Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 160–170
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Solvability for integral boundary value problems of fractional differential equation on infinite intervals

Changlong Yu*, Jufang Wang, Yanping Guo

College of Sciences, Hebei University of Science and Technology, Shijiazhuang, 050018, Hebei, P. R. China.

Communicated by Mohamed Jleli

Abstract

In this paper, we establish the solvability for integral boundary value problems of fractional differential equation with the nonlinear term dependent in a fractional derivative of lower order on infinite intervals. The existence and uniqueness of solutions for the boundary value problem are proved by means of the Schauder's fixed point theorem and Banach's contraction mapping principle. Finally, we give two examples to demonstrate the use of the main results.

Keywords: Integral boundary value problem, fractional differential equation, infinite interval, Fixed point theorem.
2010 MSC: 34A08, 34B40, 34B15.

1. Introduction

Boundary value problems on infinite intervals appear often in applied mathematics and physics. More examples and a collection of works on the existence of solutions of Boundary value problems on infinite intervals for differential, difference and integral equations may be found in the monographs [1, 15]. For some works and various techniques dealing with such boundary value problems, see [2, 5, 7, 13, 21, 22, 23, 26] and the references therein.

The fractional differential equation has emerged as a new branch in the field of differential equations for their deep back grounds. For an extensive collection of such results, we refer the readers to the monographs [9, 11, 16, 17]. There has been a significant development in nonlocal problems for fractional differential equations or inclusions, see [3, 4, 8, 10, 12, 14, 19, 20, 24, 27] and the references therein.

*Corresponding author
Email addresses: changlongyu@126.com (Changlong Yu), wangjufang1981@126.com (Jufang Wang), guoyanping65@126.com (Yanping Guo)

Received 2015-03-23

Boundary value problems for fractional differential equations on infinite intervals have been considered widely and there are some excellent results on the existence of solutions, see [6, 18, 25] and the references therein. However, to our knowledge, it is rare for works to be done on the solutions for integral boundary value problems (IBVPs) of fractional differential equations on infinite interval.

Recently, in [18], X. Su and S. Zhang considered the BVP:

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, u(t), D_{0+}^{\alpha-1}u(t)), & t \in J := [0, +\infty), \\ u(0) = 0, \quad D_{0+}^{\alpha-1}u(\infty) = u_{\infty}, & u_{\infty} \in R, \end{cases}$$

where $1 < \alpha \leq 2$, $f \in C(J \times R \times R, R)$,$D_{0+}^{\alpha}$ and $D_{0+}^{\alpha-1}$ are the standard Riemann-Liouville fractional derivatives. The existence of unbounded positive solutions was obtained by the Schauder's fixed point theorem on unbounded domain.

Motivated by the work above, in this paper, we will discuss the following IBVP:

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, u(t), D_{0+}^{\alpha-1}u(t)), & t \in J, \\ u(0) = 0, \quad D_{0+}^{\alpha-1}u(\infty) = \int_{\eta}^{+\infty} g(t)u(t)dt, \end{cases} \tag{1.1}$$

where $J = [0, +\infty)$, $1 < \alpha \leq 2$, $f \in C(J\times R\times R, R)$, $\eta \geq 0$, $g(t) \in L^1[0, +\infty)$ and $\int_{\eta}^{+\infty} g(t)t^{\alpha-1}dt < \Gamma(\alpha)$, D_{0+}^{α} and $D_{0+}^{\alpha-1}$ are the standard Riemann-Liouville fractional derivatives and $D_{0+}^{\alpha-1}u(\infty) = \lim_{t\to+\infty} D_{0+}^{\alpha-1}u(t)$.

We deal with the existence and uniqueness of solutions for BVP (1.1) by using the Schauder's fixed point theorem and Banach's contraction mapping principle and obtain multiplicity results which extend and improve the known results.

2. Preliminary results

In this section, we introduce definitions and preliminary facts which are used throughout paper. Following, let us recall some basic concepts of fractional calculus, see [9, 16, 17] and the references therein.

Definition 2.1. The Riemann-Liouville fractional integral of order $\delta > 0$ of a function $f(t)$ is defined by

$$I_{a+}^{\delta}f(t) = \frac{1}{\Gamma(\delta)}\int_a^t (t-s)^{\delta-1}f(s)ds, \qquad t > a,$$

provided that the right-hand side is pointwise defined.

