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Brunn-Minkowski type inequalities for L_p Blaschke-Minkowski homomorphisms

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Abstract

In this paper, the Brunn-Minkowski type inequalities for L_p Blaschke-Minkowski homomorphisms and L_p radial Minkowski homomorphisms are established.

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1. Introduction and preliminaries

The Brunn-Minkowski inequality is one of the most important geometric inequalities. There is a huge amount of work on its generalizations and on its connections with other areas (see [1, 5–7, 16, 18]). The excellent survey article of Gardner [5] gives a comprehensive account of various aspects and consequences of the Brunn-Minkowski inequality.

Projection bodies and intersection bodies played a critical role in the solution of the Shephard problem and the Busemann-Petty problem, respectively (see [14]). Through the work of Ludwig [12, 13], projection bodies and intersection bodies were characterized as continuous and $GL(n)$ contravariant valuations. Recently, Schuster [19, 20] introduced the Blaschke-Minkowski homomorphisms and radial Blaschke-Minkowski homomorphisms which are more general than the well-known projection body operators and intersection bodies, respectively. In order to state their definition, let $\mathcal{K}^n$ denote the space of all convex bodies in $\mathbb{R}^n$ endowed with the Hausdorff topology.

A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is called a Blaschke-Minkowski homomorphism, if it satisfies the following conditions:

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(a) Φ is continuous with respect to the Hausdorff metric.

(b) For all $K_1, K_2 \in \mathcal{K}^n$,

$$\Phi(K_1 \# K_2) = \Phi K_1 + \Phi K_2,$$

where $K_1 \# K_2$ denotes Blaschke addition (see [9]) of K_1 and K_2, and $\Phi K_1 + \Phi K_2$ is the Minkowski addition of ΦK_1 and ΦK_2.

(c) For all $K \in \mathcal{K}^n$ and every $\upsilon \in SO(n)$,

$$\Phi(\upsilon K) = \upsilon \Phi K,$$

where $SO(n)$ is the group of rotations of $\mathbb{R}^n$.

Let $\mathcal{S}^n$ denote the space of all star bodies in $\mathbb{R}^n$ endowed with the radial metric. A map $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ is called a radial Blaschke-Minkowski homomorphism if it satisfies the following conditions:

(a*) Ψ is continuous with respect to the radial metric.

(b*) For all $L_1, L_2 \in \mathcal{S}^n$,

$$\Psi(L_1 \widetilde{\#} L_2) = \Psi L_1 \tilde{+} \Psi L_2,$$

where $L_1 \widetilde{\#} L_2$ denotes the radial Blaschke addition (see [8]) of L_1 and L_2, and $\Psi L_1 \tilde{+} \Psi L_2$ is the radial Minkowski addition of ΨL_1 and ΨL_2.

(c*) For all $L \in \mathcal{S}^n$ and every $\upsilon \in SO(n)$,

$$\Psi(\upsilon L) = \upsilon \Psi L.$$

Volume inequalities for convex body and star body valued valuations are an active field of research (see [2–4, 17, 19–21, 23, 25]).

In the recent paper [22], Wang introduced the following concept of the L_p Blaschke-Minkowski homomorphisms:

A map $\Phi_p : \mathcal{K}_s^n \to \mathcal{K}_s^n$ is called an L_p Blaschke-Minkowski homomorphism, if it satisfies the following conditions:

(1) Φ_p is continuous with respect to the Hausdorff metric.

(2) For all $K_1, K_2 \in \mathcal{K}_s^n$,

$$\Phi_p(K_1 \#_p K_2) = \Phi_p K_1 +_p \Phi_p K_2,$$

where $K_1 \#_p K_2$ denotes L_p Blaschke addition of K_1 and K_2, and $\Phi_p K_1 +_p \Phi_p K_2$ is the L_p Minkowski addition of $\Phi_p K_1$ and $\Phi_p K_2$.

(3) For all $K \in \mathcal{K}_s^n$ and every $\upsilon \in SO(n)$,

$$\Phi_p(\upsilon K) = \upsilon \Phi_p K,$$

where $SO(n)$ is the group of rotations of $\mathbb{R}^n$.

In the paper [24], Wang et al. defined L_p radial Minkowski homomorphisms as follows:

A map $\Psi_p : \mathcal{S}^n \to \mathcal{S}^n$ is called an L_p radial Minkowski homomorphism, if it satisfies the following conditions:

(1*) Ψ_p is continuous with respect to the radial metric.

(2*) For all $L_1, L_2 \in \mathcal{S}^n$,

$$\Psi_p(L_1 \widetilde{+}_{n-p} L_2) = \Psi_p L_1 \widetilde{+}_p \Psi_p L_2,$$

where $L_1 \widetilde{+}_{n-p} L_2$ denotes the radial addition of L_1 and L_2, and $\Psi_p L_1 \widetilde{+}_p \Psi_p L_2$ is the radial Minkowski addition (see [8]) of $\Psi_p L_1$ and $\Psi_p L_2$.

