to_* d(gx, B)$ as $n \to \infty$. Since $B$ is approximatively compact with respect to $A$, then there exists subsequence $\{Tx_{n_k}\}$ of sequence $\{Tx_n\}$ such that converging to some element $u \in B$. Further, for each $k \in \mathbb{N}$, we have

$$\begin{aligned} d(A, B) &\leq d(gx, u) \\ &\leq d(gx, gx_{n_k+1}) \cdot d(gx_{n_k+1}, Tx_{n_k}) \cdot d(Tx_{n_k}, u) \\ &= d(gx, gx_{n_k+1}) \cdot d(A, B) \cdot d(Tx_{n_k}, u). \end{aligned} \tag{3.3}$$

Letting $k \to \infty$ in (3.3), we get $d(gx, u) = d(A, B)$ and hence $gx \in A_0$. From the fact that $A_0 \subseteq g(A_0)$, then $gx = gz$ for some $z \in A_0$. By the isometry of $g$, we get

$$d(x, z) = d(gx, gz) = 1$$

and thus $x = z$, that is, $x$ is an element of $A_0$. Since, $T(A_0) \subseteq B_0$, then there exists $x^* \in A$ such that

$$d(x^*, Tx) = d(A, B). \tag{3.4}$$

From (3.2), (3.4) and the multiplicative proximal contractive condition of $T$, we have

$$d(gx_{n+1}, x^*) \leq d(x_n, x)^\alpha$$

for all $n \in \mathbb{N}$. This yields that

$$\lim_{n\to\infty} d(gx_{n+1}, x^*) \leq \lim_{n\to\infty} d(x_n, x)^\alpha = 1^\alpha = 1. \tag{3.5}$$

This shows that the sequence $\{gx_n\}$ converges to $x^*$. By Lemma 2.6, we get that $gx = x^*$. Hence,

$$d(gx, Tx) = d(x^*, Tx) = d(A, B).$$

Next, to prove the uniqueness, suppose that there exist $x_\star \in A$ with $x \neq x_\star$ and

$$d(gx_\star, Tx_\star) = d(A, B).$$

Since $g$ is an isometry and $T$ is a multiplicative proximal contraction, it follows that

$$d(x, x_\star) = d(gx, gx_\star) \leq d(x, x_\star)^\alpha,$$

which is a contradiction. Therefore, we get $x = x_\star$. This completes the proof. □

If $g$ is the identity mapping in Theorem 3.2, then we obtain the best proximity point results as follows:

**Corollary 3.3.** *Let $(X, d)$ be a complete multiplicative metric space and $A$, $B$ be nonempty closed subsets of $X$ such that $A_0$ and $B_0$ are nonempty and $B$ is approximatively compact with respect to $A$. Let $T : A \to B$ satisfies the following conditions:*

*(a) $T$ is a multiplicative proximal contraction;*
*(b) $T(A_0) \subseteq B_0$.*

*Then there exists a unique point $x^* \in A$ such that*

$$d(x^*, Tx^*) = d(A, B).$$

*Moreover, for any fixed $x_0 \in A_0$, the sequence $\{x_n\}$ defined by*

$$d(x_n, Tx_{n-1}) = d(A, B)$$

*converges to the element $x^*$.*

For a self-mapping, Theorem 3.2 contains the following result:

**Corollary 3.4** ([2]). *Let $(X, d)$ be a complete multiplicative metric space and let $T : X \to X$ be a multiplicative Banach's contraction, then $T$ has a unique fixed point.*

Next, we give an example to illustrate Theorem 3.2.

**Example 3.5.** Let $X = \mathbb{R}^2$. Define the mapping $d : X \times X \to \mathbb{R}$ by

$$d((x_1, x_2), (y_1, y_2)) = e^{|x_1 - y_1| + |x_2 - y_2|}$$

for all $(x_1, x_2), (y_1, y_2) \in X$. It is easy to see that $(X, d)$ is a complete multiplicative metric space. Let

$$A = \{(0, x) : x \in \mathbb{R}\} \quad \text{and} \quad B = \{(1, y) : y \in \mathbb{R}\}.$$

Then $d(A,B) = e$, $A_0 = A$, $B_0 = B$ and $B$ is approximatively compact with respect to $A$. Define two mappings $T : A \to B$ and $g : A \to A$ as follows:

$$T((0,x)) = \big(1, \frac{x}{2}\big) \quad \text{and} \quad g((0,x)) = (0,-x)$$

for all $(0,x) \in A$. For all $(0,x),(0,y) \in A$, we get

$$d(g(0,x), g(0,y)) \;= d((0,-x),(0,-y)) \;= e^{|-x+y|} = e^{|x-y|} = d((0,x),(0,y)).$$

This implies that $g$ is an isometry.

Next, we show that $T$ is a multiplicative proximal contraction with $\alpha = \frac{1}{2}$. Let $(0,u),(0,v),(0,x)$ and $(0,y)$ be elements in $A$ satisfying

$$d(g(0,u), T(0,x)) = d(A,B) = e, \quad d(g(0,v), T(0,y)) = d(A,B) = e.$$

Then we have $u = -\dfrac{x}{2}$ and $v = -\dfrac{y}{2}$ and hence

$$\begin{aligned} d(g(0,u), g(0,v)) &= d((0,-u),(0,-v)) \\ &= d\bigg(\big(0,\frac{x}{2}\big),\big(0,\frac{y}{2}\big)\bigg) \\ &= \big(e^{|x-y|}\big)^{\frac{1}{2}} \\ &= \big(d((0,x),(0,y)))\big)^{\frac{1}{2}}. \end{aligned}$$

This implies that $T$ is a multiplicative proximal contraction with $\alpha = \frac{1}{2}$. Now all hypotheses in Theorem 3.2 hold and so there exists a unique point $x^*$ in $A$ such that $d(gx^*, Tx^*) = d(A,B)$. In this case, $x^* = (0,0) \in A$ is a unique element such that

$$d(gx^*, Tx^*) = d(g(0,0), T(0,0)) = d((0,0),(1,0)) = e = d(A,B).$$

## 4. Conclusions

Best proximity point results for multiplicative proximal contraction mapping in multiplicative metric spaces along with approximatively compactness was investigated under some suitable conditions. These result are a generalization of fixed point result for multiplicative Banach's contraction self-mappings due to Özavşar and Çevikel [2]. However, the best proximity point results for mappings satisfies another proximal contractive conditions in multiplicative metric spaces still open for interested mathematicians.

## Acknowledgements

The first author would like to thank Department of Mathematics statistics and computers, Faculty of Liberal Arts and Science, Kasetsart University, and also the Kasetsart University Research and Development Institute (KURDI) for financial support in this work. The second author gratefully acknowledge the financial support provided by Thammasat University under the government budget 2015, Contract No. 014/2558.

## References

[1] A. E. Bashirov, E. M. Kurpinar, A. Özyapici, *Multiplicative calculus and its applications*, J. Math. Anal. Appl., **337** (2008), 36–48. 1, 2.1

[2] M. Özavşar, A. C. Çevikel, *Fixed points of multiplicative contraction mappings on multiplicative metric spaces*, arXiv, **2012** (2012), 14 pages. 1, 2.2, 2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8, 2.9, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 3, 3.4, 4