It is well known that $I_{a+}^{\delta}f(a) = 0$, for $f(t) \in C[a, b]$,$\delta > 0$ and $I_{a+}^{\delta} : C[a, b] \to C[a, b]$ for $\delta > 0$.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\delta > 0$ of a function $f(t)$ is defined by

$$D_{a+}^{\delta}f(t) = \left(\frac{d}{dt}\right)^n I_{a+}^{n-\delta}f(t) = \frac{1}{\Gamma(n-\delta)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\delta-1}f(s)ds, \qquad t > a,$$

where n is the smallest integer greater than or equal to δ, provided that the right-hand side is pointwise defined. In particular, for $\delta = n$, $D_{a+}^{n}f(t) = f^{(n)}(t)$.

Lemma 2.3 ([9]). *In this work, we need the following composition relations:*

(a) $D_{a+}^{\delta}I_{a+}^{\delta}f(t) = f(t), \quad \delta > 0, \qquad f(t) \in L^1[0, +\infty);$

(b) $D_{a+}^{\delta}I_{a+}^{\gamma}f(t) = I_{a+}^{\gamma-\delta}f(t), \quad \gamma > \delta > 0, \qquad f(t) \in L^1[0, +\infty).$

Definition 2.4 ([13]). It holds that $f : [0, \infty) \times R^2 \to R$ is called an S-Carathéodory function if and only if

(i) for each $(u, v) \in R^2$, $t \in f(t, u, v)$ is measurable on $[0, \infty)$;

(ii) for almost every $t \in [0, \infty)$, $(u, v) \mapsto f(t, u, v)$ is continuous on R^2;

(iii) for each $r > 0$, there exist $\varphi_r(t) \in L^1[0, \infty)$, $\varphi_r(t) > 0$ on $[0, \infty)$ such that $\max\{|u|, |v|\} \leq r$ implies

$$|f(t, u, v)| \leq \varphi_r(t), \quad for \quad a.e. \quad t \in [0, \infty).$$

Lemma 2.5 ([18])**.** *For $\delta > 0$, the equation $D^{\delta}_{a^+}x(t) = 0$ is valid if and only if, $x(t) = \Sigma_{j=1}^{n} c_j(t-a)^{\delta-j}$, where $c_j \in R$, $j = 1, 2, \cdots, n$ are arbitrary constants and n is the smallest integer greater than or equal to δ.*

Lemma 2.6. *Let $y(t) \in L^1[0, +\infty)$ and $\int_{\eta}^{+\infty} g(t)t^{\alpha-1}dt \neq \Gamma(\alpha)$, then IBVP*

$$\begin{cases} D^{\alpha}_{0^+}u(t) = y(t), \qquad t \in J, \\ u(0) = 0, \quad D^{\alpha-1}_{0^+}u(\infty) = \int_{\eta}^{+\infty} g(t)u(t)dt, \end{cases} \tag{2.1}$$

has a unique soltuion

$$u(t) = \int_0^{+\infty} G(t,s)y(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)(\Gamma(\alpha) - \int_{\eta}^{+\infty} g(t)t^{\alpha-1}dt)}$$
$$\begin{cases} [H(\eta,s) - \Gamma(\alpha)]t^{\alpha-1} + (\Gamma(\alpha) - \int_{\eta}^{+\infty} g(t)t^{\alpha-1}dt)(t-s)^{\alpha-1}, & s \leq t, \\ [H(\eta,s) - \Gamma(\alpha)]t^{\alpha-1}, & s \geq t, \end{cases}$$

and

$$H(\eta,s) = \begin{cases} \int_{\eta}^{+\infty} g(t)(t-s)^{\alpha-1}dt, & s \leq \eta, \\ \int_{s}^{+\infty} g(t)(t-s)^{\alpha-1}dt, & s \geq \eta. \end{cases}$$

Proof. We may apply Lemma 2.5 to reduce the differential equation in (2.1) to the integral equation

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds.$$

In accordance with Lemma 2.3, Lemma 2.5 and the relation $D^{\alpha-1}_{0^+}t^{\alpha-1} = \Gamma(\alpha)$, we have

$$D^{\alpha-1}_{0^+}u(t) = c_1\Gamma(\alpha) + \int_0^t y(s)ds.$$

The boundary condition in (2.1) imply that $c_2 = 0$ and

$$c_1 = \frac{\int_0^{+\infty} H(\eta,s)y(s)ds - \Gamma(\alpha)\int_0^{+\infty} y(s)ds}{\Gamma(\alpha)(\Gamma(\alpha) - \beta\int_{\eta}^{+\infty} g(t)t^{\alpha-1}dt)},$$

where

$$H(\eta,s) = \begin{cases} \int_{\eta}^{+\infty} g(t)(t-s)^{\alpha-1}dt, & s \leq \eta, \\ \int_{s}^{+\infty} g(t)(t-s)^{\alpha-1}dt, & s \geq \eta. \end{cases}$$