(3⋆) For all $L \in \mathcal{S}^n$ and every $\upsilon \in SO(n)$,

$$\Psi_p(\upsilon L) = \upsilon \Psi_p L.$$

In [19], Schuster has established the following Brunn-Minkowski type inequalities.

Theorem 1.1 ([19]). *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a Blaschke-Minkowski homomorphism. If $K_1, K_2 \in \mathcal{K}^n$, then*

$$V(\Phi(K_1 + K_2))^{\frac{1}{n(n-1)}} \geq V(\Phi K_1)^{\frac{1}{n(n-1)}} + V(\Phi K_2)^{\frac{1}{n(n-1)}},$$

with equality, if and only if K_1 and K_2 are homothetic.

The operator Φ is called even, if $\Phi K = \Phi(-K)$ for all $K \in \mathcal{K}^n$.

Theorem 1.2 ([19]). *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be an even Blaschke-Minkowski homomorphism. If $K_1, K_2 \in \mathcal{K}^n$, then*

$$V(\Phi^*(K_1 + K_2))^{-\frac{1}{n(n-1)}} \geq V(\Phi^* K_1)^{-\frac{1}{n(n-1)}} + V(\Phi^* K_2)^{-\frac{1}{n(n-1)}},$$

with equality, if and only if K_1 and K_2 are homothetic. Here $\Phi^* K$ is the polar body of ΦK.

The aim of this paper is to establish Brunn-Minkowski type inequalities for L_p Blaschke-Minkowski homomorphisms and L_p radial Minkowski homomorphisms.

Theorem 1.3. *Let $\Phi_p : \mathcal{K}_s^n \to \mathcal{K}_s^n$ be an L_p Blaschke-Minkowski homomorphism. If $K_1, K_2 \in \mathcal{K}_s^n$ and $n \neq p \geq 1$, then*

$$V(\Phi_p(K_1 \#_p K_2))^{p/n} \geq V(\Phi_p K_1)^{p/n} + V(\Phi_p K_2)^{p/n}, \tag{1.1}$$

with equality in (1.1)*, if and only if $\Phi_p K_1$ and $\Phi_p K_2$ are dilates.*

Theorem 1.4. *Let $\Phi_p : \mathcal{K}_s^n \to \mathcal{K}_s^n$ be an L_p Blaschke-Minkowski homomorphism. If $K_1, K_2 \in \mathcal{K}_s^n$ and $n \neq p \geq 1$, then*

$$V(\Phi_p^*(K_1 \#_p K_2))^{-p/n} \geq V(\Phi_p^* K_1)^{-p/n} + V(\Phi_p^* K_2)^{-p/n}, \tag{1.2}$$

with equality in (1.2)*, if and only if $\Phi_p K_1$ and $\Phi_p K_2$ are dilates.*

Theorem 1.5. *Let $\Psi_p : \mathcal{S}^n \to \mathcal{S}^n$ be an L_p radial Minkowski homomorphism. If $K_1, K_2 \in \mathcal{S}_0^n$ and $0 < p < n$, then*

$$V(\Psi_p(K_1 \widetilde{+}_{n-p} K_2))^{p/n} \leq V(\Psi_p K_1)^{p/n} + V(\Psi_p K_2)^{p/n}, \tag{1.3}$$

with equality in (1.3)*, if and only if $\Psi_p K_1$ and $\Psi_p K_2$ are dilates.*

If $p < 0$ or $p > n$, then we get

$$V(\Psi_p(K_1 \widetilde{+}_{n-p} K_2))^{p/n} \geq V(\Psi_p K_1)^{p/n} + V(\Psi_p K_2)^{p/n}, \tag{1.4}$$

with equality (1.4)*, if and only if $\Psi_p K_1$ and $\Psi_p K_2$ are dilates.*

2. Notation and background material

Let $\mathcal{K}^n$ denote the set of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$, and let $\mathcal{K}_0^n$ denote the set of convex bodies that contain the origin in their interiors. The subset of $\mathcal{K}_0^n$ consisting of the centered convex bodies will be denoted by $\mathcal{K}_s^n$. S^{n-1} is the unit sphere. A convex body is uniquely determined by its support function. The support function of $K \in \mathcal{K}^n, h(K, \cdot)$, is defined on S^{n-1} by

$$h(K, u) = \max\{u \cdot x : \ x \in K\}.$$

Let δ denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

Associated with a compact subset $L \in \mathbb{R}^n$, which is star-shaped with respect to the origin, is its radial function $\rho(L,\cdot) : S^{n-1} \to \mathbb{R}$, defined by

$$\rho(L,u) = \max\{\lambda \geq 0 : \lambda u \in L\}.$$

If $\rho(L,\cdot)$ is positive and continuous, we call L a star body. Let $\mathcal{S}^n$ and $\mathcal{S}_0^n$ denote the set of star bodies and the set of star bodies (about the origin) in $\mathbb{R}^n$, respectively. Two star bodies K, L are said to be dilates (of one another), if $\rho_K(u) / \rho_L(u)$ is independent of $u \in S^{n-1}$.

