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The concept of weak (ψ, α, β) contractions in partially ordered metric spaces

Mehmet Kir*, Hukmi Kiziltunc

Department of Mathematics, Faculty of Science, Ataturk University, 25240, Erzurum-Turkey.

Abstract

In this paper, we investigate generalized weak (ψ, α, β) contractions in partially ordered sets in order to establish extensions of Banach, Kannan and Chatterjea's fixed point theorems in this setting.

Keywords: Fixed point, weak contractions, Kannan fixed point theorem, Chatterjea fixed point theorem, partially ordered set.
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1. Introduction and Preliminaries

The basic unit of analysis in order theory is the binary relation. It is well known that a relation $\Re$ on a set X is a subset of $X \times X$. We denote $(x, y) \in \Re$ by $x\Re y$. An "order" on a set X is a relation on X satisfying some additional conditions. Order relations are usually denoted by $\preceq$.

Definition 1.1 ([6])**.** A relation $\preceq$ on a set X is called a partial order if $\preceq$ is transitive, reflexive and antisymmetric; the pair $(X, \preceq)$ is called a partially ordered set or poset. In addition, a relation $\preceq$ on a set X is called a linear order if any two elements in X are comparable, that is,

$$\text{for each } x, y \in X, \text{ either } x \preceq y \text{ or } y \preceq x. \tag{1.1}$$

The pair $(X, \preceq)$ is called a linearly ordered set or chain.

In [1], Banach proved a very important result in nonlinear analysis, the contraction mapping principle. In [14], Ran and Reurings established an analogue of Banach's fixed point theorem in partially ordered sets and discussed several applications to linear and nonlinear matrix equations. Later on, Nieto and López [13] extended some of their results to study a problem regarding ordinary differential equations.

*Corresponding author
Email addresses: `mehmetkir04@gmail.com` (Mehmet Kir), `hukmu@atauni.edu.tr` (Hukmi Kiziltunc)

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Definition 1.2 ([13]). Let $(X, \preceq)$ be a poset and A be a subset of X.

(i) If for all $a \in A$ there exists an $x \in X$ such that $a \preceq x$, then x is called an upper bound for the set A. If $x \in X$ is the smallest upper bound, then x is called the least upper bound (sup) of the set A.

(ii) If for all $a \in A$ there exists an $x \in X$ such that $x \preceq a$, then x is called a lower bound for the set A. If $x \in X$ is the greatest lower bound, then x is called the largest lower bound (inf) of the set A.

(iii) A mapping $f : X \to X$ is called monotone nondecreasing if

$$x, y \in X, x \preceq y \;\Rightarrow\; fx \preceq fy$$

Definition 1.3 ([2]). Let $(X, \preceq)$ be a partially ordered set. $(X, \preceq)$ is said to be directed if every pair of elements has an upper bound, that is, for every $a, b \in X$, there exists $c \in X$ such that $a \preceq c$ and $b \preceq c$.

Theorem 1.4 ([14]). *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space, such that every pair $x, y \in X$ has a lower bound and an upper bound. If $f : X \to X$ is a continuous, monotone (i.e., either order-preserving or order-reversing) map from X into X such that*

$$\exists\; 0 < c < 1 : d\left(fx, fy\right) \leq cd\left(x, y\right), x \geq y \tag{1.2}$$

and

$$\exists x_0 \in X : x_0 \leq fx_0 \;\; or \; x_0 \geq fx_0,$$

then f has a unique fixed point $\overline{x} \in X$. Moreover, for every $x \in X$, $\lim\limits_{n\to\infty} f^n x = \overline{x}$.

In [13], Nieto and López showed that the continuity condition for the mapping f can be replaced with the requirement that if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n \preceq z$ for all $n \geq 0$. Also, to guarantee the uniqueness of the fixed point, they gave an alternative condition to the requirement that every pair $x, y \in X$ have a lower bound and an upper bound, namely that for every $x, y \in X$ there should exist $z \in X$ which is comparable to x and y.

In [5], Ćirić *et al.* introduced the concept of g-monotone mapping and proved some fixed point and common fixed point theorems for g- nondecreasing generalized nonlinear contractions in complete partially ordered metric spaces. Based on this concept, Choudhury and Kundu [4] considered (ψ, α, β)-weak contractions, proving coincidence point and common fixed point results in posets. Cherichi and Samet [2] presented new coincidence and fixed point theorems in the setting of complete ordered gauge spaces $(X, \mathcal{F}, \preceq)$ for generalized weak contractions involving two families of functions (see also [10, 12, 16]).