Hence,

$$\begin{aligned} u(t) &= \frac{\int_0^{+\infty} H(\eta,s)y(s)ds - \Gamma(\alpha)\int_0^{+\infty} y(s)ds}{\Gamma(\alpha)(\Gamma(\alpha) - \int_{\eta}^{+\infty} g(t)t^{\alpha-1}dt)}t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds \\ &= \int_0^{+\infty} G(t,s)y(s)ds. \end{aligned}$$

This proof is complete. □

Lemma 2.7. *Let* $\int_\eta^{+\infty} g(t)t^{\alpha-1}dt < \Gamma(\alpha)$,*then*

$$|G(t,s)| \leq \frac{2t^{\alpha-1}}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt}. \tag{2.2}$$

Proof. Obviously, it holds

$$0 \leq H(\eta, s) \leq \int_\eta^{+\infty} g(t)t^{\alpha-1}dt,$$

therefore, when $s \leq t$, then

$$\begin{aligned}|G(t,s)| &\leq \frac{(\int_\eta^{+\infty} g(t)t^{\alpha-1}dt + \Gamma(\alpha))t^{\alpha-1} + (\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt)t^{\alpha-1}}{\Gamma(\alpha)(\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt)} \\ &\leq \frac{2t^{\alpha-1}}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt}.\end{aligned}$$

Obviously, when $s \geq t$, (2.2) holds. This proof is complete. □

Remark 2.8. By Lemma 2.6, it is easy to get

$$D_{0^+}^{\alpha-1}u(t) = \int_0^{+\infty} G_1(t,s)y(s)ds,$$

where

$$G_1(t,s) = \frac{1}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt} \begin{cases} H(\eta,s) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt, & s \leq t, \\ H(\eta,s) - \Gamma(\alpha), & s \geq t, \end{cases}$$

and if $\int_\eta^{+\infty} g(t)t^{\alpha-1}dt < \Gamma(\alpha)$, then

$$|G_1(t,s)| \leq \frac{2\Gamma(\alpha)}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt}. \tag{2.3}$$

Define the spaces by

$$\begin{aligned} X &= \{u(t) \in C(J,R) : \sup_{t\in J} \frac{|u(t)|}{1+t^{\alpha-1}} < +\infty\}, \\ C_\infty &= \{u(t) \in X : D_{0^+}^{\alpha-1}u(t) \in C(J,R), \quad \sup_{t\in J} |D_{0^+}^{\alpha-1}u(t)| < +\infty\}, \end{aligned}$$

and the norm $\|u\|_X = \sup_{t\in J} \frac{|u(t)|}{1+t^{\alpha-1}}$, $\|u\| = \max\{\sup_{t\in J} \frac{|u(t)|}{1+t^{\alpha-1}}, \sup_{t\in J} |D_{0^+}^{\alpha-1}u(t)|\}$.

Lemma 2.9 ([18])**.** $(X, \|\cdot\|_X)$ *and* $(C_\infty, \|\cdot\|)$ *are Banach spaces.*

Define operator $T : C_\infty \to C_\infty$,

$$Tu(t) := \int_0^{+\infty} G(t,s)f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds, \tag{2.4}$$

and

$$\begin{aligned} D_{0^+}^{\alpha-1}Tu(t) &= \frac{\int_0^{+\infty} H(\eta,s)f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds - \Gamma(\alpha)\int_0^{+\infty} f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt} \\ &\quad + \int_0^t f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds \\ &= \int_0^{+\infty} G_1(t,s)f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds. \end{aligned} \tag{2.5}$$

In this paper, our basic space is C_∞. Note that the Arzela-Ascoli theorem fails to work in C_∞. Therefore, we need the following compactness criterion.

Lemma 2.10 ([18]). *Let $Z \subseteq Y$ be a bounded set, then Z is relatively compact in Y if the following conditions hold:*

(i) For any $u(t) \in Z$, $\frac{u(t)}{1+t^{\alpha-1}}$ and $D_{0^+}^{\alpha-1}u(t)$ are equicontinuous on any compact interval of J;
(ii) Given $\varepsilon > 0$, there exists a constant $T = T(\varepsilon) > 0$ such that

$$\left|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}\right| < \varepsilon \quad \text{and} \quad |D_{0^+}^{\alpha-1}u(t_1) - D_{0^+}^{\alpha-1}u(t_2)| < \varepsilon$$

for any $t_1, t_2 \geq T$ and $u(t) \in Z$.

3. Main results

In this section, we apply fixed point theorem to IBVP (1.1). First, we give the uniqueness result based on Banach's contraction mapping principle.

