Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2520–2526
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

An integrating factor approach to the Hyers-Ulam stability of a class of exact differential equations of second order

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Communicated by R. Saadati

Abstract

Using the integrating factor method, this paper deals with the Hyers-Ulam stability of a class of exact differential equations of second order. As a direct application of the main result, we also obtain the Hyers-Ulam stability of a special class of Cauchy-Euler equations of second order.

Keywords: Integrating factor method, Hyers-Ulam stability, exact differential equation, Cauchy-Euler equation.
2010 MSC: 34D20, 34G10.

1. Introduction

The Ulam stability (it consists primarily of Hyers-Ulam stability and Hyers-Ulam-Rassias stability) has gradually become an important research branch in the theory of differential equations. Precisely, for an n-th order differential equation

$$F(x, y, y', \cdot, y^n) = 0, \qquad x \in T,$$

where T is a subinterval of the real line $\mathbb{R}$. We say that it has the Hyers-Ulam stability or it is stable in the sense of Hyers-Ulam if for a given $\epsilon > 0$ and an n times differentiable function f satisfies the differential inequality

$$|F(x, y, y', \cdot, y^n)| \leq \epsilon$$

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Received 2015-10-27

for all $x \in T$, then there exists a solution $g : T \to \mathbb{R}$ of the preceding differential equation such that $|f(x) - g(x)| \leq K(\epsilon)$ for all $x \in T$, where $K(\epsilon)$ depends only on ϵ and $\lim_{\epsilon \to 0} K(\epsilon) = 0$. More generally, if the ϵ and $K(\epsilon)$ mentioned above are replaced by two control functions $\varphi(x)$ and $\Phi(x)$, respectively, then we say that the preceding differential equation has the Hyers-Ulam-Rassias stability(or generalized Hyers-Ulam stability).

Obloza [16] seems to be the first author who is devoted to the study of the Ulam stability of differential equations. A few years later, Alsina and Ger [2] considered the Hyers-Ulam stability of the differential equation $y' = y$. Soon afterwards, such stability results of the differential equation $y' = \lambda y$ in various abstract spaces have been obtained by Miura and Takahasi et al. [12, 13, 22]. Since then, many interesting results on the Ulam stability of different types of differential equations have been established by various authors [1, 3, 4, 5, 7, 8, 9, 10, 11, 14, 15, 17, 18, 19, 20, 21, 23].

Note that various methods have been used to study the Ulam stability of differential equations, such as the direct method, integrating factor method, power series method, Laplace transform methods, the fixed point technique and so on. In [23], the authors investigated the Hyers-Ulam stability of linear differential equation of first order by using the integrating factor method. Inspired by this method, this paper aims to consider the Hyers-Ulam stability of a class of exact differential equations of second order.

2. Preliminaries

Definition 2.1 ([6])**.** A second order differential equation

$$P(x)y''(x) + Q(x)y'(x) + R(x) = 0 \tag{2.1}$$

is said to be *exact* if $P''(x) - Q'(x) + R(x) = 0$.

By a simple calculation, we know that (2.1) can be written as in the following form

$$[P(x)y'(x)]' + [W(x)y(x)]' = 0, \tag{2.2}$$

where $W(x) = Q(x) - P'(x)$. Integrating both sides of (2.2) gives the following first order linear differential equation

$$P(x)y' + W(x)y = P(x)y' + (Q(x) - P'(x))y = c, \tag{2.3}$$

where c is the integration constant. For any integration constant c, it is easy to see that the solution of (2.3) must be the solution of (2.2), and hence the solution of (2.1).

Throughout this paper, let $I = (a, b)$, $-\infty < a < b < +\infty$, denote the open interval of the real line $\mathbb{R}$. Moreover, we denote by $C[I, \mathbb{R}]$, $C^1[I, \mathbb{R}]$ and $C^2[I, \mathbb{R}]$ the set of all real continuous functions on I, the set of all differentiable functions which have a continuous derivative on I and the set of all differentiable functions which have a second continuous derivative on I, respectively.

3. Main results

In this section, we shall prove the Ulam stability of the second order exact differential equation (2.1) on a bounded open interval.