In this work, we investigate generalized weak (ψ, α, β) contractions in posets in order to establish analogues of the Banach, Kannan [9] and Chatterjea [3] fixed point theorems in this setting.

2. Main Results

For simplicity, we will make the following notations:

- Ψ is the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the properties:

  (ψ_1) ψ is continuous and monotone nondecreasing;
  (ψ_2) $\psi\left(t\right) = 0$ if and only if $t = 0$.

- Φ is the set of functions $\alpha : [0, \infty) \to [0, \infty)$ satisfying:

  (α_1) α is continuous;
  (α_2) $\alpha\left(t\right) = 0$ if and only if $t = 0$.

- Γ_1 is the set of functions $\beta : [0, \infty) \to [0, \infty)$ satisfying:

  (β_1) β is lower semi-continuous;

(β_2) $\beta(t) = 0$ if and only if $t = 0$.

- Γ_2 is the set of functions $\beta : [0,\infty)^2 \to [0,\infty)$ satisfying:

(β_1') β is continuous;
(β_2') β is monotone increasing in both arguments;
(β_3') $\beta(0,0) = 0$ and $\beta(\varepsilon, 0) = 0$ implies $\varepsilon = 0$.

Definition 2.1 ([11, 15])**.** Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be subsequentially convergent if every sequence $\{y_n\}$ with the property that $\{Ty_n\}$ is convergent has a convergent subsequence.

We will denote by $Sb{-}COP(X)$ the set of all mappings $T : X \to X$ which are subsequentially convergent, continuous, one to one and preserve the order.

Theorem 2.2. *Let $(X,\preceq)$ be a partially ordered set endowed with a metric d such that (X,d) is a complete metric space. Let $T \in Sb-COP(X)$ and $f : X \to X$ be a monotone nondecreasing mapping with the property that, for all $x,y \in X$ with $x \preceq y$,*

$$\psi(d(Tfx,Tfy)) \leq \alpha(d(Tx,Ty)) - \beta(d(Tx,Ty)) \tag{2.1}$$

where $\psi \in \Psi$, $\alpha \in \Phi$, $\beta \in \Gamma_1$ are such that

$$\psi(t_1) \leq \alpha(t_2) \Rightarrow t_1 \leq t_2 \tag{2.2}$$

and

$$\psi(t) - \alpha(t) + \beta(t) > 0, \ \forall t > 0. \tag{2.3}$$

Also, suppose that either
(C1) f is continuous, or
(C2) if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n \preceq z$ for all $n \geq 0$.
If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if for all $(x,y) \in X \times X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$ (i.e. $(X,\preceq)$ is directed), then the fixed point is unique.

Proof. Let $x_0 \in X$ be an arbitrary point and $x_n = fx_{n-1} = f^n x_0$, $n = 1,2,3,\ldots$. As f is nondecreasing with $x_0 \preceq fx_0$ and $T \in Sb-COP(X)$, we have

$$Tx_0 \preceq Tfx_0 \preceq Tf^2x_0 \preceq Tf^3x_0 \preceq \cdots \preceq Tf^nx_0 \preceq \cdots \tag{2.4}$$

that is, $Tx_n \preceq Tx_{n+1}$ for all n. The rest of the proof consists of four steps.

Step 1. We will show that $\lim\limits_{n\to\infty} d(Tx_{n+1}, Tx_n) = 0$.

For convenience, let $D_n := d(Tx_n, Tx_{n+1})$ for all $n \geq 0$. From (2.1) we get

$$\psi(D_{n+1}) = \psi(d(Tx_{n+1}, Tx_{n+2})) \leq \alpha(D_n) - \beta(D_n) \leq \alpha(D_n). \tag{2.5}$$

Using the condition (2.2), we obtain that $\{D_n\}$ is a monotone decreasing sequence of non-negative real numbers and consequently, there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} D_n = r$. Letting $n \to \infty$ in (2.5), we get

$$\psi(r) \leq \alpha(r) - \beta(r) \ \text{ as } n \to \infty.$$

By hypothesis (2.3), this implies that $\lim\limits_{n\to\infty} D_n = 0$.

Step 2. We show that $\{Tx_n\}$ is a Cauchy sequence.