Theorem 3.1. *Let $f : J \times R^2 \to R$ be an S-Caratheodory function and there exist $L_1(t), L_2(t) \in L^1[0, +\infty)$ such that*

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq L_1(t)|u_1 - u_2| + L_2(t)|v_1 - v_2|, \quad t \in I, \quad (u_1, v_1), (u_2, v_2) \in R^2.$$

In addition, suppose that $\Lambda < 1$ holds, where

$$\Lambda = \frac{2}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt} \int_0^{+\infty} [(1 + t^{\alpha-1})L_1(t) + L_2(t)]dt.$$

Then IBVP (1.1) has a unique solution.

Proof. Let us choose

$$r \geq \frac{2\int_0^{+\infty} \varphi_r(t)dt}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt - 2\int_0^{+\infty}[(1 + t^{\alpha-1})L_1(t) + L_2(t)]dt}.$$

Now, we show that $TB_r \subset B_r$, where $B_r = \{u \in C_\infty : \|u\| \leq r\}$.
For each $u \in B_r$, we have

$$\begin{aligned}
\frac{|Tu(t)|}{1+t^{\alpha-1}} &= \int_0^{+\infty} \frac{|G(t,s)|}{1+t^{\alpha-1}} \mid f(s, u(s), D_{0^+}^{\alpha-1}u(s)) \mid ds \\
&\leq \frac{2}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt} \int_0^{+\infty} |f(s, u(s), D_{0^+}^{\alpha-1}u(s))|ds \\
&\leq \frac{2}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt} \int_0^{+\infty} \Big[|f(s, u(s), D_{0^+}^{\alpha-1}u(s)) - f(s, 0, 0)| + |f(s, 0, 0)|\Big] ds \\
&\leq \frac{2}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt} \left(||u|| \int_0^{+\infty} [(1 + t^{\alpha-1})L_1(t) + L_2(t)]dt + \int_0^{+\infty} \varphi_r(t)dt\right) \\
&\leq r,
\end{aligned}$$

and

$$|D_{0^+}^{\alpha-1}Tu(t)| \leq \frac{2\Gamma(\alpha)}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt} \int_0^{+\infty} |f(s, u(s), D_{0^+}^{\alpha-1}u(s))|ds \leq \Gamma(\alpha)r \leq r.$$

Hence, we obtain that $||Tu|| \leq r$, so $TB_r \subset B_r$.

Next, for $u, v \in C_\infty$ and for each $t \in J$, we have

$$\begin{aligned}
\left|\frac{Tu(t)}{1+t^{\alpha-1}} - \frac{Tv(t)}{1+t^{\alpha-1}}\right| &= \frac{1}{1+t^{\alpha-1}}\left|\int_0^{+\infty} G(t,s)(f(s,u(s),D_{0^+}^{\alpha-1}u(s)) - f(s,v(s),D_{0^+}^{\alpha-1}v(s)))ds\right| \\
&\leq \frac{2}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\int_0^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s)) - f(s,v(s),D_{0^+}^{\alpha-1}v(s))|ds \\
&\leq \frac{2}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\int_0^{+\infty} \Big[L_1(s)|u_1(s) - u_2(s)| + L_2(s)|v_1(s) - v_2(s)|\Big]ds \\
&\leq \frac{2}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\int_0^{+\infty} [(1+t^{\alpha-1})L_1(t) + L_2(t)]dt||u-v|| \\
&\leq \Lambda||u-v|| < ||u-v||,
\end{aligned}$$

and

$$\begin{aligned}
|D_{0^+}^{\alpha-1}Tu(t) - D_{0^+}^{\alpha-1}Tv(t)| &= \left|\int_0^{+\infty} G_1(t,s)(f(s,u(s),D_{0^+}^{\alpha-1}u(s)) - f(s,v(s),D_{0^+}^{\alpha-1}v(s)))ds\right| \\
&\leq \frac{2\Gamma(\alpha)}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\int_0^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s)) - f(s,v(s),D_{0^+}^{\alpha-1}v(s))|ds \\
&< \Gamma(\alpha)||u-v|| \leq ||u-v||.
\end{aligned}$$

Therefore, we obtain that $||Tu - Tv|| < ||u - v||$, T is a contraction map. Thus, the conclusion of the theorem follows by Bananch's contraction mapping principle. □ □

The next existence result is based on the Schauder's fixed-point theorem.

Theorem 3.2. *Assume that there exists nonnegative functions* $p(t), q(t), r(t) \in L^1(J, R^+)$ *with* $t^{\alpha-1}p(t) \in L^1(J, R^+)$ *such that*

$$|f(t,u,v)| \leq p(t)|u| + q(t)|v| + r(t), \quad t \in J, \quad (u,v) \in R^2. \tag{3.1}$$

Then BVP (1.1) *has at least one solution provided*

$$\int_0^{+\infty} \Big[(1+t^{\alpha-1})p(t) + q(t)\Big]dt < \frac{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt}{2}.$$

Proof. First of all, by the continuity of f, we can conclude that $Tu(t)$ and $D_{0^+}^{\alpha-1}Tu(t)$ are continuous on J.