Theorem 3.1. *Let $P \in C^2[I, \mathbb{R}]$, $Q \in C^1[I, \mathbb{R}]$ and $R \in C[I, \mathbb{R}]$ with $P(x) \neq 0$ on I and $P''(x) - Q'(x) + R(x) = 0$. Let $W(x) = Q(x) - P'(x)$ such that $|W(x)| \geq \alpha$ on I for some $\alpha > 0$ that is independent of x. For a given $\epsilon > 0$, if $f \in C^2[I, \mathbb{R}]$ satisfies the differential inequality*

$$|P(x)f''(x) + Q(x)f'(x) + R(x)f(x)| \leq \epsilon \tag{3.1}$$

for all $x \in I$, then there exists a solution $h \in C^2[I, \mathbb{R}]$ of (2.1) such that

$$|f(x) - h(x)| \leq 3\epsilon(b - a)|N|e^{\int_a^b \left|\frac{W(s)}{P(s)}\right| ds} \tag{3.2}$$

for all $x \in I$, where N is an integer with the minimum absolute value such that $NW(x) \geq 1$ and

$$h(x) = e^{-\int_a^x \frac{W(s)}{P(s)}ds}\Big[\Big(f(b_1) + \epsilon|N|(b_1 - a)\Big)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds} - \int_x^{b_1} \frac{M}{P(t)} e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big],$$

b_1 is any fixed point in I such that $y(b_1)$ is finite.

Proof. We may assume that $P(x) > 0$ for all $x \in I$. By the inequality (3.1), we get

$$-\epsilon \leq P(x)f''(x) + Q(x)f'(x) + R(x)f(x) \leq \epsilon$$

for all $x \in I$. Since $P''(x) - Q'(x) + R(x) = 0$, we can obtain

$$-\epsilon \leq [P(x)f'(x)]' + [(Q(x) - P'(x))f(x)]' = [P(x)f'(x)]' + [W(x)f(x)]' \leq \epsilon \tag{3.3}$$

for all $x \in I$. Integrating both sides of the inequality (3.3) from a to x, it follows that

$$-\epsilon(x - a) \leq P(x)f'(x) - P(a)f'(a) + W(x)f(x) - W(a)f(a) \leq \epsilon(x - a)$$

for all $x \in I$. Setting $M = P(a)f'(a) + W(a)f(a)$. Then, we have

$$-\epsilon(x - a) \leq P(x)f'(x) + W(x)f(x) - M \leq \epsilon(x - a) \tag{3.4}$$

for all $x \in I$. Multiplying both sides of the inequality (3.4) by the function $\frac{1}{P(x)}e^{\int_a^x \frac{W(s)}{P(s)}ds}$, we can infer that

$$\begin{aligned} -\epsilon(x - a)\frac{1}{P(x)}e^{\int_a^x \frac{W(s)}{P(s)}ds} &\leq f'(x)e^{\int_a^x \frac{W(s)}{P(s)}ds} + f(x)\frac{W(x)}{P(x)}e^{\int_a^x \frac{W(s)}{P(s)}ds} - \frac{M}{P(x)}e^{\int_a^x \frac{W(s)}{P(s)}ds} \\ &\leq \epsilon(x - a)\frac{1}{P(x)}e^{\int_a^x \frac{W(s)}{P(s)}ds} \end{aligned} \tag{3.5}$$

for all $x \in I$. Since $|W(x)| \geq \alpha > 0$, there exists an integer number N with the minimum absolute value such that $NW(x) \geq 1$. Then, it follows from (3.5) that

$$\begin{aligned} -\epsilon N(x - a)\frac{W(x)}{P(x)}e^{\int_a^x \frac{W(s)}{P(s)}ds} &\leq f'(x)e^{\int_a^x \frac{W(s)}{P(s)}ds} + f(x)\frac{W(x)}{P(x)}e^{\int_a^x \frac{W(s)}{P(s)}ds} - \frac{M}{P(x)}e^{\int_a^x \frac{W(s)}{P(s)}ds} \\ &\leq \epsilon N(x - a)\frac{W(x)}{P(x)}e^{\int_a^x \frac{W(s)}{P(s)}ds}. \end{aligned} \tag{3.6}$$

Choosing $b_1 \in I$ such that $f(b_1)$ is finite. For any $x \in (a, b_1]$, by integrating both sides of (3.6) from x to b_1 with respect to t, we have

$$\begin{aligned} -\epsilon N\int_x^{b_1}(t - a)\frac{W(t)}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt &\leq f(b_1)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds} - f(x)e^{\int_a^x \frac{W(s)}{P(s)}ds} - \int_x^{b_1} \frac{M}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt \\ &\leq \epsilon N\int_x^{b_1}(t - a)\frac{W(t)}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt. \end{aligned}$$