We proceed by contradiction. Suppose that $\{Tx_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{Tx_{m(k)}\}$ and $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ with $n(k) > m(k) > k$ such that

$$d\left(Tx_{m(k)}, Tx_{n(k)}\right) \geq \varepsilon. \tag{2.6}$$

Furthermore, we can choose $n(k)$ to be the smallest integer with $n(k) > m(k)$, for all k. As a consequence,

$$d\left(Tx_{m(k)}, Tx_{n(k)-1}\right) < \varepsilon.$$

From (2.6) we have

$$\varepsilon \leq d\left(Tx_{m(k)}, Tx_{n(k)}\right) \leq d\left(Tx_{m(k)}, Tx_{n(k)-1}\right) + d\left(Tx_{n(k)}, Tx_{n(k)-1}\right), \tag{2.7}$$

whence, by letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} d\left(Tx_{m(k)}, Tx_{n(k)}\right) = \varepsilon.$$

Also,

$$d\left(Tx_{m(k)}, Tx_{n(k)-1}\right) \leq d\left(Tx_{m(k)}, Tx_{n(k)}\right) + d\left(Tx_{n(k)}, Tx_{n(k)-1}\right), \tag{2.8}$$

implying for $k \to \infty$ that

$$\lim_{k\to\infty} d\left(Tx_{m(k)}, Tx_{n(k)-1}\right) = \varepsilon.$$

Moreover, by letting $k \to \infty$ in the inequality

$$d\left(Tx_{m(k)-1}, Tx_{n(k)}\right) \leq d\left(Tx_{m(k)-1}, Tx_{m(k)}\right) + d\left(Tx_{m(k)}, Tx_{n(k)}\right), \tag{2.9}$$

we obtain

$$\lim_{k\to\infty} d\left(Tx_{m(k)-1}, Tx_{n(k)}\right) = \varepsilon.$$

Finally, from

$$d\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right) \leq d\left(Tx_{m(k)-1}, Tx_{m(k)}\right) + d\left(Tx_{m(k)}, Tx_{n(k)-1}\right) \tag{2.10}$$

we get

$$\lim_{k\to\infty} d\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right) = \varepsilon.$$

From (2.1), we have

$$\psi(\varepsilon) \leq \psi\left(d\left(Tx_{m(k)}, Tx_{n(k)}\right)\right) \leq \alpha\left(d\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right)\right) - \beta\left(d\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right)\right) \tag{2.11}$$

whence by letting $k \to \infty$ we get

$$\psi(\varepsilon) \leq \alpha(\varepsilon) - \beta(\varepsilon). \tag{2.12}$$

This inequality together with hypothesis (2.3) imply that $\varepsilon = 0$, contradicting our initial assumption. Therefore we conclude that $\{Tx_n\}$ is a Cauchy sequence.

Step 3. We prove that there exists an element $p \in X$ such that $\lim_{n\to\infty} x_n = p$ and p is a fixed point of f.

Indeed, as $\{Tx_n\}$ is a Cauchy sequence in the complete metric space (X, d), there exists $v \in X$ such that $\lim_{n\to\infty} Tx_n = v$. Since $T \in Sb - COP(X)$, $\{x_n\}$ has a convergent subsequence $\left\{x_{n(k)}\right\}$ and there exists $p \in X$ such that

$$\lim_{k\to\infty} x_{n(k)} = p.$$

Also, T is continuous and $x_{n(k)} \to p$, therefore $Tx_{n(k)} \to Tp$ as $k \to \infty$. Since T preserves the order, that is, $Tx_{n(k)} \preceq Tp$, we have

$$\lim_{k\to\infty} d\left(Tx_{n(k)}, Tp\right) = 0.$$

We show that $p \in X$ is a fixed point of f. We have two cases.

Case 1: If (C1) holds, then from the continuity of f we have

$$Tp = \lim_{k\to\infty} Tx_{n(k)} = \lim_{k\to\infty} Tfx_{n(k)-1} = Tfp.$$

Using the fapt that T is one to one, we obtain $fp = p$, i.e. $p \in X$ is a fixed point of f.