In what follows, we divide the proof into several steps.

Setp 1. Choose

$$R \geq \frac{2\int_0^{+\infty} r(s)ds}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt - 2\int_0^{+\infty}[(1+s^{\alpha-1})p(s) + q(s)]ds},$$

and let

$$U = \{u(t) \in C_\infty : ||u|| \leq R\}.$$

Then, $T : U \to U$.

Indeed, for any $u(t) \in U$, by (2.2)-(2.5) and the condition of Theorem 3.2, we can get

$$\frac{|Tu(t)|}{1+t^{\alpha-1}} \leq \frac{2}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\int_0^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds$$

$$\begin{aligned}
&\le \frac{2}{\Gamma(\alpha)-\int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\int_0^{+\infty}\Big[p(s)|u(s)|+q(s)|D_{0^+}^{\alpha-1}u(s)|+r(s)\Big]ds\\
&\le \frac{2}{\Gamma(\alpha)-\int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\Big[||u||\int_0^{+\infty}[(1+s^{\alpha-1})p(s)+q(s)]ds+\int_0^{+\infty}r(s)ds\Big]\\
&\le R,
\end{aligned}$$

and

$$\begin{aligned}
|D_{0^+}^{\alpha-1}Tu(t)| &\le \frac{2\Gamma(\alpha)}{\Gamma(\alpha)-\int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\int_0^{\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\
&\le \frac{2\Gamma(\alpha)}{\Gamma(\alpha)-\int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\Big[\int_0^{+\infty}[(1+s^{\alpha-1})p(s)+q(s)]ds+\int_0^{+\infty}r(s)ds\Big]\\
&\le \Gamma(\alpha)R\le R.
\end{aligned}$$

Hence, $||Tu||\le R$ and this show that $T:U\to U$.

Setp 2. $T:U\to U$ is continuous operator.

Let $u_n,u\in U,\ \ n=1,2,\cdots$, and $||u_n-u||\to 0$ as $n\to\infty$. Then by by (2.2)-(2.5) and the condition of Theorem 3.2, we obtain that

$$\begin{aligned}
&\left|\frac{Tu_n(t)}{1+t^{\alpha-1}}-\frac{Tu(t)}{1+t^{\alpha-1}}\right|\\
&\quad=\frac{1}{1+t^{\alpha-1}}\left|\int_0^{+\infty}G(t,s)(f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s))-f(s,u(s),D_{0^+}^{\alpha-1}u(s)))ds\right|\\
&\quad\le \frac{2}{\Gamma(\alpha)-\int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\int_0^{+\infty}|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s))-f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\
&\quad\le \frac{2}{\Gamma(\alpha)-\int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\Big[\int_0^{+\infty}|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s))|ds+\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\Big]\\
&\quad\le \frac{2}{\Gamma(\alpha)-\int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\Big[||u_n||\int_0^{+\infty}[(1+s^{\alpha-1})p(s)+q(s)]ds+\int_0^{+\infty}r(s)ds\\
&\qquad+||u||\int_0^{+\infty}[(1+s^{\alpha-1})p(s)+q(s)]ds+\int_0^{+\infty}r(s)ds\Big]\\
&\quad\le \frac{4R\int_0^{+\infty}[(1+s^{\alpha-1})p(s)+q(s)]ds+4\int_0^{+\infty}r(s)ds}{\Gamma(\alpha)-\int_\eta^{+\infty} g(t)t^{\alpha-1}dt},
\end{aligned}$$

and

$$\begin{aligned}
&|D_{0^+}^{\alpha-1}Tu_n(t)-D_{0^+}^{\alpha-1}Tu(t)|\\
&\qquad\le \frac{2\Gamma(\alpha)}{\Gamma(\alpha)-\int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\int_0^{+\infty}|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s))-f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\
&\qquad\le \frac{4\Gamma(\alpha)}{\Gamma(\alpha)-\int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\Big[R\int_0^{+\infty}[(1+s^{\alpha-1})p(s)+q(s)]ds+\int_0^{+\infty}r(s)ds\Big].
\end{aligned}$$

Therefore, by Lebesgue's dominated convergence theorem, we have $||Tu_n-Tu||\to 0$ as $n\to\infty$. Hence, T is continuous.

Setp 3. Let V be a subset of U. We apply Lemma 2.10 to verify that TV is relatively compact.