Integrating by parts leads to

$$\begin{aligned} &-\epsilon N\Big((b_1 - a)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds} - (x - a)e^{\int_a^x \frac{W(s)}{P(s)}ds} - \int_x^{b_1} e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big) \\ &\leq f(b_1)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds} - f(x)e^{\int_a^x \frac{W(s)}{P(s)}ds} - \int_x^{b_1} \frac{M}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt \\ &\leq \epsilon N\Big((b_1 - a)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds} - (x - a)e^{\int_a^x \frac{W(s)}{P(s)}ds} - \int_x^{b_1} e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big). \end{aligned} \tag{3.7}$$

For the sake of clarity, we divide it into two cases to estimate the above equality:
Case I: when $N > 0$, we have

$$\begin{aligned}
&\epsilon N\Big((x-a)e^{\int_a^x \frac{W(s)}{P(s)}ds} + \int_x^{b_1} e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big) \\
&\quad \le \Big(f(b_1) + \epsilon N(b_1 - a)\Big)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds} - f(x)e^{\int_a^x \frac{W(s)}{P(s)}ds} - \int_x^{b_1} \frac{M}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt \\
&\quad \le \epsilon N\Big(2(b_1 - a)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds} - (x-a)e^{\int_a^x \frac{W(s)}{P(s)}ds} - \int_x^{b_1} e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big).
\end{aligned}$$

Further, we can obtain that

$$\begin{aligned}
\epsilon N(x-a)e^{\int_a^x \frac{W(s)}{P(s)}ds} &\le \Big(f(b_1) + \epsilon N(b_1 - a)\Big)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds} - f(x)e^{\int_a^x \frac{W(s)}{P(s)}ds} - \int_x^{b_1} \frac{M}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt \\
&\le \epsilon N\Big(2(b_1 - a)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds} - (x-a)e^{\int_a^x \frac{W(s)}{P(s)}ds}\Big).
\end{aligned}$$

Then, we conclude that

$$\begin{aligned}
\epsilon N(x-a) &\le e^{-\int_a^x \frac{W(s)}{P(s)}ds}\Big[\Big(f(b_1) + \epsilon N(b_1 - a)\Big)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds} - \int_x^{b_1} \frac{M}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big] - f(x) \\
&\le \epsilon N\Big(2(b_1 - a)e^{\int_x^{b_1} \frac{W(s)}{P(s)}ds} - (x-a)\Big).
\end{aligned}$$

Case II: when $N < 0$, we get

$$\begin{aligned}
&-\epsilon N\Big(2(b_1 - a)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds} - (x-a)e^{\int_a^x \frac{W(s)}{P(s)}ds} - \int_x^{b_1} e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big) \\
&\quad \le \Big(f(b_1) - \epsilon N(b_1 - a)\Big)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds} - f(x)e^{\int_a^x \frac{W(s)}{P(s)}ds} - \int_x^{b_1} \frac{M}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt \\
&\quad \le -\epsilon N\Big((x-a)e^{\int_a^x \frac{W(s)}{P(s)}ds} + \int_x^{b_1} e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big).
\end{aligned}$$

Thus, we can infer that

$$\begin{aligned}
&\epsilon N\Big((x-a)e^{\int_a^x \frac{W(s)}{P(s)}ds} + \int_x^{b_1} e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big) \\
&\quad \le \Big(f(b_1) - \epsilon N(b_1 - a)\Big)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds} - f(x)e^{\int_a^x \frac{W(s)}{P(s)}ds} - \int_x^{b_1} \frac{M}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt \\
&\quad \le -\epsilon N\Big((x-a)e^{\int_a^x \frac{W(s)}{P(s)}ds} + \int_x^{b_1} e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big).
\end{aligned}$$

Therefore, we can obtain

$$\begin{aligned}
&\epsilon N\Big((x-a) + e^{\int_a^x -\frac{W(s)}{P(s)}ds}\int_x^{b_1} e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big) \\
&\quad \le e^{\int_a^x -\frac{W(s)}{P(s)}ds}\Big[\Big(f(b_1) - \epsilon N(b_1 - a)\Big)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds} - \int_x^{b_1} \frac{M}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big] - f(x) \\
&\quad \le -\epsilon N\Big((x-a) + e^{\int_a^x -\frac{W(s)}{P(s)}ds}\int_x^{b_1} e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big).
\end{aligned}$$