Case 2: Assume (C2) holds. Since $\{Tx_{n(k)}\}$ converges to $Tp \in X$, for all $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that for all $n(k) > N_1$ we have

$$d\left(Tx_{n(k)}, Tp\right) < \frac{\varepsilon}{2}.$$

Also, as $\{Tx_{n(k)}\}$ converges to Tp, from (C2) we get $Tx_{n(k)} \preceq Tp$ and we have

$$\psi\left(d\left(Tf^{n(k)+1}x, Tfp\right)\right) \leq \alpha\left(d\left(Tf^{n(k)}x, Tp\right)\right) - \beta\left(d\left(Tf^{n(k)}x, Tp\right)\right). \tag{2.13}$$

Letting $k \to \infty$ in (2.13), we get

$$\psi\left(d\left(Tp, Tfp\right)\right) \leq \alpha(0) - \beta(0). \tag{2.14}$$

The inequality (2.14) implies that $Tp = Tfp$. As T is one to one, it follows that $p \in X$ is a fixed point of f.

Step 4. Finally, we show that, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, then the fixed point is unique.

For this, let $p' \in X$ be another fixed point of f. By (1.1), there exists an element z in X such that z is comparable to p and p'. The monotonicity of f implies that fz is comparable to $p = fp$ and $p' = fp'$. As $T \in Sb - COP(X)$, Tfz is comparable to Tp and Tp'. Also, as $\psi \in \Psi$, we have

$$\psi\left(d\left(Tp, Tp'\right)\right) \leq \psi\left(d\left(Tfp, Tfp'\right)\right) \leq \alpha\left(d\left(Tp, Tp'\right)\right) - \beta\left(d\left(Tp, Tp'\right)\right).$$

From the condition (2.3), we get

$$d\left(Tp, Tp'\right) = 0.$$

Since T is one to one it follows that $p = p'$. This completes the proof. $\square$

Corollary 2.3 ([8]). *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be monotone nondecreasing. Suppose that, for all $x, y \in X$ with $x \preceq y$,*

$$\psi\left(d\left(fx, fy\right)\right) \leq \psi\left(d\left(x, y\right)\right) - \phi\left(d\left(x, y\right)\right),$$

where ψ and ϕ are altering distance functions. Also, suppose that either (C1) or (C2) holds. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if $(X, \preceq)$ is directed, then the fixed point is unique.

Corollary 2.4 ([7]). *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping. Suppose that for all $x, y \in X$ with $x \preceq y$,*

$$d\left(fx, fy\right) \leq \beta\left(d\left(x, y\right)\right) d\left(x, y\right),$$

where $\beta : [0, \infty) \to [0, 1)$ and $\beta(t_n) \to 1$ implies $t_n \to 0$. Also, suppose that either condition (C1) or (C2) holds. If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if $(X, \preceq)$ is directed, then the fixed point is unique.

Example 2.5. Let $X = [1, \infty) \times [1, \infty)$ and consider the usual order given by $(a, b) \preceq (c, d)$ if and only if $a \leq c$ and $b \leq d$. Then $(X, \preceq)$ is a partially ordered set. Let X be endowed with the Euclidean distance. We define a mapping $f : X \to X$ by $f(x, y) = \left(4\sqrt[3]{x}, 4\sqrt[3]{y}\right)$. It is clear that f does not satisfy the contractive condition (1.2).

Now we define $T : X \to X$ by $T(x, y) = (\ln ex, \ln ey)$, and set $\psi(t) = t$, $\alpha(t) = \frac{t}{3}$, $\beta(t) = 0$. Then we have

$$\begin{aligned} \psi\left(d\left(Tf(a, b), Tf(c, d)\right)\right) &= d\left(Tf(a, b), Tf(c, d)\right) \\ &= \frac{1}{3}\sqrt{\left(\ln\frac{a}{c}\right)^2 + \left(\ln\frac{b}{d}\right)^2} \\ &= \frac{1}{3} d\left(T(a, b), T(c, d)\right). \end{aligned} \tag{2.15}$$

According to Theorem 2.2, f has a fixed point. Indeed, $(8,8) \in X$ is a fixed point of f. As X is directed, this fixed point is unique.