Let $I \in J$ be a compact interval, $t_1, t_2 \in I$ and $t_1 < t_2$. Then for any $u(t) \in V$, it is easy to know $f(t, u(t), D_{0^+}^{\alpha-1}u(t))$ is bounded on I, so we can obtain that

$$\begin{aligned}
&\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}} - \frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right| \\
&\le \frac{\left|\int_0^{+\infty} H(\eta,s) f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds - \Gamma(\alpha)\int_0^{+\infty} f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\right|}{\Gamma(\alpha)\left[\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt\right]}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}} f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds\right| \\
&\le \frac{\int_0^{+\infty} H(\eta,s)|f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds + \Gamma(\alpha)\int_0^{+\infty} |f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds}{\Gamma(\alpha)\left[\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt\right]}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right| \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}|f(s,u(s),D_{0^+}^{\alpha-1}u(s))ds \\
&\le \frac{2\left[R\int_0^{+\infty}[(1+s^{\alpha-1})p(s)+q(s)]ds + \int_0^{+\infty} r(s)ds\right]}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right| \\
&\quad + \frac{\max|f(s,u(s),D_{0^+}^{\alpha-1}u(s))|}{\Gamma(\alpha)}\left[\int_{t_1}^{t_2}\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}ds + \int_0^{t_1}\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right|ds\right], \\
&\to 0, \text{ uniformly as } t_1 \to t_2,
\end{aligned}$$

and

$$|D_{0^+}^{\alpha-1}Tu(t_2) - D_{0^+}^{\alpha-1}Tu(t_1)| \le \int_{t_1}^{t_2} |f(t,u(t),D_{0^+}^{\alpha-1}u(t))|ds \to 0, \text{uniformly as } t_1 \to t_2.$$

Then it is easy to see that $\frac{Tu(t)}{1+t^{\alpha-1}}$ and $D_{0^+}^{\alpha-1}Tu(t)$ are equicontinuous on I.

Now, we show that for any $u(t) \in V$, $\frac{Tu(t)}{1+t^{\alpha-1}}$ and $D_{0^+}^{\alpha-1}Tu(t)$ satisfy the condition (ii) of Lemma 2.10. Observing that by the condition of Theorem 3.2, we have

$$\begin{aligned}
\int_0^{+\infty} |f(t,u(t),D_{0^+}^{\alpha-1}u(t))|dt &\le ||u||\int_0^{+\infty}[(1+t^{\alpha-1})p(t)+q(t)]dt + \int_0^{+\infty} r(t)dt \\
&\le \frac{2}{\Gamma(\alpha) - \int_\eta^{+\infty} g(t)t^{\alpha-1}dt}\left[||u||\int_0^{+\infty}[(1+t^{\alpha-1})p(t)+q(t)]dt + \int_0^{+\infty} r(t)dt\right] \\
&\le R,
\end{aligned}$$

we know that for given $\varepsilon > 0$, there exists a constant $L > 0$, such that

$$\int_L^{+\infty} |f(t,u(t),D_{0^+}^{\alpha-1}u(t))|dt < \varepsilon. \tag{3.2}$$

On the other hand, since $\lim_{t\to+\infty} \frac{t^{\alpha-1}}{1+t^{\alpha-1}} = 1$, there exists a constant $T_1 > 0$, such that for any $t_1, t_2 \ge T_1$, we have

$$\left|\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\right| < \varepsilon. \tag{3.3}$$

Similarly, $\lim\limits_{t\to+\infty}\frac{(t-L)^{\alpha-1}}{1+t^{\alpha-1}}=1$ and thus there exists a constant $T_2>L>0$, such that for any $t_1,t_2\ge T_2$ and $0\le s\le L$,

$$\left|\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}\right|<\varepsilon. \tag{3.4}$$

Now, choose $T_0>\max\{T_1,T_2\}$, then for any $t_1,t_2\ge T_0$, by (3.2)-(3.4), we can obtain that

$$\begin{aligned}\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}}-\frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right| &\le \frac{2\Big[R\int_0^{+\infty}[(1+s^{\alpha-1})p(s)+q(s)]ds+\int_0^{+\infty}r(s)ds\Big]}{\Gamma(\alpha)-\int_\eta^{+\infty}g(t)t^{\alpha-1}dt}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^L\left|\frac{(t_2-s)^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{(t_1-s)^{\alpha-1}}{1+t_1^{\alpha-1}}\right||f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\left[\int_L^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds+\int_L^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s))|ds\right]\\ &\le \frac{2\Big[R\int_0^{+\infty}[(1+s^{\alpha-1})p(s)+q(s)]ds+\int_0^{+\infty}r(s)ds\Big]}{\Gamma(\alpha)-\int_\eta^{+\infty}g(t)t^{\alpha-1}dt}\varepsilon\\ &\quad+\frac{\max\limits_{t\in[0,L],u\in V}|f(s,u(s),D_{0^+}^{\alpha-1}u(s))|}{\Gamma(\alpha)}L\varepsilon+\frac{2}{\Gamma(\alpha)}\varepsilon,\end{aligned}$$

and

$$\begin{aligned}|D_{0^+}^{\alpha-1}Tu(t_2)-D_{0^+}^{\alpha-1}Tu(t_1)| &\le \int_{t_1}^{t_2}|f(t,u(t),D_{0^+}^{\alpha-1}u(t))|ds\\ &\le \int_L^{+\infty}|f(t,u(t),D_{0^+}^{\alpha-1}u(t))|ds<\varepsilon.\end{aligned}$$

Consequently, lemma 2.10 yields that TV is relatively compact.