Using the integral mean value theorem, we get

$$\begin{aligned}
&\epsilon N\Big((x-a)+(b_1-x)e^{\int_x^\xi \frac{W(s)}{P(s)}ds}\Big)\\
&\quad\le e^{-\int_a^x \frac{W(s)}{P(s)}ds}\Big[\Big(f(b_1)-\epsilon N(b_1-a)\Big)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds}-\int_x^{b_1}\frac{M}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big]-f(x)\\
&\quad\le -\epsilon N\Big((x-a)+(b_1-x)e^{\int_x^\xi \frac{W(s)}{P(s)}ds}\Big),
\end{aligned}$$

where $\xi\in[x,b_1]$.

Based on the above two cases, we conclude that

$$\begin{aligned}
&-\epsilon|N|\Big((x-a)+2(b_1-a)e^{\int_x^{b_1} \frac{W(s)}{P(s)}ds}\Big)\\
&\quad\le e^{-\int_a^x \frac{W(s)}{P(s)}ds}\Big[\Big(f(b_1)+\epsilon|N|(b_1-a)\Big)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds}-\int_x^{b_1}\frac{M}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big]-f(x)\\
&\quad\le \epsilon|N|\Big((x-a)+2(b_1-a)e^{\int_x^{b_1} \frac{W(s)}{P(s)}ds}\Big).
\end{aligned}\tag{3.8}$$

By an argument similar to the above, for any $x\in[b_1,b)$, integrating both sides of (3.6) from b_1 to x with respect to t, we can infer that

$$\begin{aligned}
&-\epsilon N\Big((x-a)e^{\int_a^x \frac{W(s)}{P(s)}ds}-(b_1-a)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds}-\int_{b_1}^x e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big)\\
&\quad\le f(x)e^{\int_a^x \frac{W(s)}{P(s)}ds}-f(b_1)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds}-\int_x^{b_1}\frac{M}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\\
&\quad\le \epsilon N\Big((x-a)e^{\int_a^x \frac{W(s)}{P(s)}ds}-(b_1-a)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds}-\int_{b_1}^x e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big).
\end{aligned}$$

Similarly, by dividing the above inequality into two cases, we have
Case I: when $N>0$,

$$\begin{aligned}
&-\epsilon N\Big((x-a)-\int_{b_1}^x e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big)\\
&\quad\le f(x)-e^{-\int_a^x \frac{W(s)}{P(s)}ds}\Big[\Big(f(b_1)+\epsilon N(b_1-a)\Big)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds}-\int_x^{b_1}\frac{M}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big]\\
&\quad\le \epsilon N\Big((x-a)-2(b_1-a)e^{\int_x^{b_1} \frac{W(s)}{P(s)}ds}-\int_{b_1}^x e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big).
\end{aligned}$$

Furthermore, we can obtain

$$\begin{aligned}
-\epsilon N(x-a)&\le f(x)-e^{-\int_a^x \frac{W(s)}{P(s)}ds}\Big[\Big(f(b_1)+\epsilon N(b_1-a)\Big)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds}-\int_x^{b_1}\frac{M}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big]\\
&\le \epsilon N(x-a).
\end{aligned}$$

Case II: when $N<0$,

$$\begin{aligned}
&\epsilon N\Big(2(b_1-a)e^{-\int_{b_1}^x \frac{W(s)}{P(s)}ds}+e^{-\int_a^x \frac{W(s)}{P(s)}ds}\int_{b_1}^x e^{\int_a^t \frac{W(s)}{P(s)}ds}dt-(x-a)\Big)\\
&\quad\le f(x)-e^{-\int_a^x \frac{W(s)}{P(s)}ds}\Big[\Big(f(b_1)-\epsilon N(b_1-a)\Big)e^{\int_a^{b_1} \frac{W(s)}{P(s)}ds}-\int_x^{b_1}\frac{M}{P(t)}e^{\int_a^t \frac{W(s)}{P(s)}ds}dt\Big]\\
&\quad\le -\epsilon N\Big(e^{-\int_a^x \frac{W(s)}{P(s)}ds}\int_{b_1}^x e^{\int_a^t \frac{W(s)}{P(s)}ds}dt-(x-a)\Big).
\end{aligned}$$