Theorem 2.6. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X,d) be a complete metric space. Let $T \in Sb - COP(X)$ and $f : X \to X$ be a monotone nondecreasing mapping satisfying the inequality*

$$\psi\left(d\left(Tfx, Tfy\right)\right) \leq \alpha\left(\frac{1}{2}\left(d\left(Tx, Tfx\right) + d\left(Ty, Tfy\right)\right)\right) - \beta\left(d\left(Tx, Tfx\right), d\left(Ty, Tfy\right)\right) \tag{2.16}$$

for all $x, y \in X$ with $x \preceq y$, where $\psi \in \Psi$, $\alpha \in \Phi$, $\beta \in \Gamma_2$ are such that

$$\psi\left(t_1\right) \leq \alpha\left(t_2\right) \Rightarrow t_1 \leq t_2, \tag{2.17}$$

and for all $t > 0$,

$$\psi\left(t\right) - \alpha\left(t\right) + \beta\left(t\right) > 0. \tag{2.18}$$

Also, suppose that either

(C1) f is continuous, or

(C2) if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n \preceq z$ for all $n \geq 0$.

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if $(X, \preceq)$ is directed, then the fixed point is unique.

Proof. Let $x_0 \in X$ be an arbitrary point such that $x_n = fx_{n-1} = f^n x_0$, $n = 1, 2, 3, ...$ As f is nondecreasing with $x_0 \preceq fx_0$ and $T \in Sb - COP(X)$, we have

$$Tx_0 \preceq Tfx_0 \preceq Tf^2x_0 \preceq Tf^3x_0 \preceq \cdots \preceq Tf^nx_0 \preceq \cdots,$$

that is, $Tx_n \preceq Tx_{n+1}$ for all n.The proof consists of four steps.

Step 1. We show that $\lim_{n\to\infty} d\left(Tx_{n+1}, Tx_n\right) = 0$.

For convenience, we denote $D_{n+1} := d\left(Tx_{n+1}, Tx_{n+2}\right)$ for all $n \geq 0$. From (2.16) we obtain

$$\psi\left(D_{n+1}\right) = \psi\left(d\left(Tfx_n, Tfx_{n+1}\right)\right) \leq \alpha\left(\frac{1}{2}\left(D_n + D_{n+1}\right)\right) - \beta\left(D_n, D_{n+1}\right)) \leq \alpha\left(\frac{1}{2}\left(D_n + D_{n+1}\right)\right). \tag{2.19}$$

By (2.17), we have $D_{n+1} \leq D_n$ for all n, that is, $\{D_n\}$ is a monotone decreasing sequence of non-negative real numbers. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} D_n = r$. By (2.19), $\psi\left(r\right) \leq \alpha\left(r\right) - \beta\left(r, r\right)$ as $n \to \infty$. Taking into account the hypothesis (2.18), it follows that $\lim_{n\to\infty} D_n = 0$.

Step 2. We prove that $\{Tx_n\}$ is a Cauchy sequence in X.

If we suppose the contrary, then there exists $\varepsilon > 0$ for which we can find subsequences $\left\{Tx_{m(k)}\right\}$ and $\left\{Tx_{n(k)}\right\}$ of $\{Tx_n\}$ with $n\left(k\right) > m\left(k\right) > k$ such that

$$d\left(Tx_{m(k)}, Tx_{n(k)}\right) \geq \varepsilon.$$

From (2.16), we have

$$\begin{aligned} \psi\left(\varepsilon\right) &\leq \psi\left(d\left(Tx_{m(k)}, Tx_{n(k)}\right)\right) = \psi\left(d\left(Tfx_{m(k)-1}, Tfx_{n(k)-1}\right)\right) \\ &\leq \alpha\left(\frac{1}{2}\left(D_{m(k)-1} + D_{n(k)-1}\right)\right) - \beta\left(D_{m(k)-1}, D_{n(k)-1}\right), \end{aligned} \tag{2.20}$$

whence by letting $k \to \infty$ we obtain

$$\psi\left(\varepsilon\right) \leq \alpha\left(0\right) - \beta\left(0, 0\right).$$

Due to the hypothesis (2.18), the inequality (2.20) implies that $\varepsilon = 0$, a contradiction. Therefore we conclude that $\{Tx_n\}$ is Cauchy sequence.

Step 3. By proceeding similarly as in Step 3 of the proof of Theorem 2.2, we obtain that there exists $p \in X$ such that $\{Tx_n\}$ converges to $Tp \in X$.

Next, we will show that p is a fixed point of f. We have two cases.

Case 1: If (C1) holds, from the continuity of f, we have

$$Tp = \lim_{k\to\infty} Tx_{n(k)} = \lim_{k\to\infty} Tfx_{n(k)-1} = Tfp,$$

and since T is one to one it follows that $p \in X$ is a fixed point of f.