Therefore, by Schauder's fixed point theorem, we conclude that the BVP (1.1) has at least one solutions in U and the proof is finished. □ □

4. Example

Example 4.1. Consider the following IBVP for fractional differential equation on infinite intervals:

$$\begin{cases} D_{o^+}^{\frac{3}{2}}u(t)=e^{-t}+\frac{1}{10(1+t^2)(1+\sqrt{t})}\sin(u(t))+\frac{1}{4(1+e^t)}\arctan(D_{0^+}^{\frac{1}{2}}u(t)), & t\in J,\\ u(0)=0, \quad D_{0^+}^{\frac{1}{2}}u(+\infty)=\int_1^{+\infty}\frac{1}{10}t^{-\frac{1}{2}}e^{-t}u(t)dt. \end{cases} \tag{4.1}$$

Here, $\alpha=\frac{3}{2}$, $\eta=1$, $f(t,u,v)=e^{-t}+\frac{1}{10(1+t^2)(1+\sqrt{t})}\sin(u)+\frac{1}{4(1+e^t)}\arctan(v)$, $L_1(t)=\frac{1}{10(1+t^2)(1+\sqrt{t})}$, $L_2(t)=\frac{1}{4(1+e^t)}$ and $g(t)=\frac{1}{10}t^{-\frac{1}{2}}e^{-t}$. With the aid of simple computation, we can obtain that

$$\left|f(t,u_1,v_1)-f(t,u_2,v_2)\right|\le\frac{1}{10(1+t^2)(1+\sqrt{t})}|u_1-u_2|+\frac{1}{4(1+e^t)}|v_1-v_2|,$$

$\int_\eta^{+\infty}g(t)t^{\alpha-1}dt=\int_1^{+\infty}\frac{1}{10}e^{-t}dt<\Gamma(\frac{3}{2})$ and $\Lambda\approx 0.77785<1$. In the view of Theorem 3.1, then IBVP (4.1) has a unique solution.

Example 4.2. Consider the following IBVP for fractional differential equation on infinite intervals:

$$\begin{cases} D_{o^+}^{\frac{5}{4}}u(t) = \frac{\ln(1+|u(t)|)}{10(1+t^2)(1+\sqrt[4]{t})} + \frac{1}{10e^t}\sin(|D_{0^+}^{\frac{1}{4}}u(t)|) + te^{-t^2}, \quad t \in J, \\ u(0) = 0, \quad D_{0^+}^{\frac{1}{4}}u(+\infty) = \int_1^{+\infty} \frac{1}{2}t^{\frac{3}{4}}e^{-t}u(t)dt. \end{cases} \tag{4.2}$$

Here, $\alpha = \frac{5}{4}$, $\eta = 1$, $f(t,u,v) = \frac{\ln(1+|u|)}{10(1+t^2)(1+\sqrt[4]{t})} + \frac{1}{10e^t}\sin(|v|) + te^{-t^2}$ and $g(t) = \frac{1}{2}e^{-t}$. Obviously,

$$|f(t,u,v)| \leq \frac{|u|}{10(1+t^2)(1+\sqrt[4]{t})} + \frac{1}{10e^t}|v| + te^{-t^2},$$

With the aid of simple computation, we find that

$$\int_0^{+\infty} \left[(1+t^{\frac{1}{4}})\frac{1}{10(1+t^2)} + \frac{1}{10e^t}\right]dt = \frac{\pi}{20} + \frac{1}{10} \approx 0.25708,$$

and

$$\frac{\Gamma(\frac{5}{4}) - \frac{1}{2}\int_1^{+\infty} te^{-t}dt}{2} \approx \frac{0.906402 - \frac{1}{2} \times 0.73576}{2} = 0.26926,$$

Hence, the conditions of Theorem 3.2 we satisfied, so IBVP (4.2) has at least one solution.

Acknowledgements:

This paper is supported by the Natural Science Foundation of China (11201112), the Natural Science Foundation of Hebei Province (A2013208147), (A2014208152), and (A2015208114), and the Foundation of Hebei Education Department (Z2014062) and (QN2015175).