Using the integral mean value theorem, it follows that

$$\begin{aligned}\epsilon N(3(x-a)e^{\int_{b_1}^{x}\frac{|W(s)|}{P(s)}ds}) &\le \epsilon N\Big(2(b_1-a)e^{-\int_{b_1}^{x}\frac{W(s)}{P(s)}ds}+e^{-\int_{\eta}^{x}\frac{W(s)}{P(s)}ds}(x-b_1)-(x-a)\Big)\\ &\le f(x)-e^{-\int_a^x\frac{W(s)}{P(s)}ds}\Big[\Big(f(b_1)-\epsilon N(b_1-a)\Big)e^{\int_a^{b_1}\frac{W(s)}{P(s)}ds}-\int_x^{b_1}\frac{M}{P(t)}e^{\int_a^t\frac{W(s)}{P(s)}ds}dt\Big]\\ &\le -\epsilon N\Big(e^{-\int_{\eta}^{x}\frac{W(s)}{P(s)}ds}(x-b_1)-(x-a)\Big)\\ &\le -\epsilon N(3(x-a)e^{\int_{b_1}^{x}\frac{|W(s)|}{P(s)}ds}),\end{aligned}$$

where $\eta \in [b_1, x]$.

Taking all these two cases, we can obtain

$$\begin{aligned}&-\epsilon|N|(3(x-a)e^{\int_{b_1}^{x}\frac{|W(s)|}{P(s)}ds})\\ &\quad\le f(x)-e^{-\int_a^x\frac{W(s)}{P(s)}ds}\Big[\Big(f(b_1)+\epsilon|N|(b_1-a)\Big)e^{\int_a^{b_1}\frac{W(s)}{P(s)}ds}-\int_x^{b_1}\frac{M}{P(t)}e^{\int_a^t\frac{W(s)}{P(s)}ds}dt\Big]\\ &\quad\le \epsilon|N|(3(x-a)e^{\int_{b_1}^{x}\frac{|W(s)|}{P(s)}ds}).\end{aligned}\tag{3.9}$$

From (3.8) and (3.9), it follows that

$$|f(x)-h(x)| \le 3\epsilon(b-a)|N|e^{\int_a^b\frac{|W(s)|}{P(s)}ds}$$

for all $x \in I$, where

$$h(x)=e^{-\int_a^x\frac{W(s)}{P(s)}ds}\Big[\Big(f(b_1)+\epsilon|N|(b_1-a)\Big)e^{\int_a^{b_1}\frac{W(s)}{P(s)}ds}-\int_x^{b_1}\frac{M}{P(t)}e^{\int_a^t\frac{W(s)}{P(s)}ds}dt\Big].$$

By a tedious calculation, it is easy to verify that

$$P(x)h'(x)+W(x)h(x)+M=0, \qquad x\in I,$$

which implies that $h(x)$ is also a solution of (2.1), since M is a constant. This completes the proof of the theorem. □

Remark 3.2. Theorem 3.1 shows that the error estimation is independent of the choice of the point b_1. That is to say, the upper bound of the error is always valid for the solution $h(x)$ of (2.1) obtained by using any initial value b_1.

As a direct consequence of Theorem 3.1, we can obtain the Hyers-Ulam stability result of a special class of Cauchy-Euler equations.

Corollary 3.3. *Let $0 \notin I = (a, b)$ and let $A, B, C \in \mathbb{R}$ with $A \neq 0$, $B = 2A + C$ and $B - 2A = C \neq 0$. For a given $\epsilon > 0$, if $f \in C^2[I, \mathbb{R}]$ satisfies the differential inequality*

$$|Ax^2f''(x)+Bxf'(x)+Cf(x)| \le \epsilon$$

for all $x \in I$, then there exists a solution $h \in C^2[I, \mathbb{R}]$ of the Cauchy-Euler equation

$$Ax^2y''(x)+Bxy'(x)+Cy(x)=0,$$

such that

$$|f(x)-h(x)| \le 3\epsilon(b-a)Ne^{|\frac{B-2A}{A}||\ln|a|-\ln|b||}$$

for all $x \in I$, where $N = \lceil\frac{1}{\beta}\rceil$ (the smallest integer not less than $\frac{1}{\beta}$), $\beta = \min\{|(B-2A)a|, |(B-2A)b|\}$.

Acknowledgements

This work was supported by 'Qing Lan' Talent Engineering Funds by Tianshui Normal University, the Research Project of Higher Learning of Gansu Province (No. 2014B-080) and the Key Subjects Construction of Tianshui Normal University.

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