Case 2: Suppose (C2) holds. Since $\left\{Tx_{n(k)}\right\}$ converges to $Tp \in X$, for all $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that for all $n\,(k) > N_1$ we have

$$d\left(Tx_{n(k)}, Tu\right) < \frac{\varepsilon}{2}.$$

As $\left\{Tx_{n(k)}\right\}$ converges to Tp, from (C2) we obtain $Tx_{n(k)} \preceq Tp$ and we have

$$\psi\left(d\left(Tx_{n(k)+1}, Tfp\right)\right) \leq \alpha\left(\frac{1}{2}\left(D_{n(k)} + d\left(Tp, Tfp\right)\right)\right) - \beta\left(D_{n(k)}, d\left(Tp, Tfp\right)\right). \tag{2.21}$$

Letting $k \to \infty$ in the above relation, we have

$$\psi\left(d\left(Tp, Tfp\right)\right) \leq \alpha\left(\frac{1}{2} d\left(Tp, Tfp\right)\right) - \beta\left(0, d\left(Tp, Tfp\right)\right). \tag{2.22}$$

This inequality implies that $Tp = Tfp$. As T is one to one, it follows that $p \in X$ is a fixed point of f.

Step 4. Finally, we prove that, under the assumption that $(X, \preceq)$ is directed, the fixed point is unique.

Indeed, let $p' \in X$ be another fixed point of f. From (1.1), there exists an element $z \in X$ such that z is comparable to p and p'. The monotonicity of f implies that fz is comparable to $p = fp$ and $p' = fp'$. As $T \in Sb - COP\,(X)$, Tfz is comparable to Tp and Tp'. Also, it is easy to obtain that $\psi\left(d\left(Tp, Tp'\right)\right) \leq 0$. As T is one to one, we get that $p = p'$, and the proof is completed. □

Remark 2.7. It is clear that Theorem 2.6 is an extension of the Kannan fixed point theorem to the context of partially ordered metric spaces.

In Theorem 2.6, if we consider $\psi\,(t) = \alpha\,(t)$, then we obtain the following result which is more general than Theorem 2 in [15].

Corollary 2.8. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $f : X \to X$ be a monotone nondecreasing mapping and $T \in Sb - COP\,(X)$. Suppose that for all $x, y \in X$ with $x \preceq y$,*

$$\psi(d\left(Tfx, Tfy\right)) \leq \psi\left(\frac{1}{2}\left(d\left(Tx, Tfx\right) + d\left(Ty, Tfy\right)\right)\right) - \beta\left(d\left(Tx, Tfx\right), d\left(Ty, Tfy\right)\right).$$

Also, suppose that

(C1) f is continuous, or

(C2) if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n \preceq z$ for all $n \geq 0$.

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if $(X, \preceq)$ is directed, then the fixed point is unique.

Theorem 2.9. *Let $(X, \preceq)$ be a partially ordered set endowed with a metric d and (X, d) be a complete metric space. Let $T \in Sb - COP\,(X)$ and $f : X \to X$ be a monotone nondecreasing mapping satisfying*

$$\psi\left(d\left(Tfx, Tfy\right)\right) \leq \alpha\left(\frac{1}{2}\left(d\left(Tx, Tfy\right) + d\left(Ty, Tfx\right)\right)\right) - \beta\left(d\left(Tx, Tfy\right), d\left(Ty, Tfx\right)\right) \tag{2.23}$$

for all $x, y \in X$ with $x \preceq y$, where $\psi \in \Psi$, $\alpha \in \Phi$, $\beta \in \Gamma_2$ are such that

$$\psi(t_1) \leq \alpha(t_2) \Rightarrow t_1 \leq t_2, \tag{2.24}$$

and for all $t > 0$,

$$\psi(t) - \alpha(t) + \beta(t) > 0. \tag{2.25}$$

Also, suppose that either

(C1) f is continuous, or

(C2) if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n \preceq z$ for all $n \geq 0$.

If there exists $x_0 \in X$ with $x_0 \preceq fx_0$, then f has a fixed point in X. Moreover, if $(X, \preceq)$ is directed, then the fixed point is unique.

Proof. Let $x_0 \in X$ be an arbitrary point and let $x_n = fx_{n-1} = f^n x_0$, $n = 1, 2, 3, \ldots$. As f is nondecreasing, $x_0 \preceq fx_0$ and $T \in Sb - COP(X)$, we have that $Tx_n \preceq Tx_{n+1}$ for all n. Again, the proof consists of four steps.