References

[1] R. P. Agarwal, D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publisher, Dordrecht, (2001). 1

[2] R. P. Agarwal, D. O'Regan, *Nonlinear boundary value problems on the semi-infinite interval: An upper and lower solution approach*, Mathematika, **49** (2002), 129–140. 1

[3] R. P. Agarwal, V. Lakshmikantham, J. J. Nieto, *On the concept of solution for fractional differential equations with uncertainty*, Nonlinear Anal., **72** (2010), 2859–2862. 1

[4] B. Ahmad, *Existence of solutions for irregular value problems of nonlinear fractional differential equations*, Appl. Math. Lett., **23** (2010), 390–394. 1

[5] B. Ahmad, A. Alsaedi, B. S. Alghamdi, *Analytic approximation of solutions of the forced Duffing equation with integral boundary conditions*, Nonlinear Anal. Real World Appl., **9** (2008), 1727–1740. 1

[6] A. Arara, M. Benchohra, N. Hamidi, J. J. Nieto, *Fractional order differential equations on an unbounded domain*, Nonlinear Anal., **72** (2010), 580–586. 1

[7] J. V. Baxley, *Existence and uniqueness for nonlinear boundary value problems on infinite interval*, J. Math. Anal. Appl., **147** (1990), 122–133. 1

[8] J. Caballero, J. Harjani, K. Sadarangani, *Positive solutions for a class of singular fractional boundary value problem*, Comput. Math. Appl., **62** (2011), 1325–1332. 1

[9] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Elsevier Science B.V, Amsterdam, (2006). 1, 2, 2.3

[10] V. Lakshmikantham, *Theory of fractional functional differential equations*, Nonlinear Anal., **69** (2008), 3337–3343. 1

[11] V. Lakshmikantham, S. Leela, J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic, Cambridge, UK, (2009). 1

[12] V. Lakshmikantham, A. S. Vatsala, *General uniqueness and monotone iterative technique for fractional differential equations*, Appl. Math. Lett., **21** (2008), 828–834. 1

[13] H. Lian, W. Ge, *Solvability for second-order three-point boundary value problems on a half-line*, Appl. Math. Lett., **19** (2006), 1000–1006. 1, 2.4

[14] K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equation*, Wiley, New York, (1993). 1

[15] D. O'Regan, *Theory of Singular Boundary Value Problems*, World Scientific, River Edge, New Jersey, USA, (1994). 1

[16] I. Podlubny, *Fractional differential equations, in: Mathematics in Science and Engineering*, **158**, Academic Press, Inc., San Diego, CA, (1999). 1, 2

[17] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integral and Derivative, in:Theory and Applications*, Gordon and Breach, Switzerland, (1993). 1, 2

[18] X. Su, S. Zhang, *Unbounded solutions to a boundary value problem of fractional order on the half-line*, Comput. Math. Appl., **61** (2011), 1079–1087. 1, 2.5, 2.9, 2.10

[19] Y. Wang, L. Liu, *Positive solutions for fractional m-point boundary value problem in Banach spaces*, Acta. Math. Sci., **32** (2012), 246–256. 1

[20] L. Wang, X. Zhang, *Positive solutions of m-point boundary value problems for a class of nonlinear fractional differential equations*, J. Appl. Math. Comput., **42** (2013), 387–399. 1

[21] J. R. L. Webb, G. Infante, *Positive solutions of nonlocal boundary value problems involving integral conditions*, Nonlinear Differ. Equ. Appl., **15** (2008), 45–67. 1

[22] C. Yu, Z. Li, H. Wei, J. Wang, *Existence and uniqueness of solutions for second-order m-point boundary value problems at resonance on infinite interval*, J. Hebei Univ. Sci. Tech., **34** (2013), 7–14. 1

[23] C. Yu, J. Wang, Y. Guo, *Positive solutions for boundary-value problems with integral boundary conditions on infinite interval*, Electr. J. Differ. Equ., **2012** (2012), 9 pages. 1

[24] S. Zhang, *The existence of a positive solution for a nonlinear fractional differential equation*, J. Math. Anal. Appl., **252** (2000), 804–812. 1

[25] X. Zhao, W. Ge, *Unbounded solutions for a fractional boundary value problem on the infinite interval*, Acta. Appl. Math., **109** (2010), 495–505. 1

[26] X. Zhang, W. Ge, *Symmetric positive solutions of boundary value problems with integral boundary conditions*, Appl. Math. Comput., **219** (2012), 3553–3564. 1

[27] Y. Zhou, F. Jiao, *Nonlocal Cauchy problem for fractional evolution equations*, Nonlinear Anal. Real World Appl., **11** (2010), 4465–4475. 1