Step 1. We show that $\lim\limits_{n\to\infty} d(Tx_{n+1}, Tx_n) = 0$.

Let $D_{n+1} := d(Tx_{n+1}, Tx_{n+2})$ for all $n \geq 0$. From (2.23) we obtain

$$\begin{aligned}\psi(D_{n+1}) = \psi(d(Tfx_n, Tfx_{n+1})) &\leq \alpha\left(\frac{1}{2} d(Tx_n, Tx_{n+2})\right) - \beta(d(Tx_n, Tx_{n+2}), 0)) \\ &\leq \alpha\left(\frac{1}{2}(D_n + D_{n+1})\right) - \beta(D_n, 0) \leq \alpha\left(\frac{1}{2}(D_n + D_{n+1})\right).\end{aligned} \tag{2.26}$$

The hypothesis (2.24) implies that $d(Tx_{n+1}, Tx_{n+2}) \leq d(Tx_n, Tx_{n+1})$, that is, $\{d(Tx_n, Tx_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Consequently, there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(Tx_n, Tx_{n+1}) = r$.

Passing to the limit for $n \to \infty$ in (2.26), we obtain that $\psi(r) \leq \alpha(r) - \beta(r, 0)$, whence, by (2.25), it follows that $r = 0$.

Step 2. We show that $\{Tx_n\}$ is a Cauchy sequence in X.

As in the proof of Theorem 2.2, we see that if $\{Tx_n\}$ is not a Cauchy sequence, then there exists $\varepsilon > 0$ such that $d\left(Tx_{m(k)}, Tx_{n(k)}\right) \geq \varepsilon$ and the sequences $\left\{d\left(Tx_{m(k)-1}, Tx_{n(k)}\right)\right\}$, $\left\{d\left(Tx_{m(k)}, Tx_{n(k)-1}\right)\right\}$ converge to ε.

From (2.23), we have

$$\begin{aligned}\psi(\varepsilon) &\leq \psi\left(d\left(Tx_{m(k)}, Tx_{n(k)}\right)\right) \\ &\leq \alpha\left(\frac{1}{2}\left(d\left(Tx_{m(k)-1}, Tx_{n(k)}\right) + d\left(Tx_{n(k)-1}, Tx_{m(k)}\right)\right)\right) \\ &\quad - \beta\left(d\left(Tx_{m(k)-1}, Tx_{n(k)}\right), d\left(Tx_{n(k)-1}, Tx_{m(k)}\right)\right).\end{aligned} \tag{2.27}$$

For $k \to \infty$ in (2.27), we obtain

$$\psi(\varepsilon) \leq \alpha(0) - \beta(0, 0),$$

which, together with (2.25), implies that $\varepsilon = 0$, a contradiction. We conclude that $\{Tx_n\}$ is Cauchy sequence.

Steps 3 and 4. By using similar methods as in Theorems 2.2 and 2.6, we infer that there exists $p \in X$ such that $\{x_n\}$ converges to p and p is a fixed point of f. Furthermore, if $(X, \preceq)$ is directed, then the fixed point is unique. □

Remark 2.10. It is clear that Theorem 2.9 extends the Chatterjea fixed point theorem to the context of partially ordered metric spaces.

In Theorem 2.9, if we consider $\psi(t) = \alpha(t)$, then we obtain the following result which is more general than Theorem 1 in [15].

Corollary 2.11. *Let $(X,\preceq)$ be a partially ordered set endowed with a metric d and (X,d) be a complete metric space. Let $f:X\to X$ be a monotone nondecreasing mapping and $T\in Sb-COP(X)$. Suppose that for all $x,y\in X$ with $x\preceq y$,*

$$\psi(d(Tfx,Tfy))\leq\psi\left(\frac{1}{2}\left(d\left(Tx,Tfy\right)\right)+d\left(Ty,Tfx\right)\right)\right)-\beta\left(d\left(Tx,Tfy\right),d\left(Ty,Tfx\right)\right).$$

Also, suppose that

(C1) f is continuous, or

(C2) if any nondecreasing sequence $\{x_n\}$ in X converges to z, then $x_n\preceq z$ for all $n\geq 0$.

If there exists $x_0\in X$ with $x_0\preceq fx_0$, then f has a fixed point in X. Moreover, if $(X,\preceq)$ is directed, then the fixed point is unique.

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