If $K \in \mathcal{K}_0^n$, then the polar body of K, K^*, is defined by

$$K^* := \{x \in \mathbb{R}^n : \quad x \cdot y \leq 1, \forall y \in K\}. \tag{2.1}$$

From (2.1), it follows that $(K^*)^* = K$ and

$$h_{K^*} = \frac{1}{\rho_K}, \quad \rho_{K^*} = \frac{1}{h_K}.$$

Let $K_1, K_2 \in \mathcal{K}_0^n, p \geq 1$, and $\lambda_1, \lambda_2 \geq 0$ (not both 0). The L_p Minkowski sum $\lambda_1 \cdot K_1 +_p \lambda_2 \cdot K_2$ is the convex body whose support function is given by (see [15])

$$h(\lambda_1 \cdot K_1 +_p \lambda_2 \cdot K_2, \cdot)^p = \lambda_1 h(K_1,\cdot)^p + \lambda_2 h(K_2,\cdot)^p.$$

For $p \geq 1$, the L_p-mixed volume $V_p(K,L)$ of $K, L \in \mathcal{K}_o^n$, can be defined by

$$\frac{n}{p} V_p(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

In [15], Lutwak has shown that for $p \geq 1$, and each $K \in \mathcal{K}_o^n$, there exists a positive Borel measure $S_p(K,\cdot)$ on S^{n-1}, such that the L_p-mixed volume $V_p(K,L)$ has the following integral representation:

$$V_p(K,L) = \frac{1}{n} \int_{S^{n-1}} h^p(L,u) dS_p(K,u),$$

for all $L \in \mathcal{K}_o^n$. The L_p-Minkowski inequality states that for $K, L \in \mathcal{K}_o^n$ and $p \geq 1$

$$V_p(K,L) \geq V(K)^{(n-p)/n} V(L)^{p/n},$$

with equality, if and only if K and L are dilates.

For $n \neq p \geq 1$ and $K, L \in \mathcal{K}_s^n$, the L_p-Blaschke addition $K \tilde{+}_p L \in \mathcal{K}_s^n$ was defined in [15] by

$$S_p(K \#_p L, \cdot) = S_p(K,\cdot) + S_p(L,\cdot).$$

Let $K, L \in \mathcal{S}^n$, and $p \in \mathbb{R}$ and $p \neq 0$. The L_p radial addition $K \tilde{+}_p \varepsilon \cdot L$ is the star body defined by

$$\rho(K \tilde{+}_p \varepsilon \cdot L, \cdot)^p = \rho(K,\cdot)^p + \varepsilon \rho(L,\cdot)^p$$

The L_p dual mixed volume $V_p(K,L)$ of $K, L \in \mathcal{K}_o^n$, can be defined by

$$\frac{n}{p} \widetilde{V}_p(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K \widetilde{+}_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

The definition above and the polar coordinate formula for volume give the following integral representation of the dual mixed volume $\widetilde{V}_p(K,L)$

$$\widetilde{V}_p(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n-p} \rho(L,u)^p dS(u).$$

3. Proof of the main results

In this section, we give the proofs of our main results Theorem 1.3–1.5. First, we need the following lemma.

Lemma 3.1 ([10]). *Let $K, L \in \mathcal{S}^n$, if $0 < p < n$, then*

$$\widetilde{V}_p(K, L) \leq V(K)^{(n-p)/n} V(L)^{p/n},$$

with equality, if and only if K and L are dilates. If $p < 0$ or $p > n$, then

$$\widetilde{V}_p(K, L) \geq V(K)^{(n-p)/n} V(L)^{p/n},$$

with equality, if and only if K and L are dilates.

Proof of Theorem 1.3. Let $K, L \in \mathcal{K}_s^n$ and $n \neq p \geq 1$. From the definition of L_p Blaschke-Minkowski homomorphisms and the L_p-Minkowski inequality, for any $M \in \mathcal{K}_0^n$, it follows that

$$\begin{aligned} V_p(M, \Phi_p(K_1 \#_p K_2)) &= V_p(M, \Phi_p K_1 +_p \Phi_p K_2) \\ &= V_p(M, \Phi_p K_1) + V_p(M, \Phi_p K_2) \\ &\geq V(M)^{(n-p)/n} (V(\Phi_p K_1)^{p/n} + V(\Phi_p K_2)^{p/n}), \end{aligned}$$

with equality, if and only if M, $\Phi_p K_1$ and $\Phi_p K_2$ are dilates.

By taking $M = \Phi_p(K_1 \#_p K_2)$, we get

$$V(\Phi_p(K_1 \#_p K_2))^{p/n} \geq V(\Phi_p K_1)^{p/n} + V(\Phi_p K_2)^{p/n},$$

with equality, if and only if $\Phi_p K_1$ and $\Phi_p K_2$ are dilates.

Therefore we have proved inequality (1.1). □

Proof of Theorem 1.4. Let $K, L \in \mathcal{K}_s^n$ and $n \neq p \geq 1$. From the polar coordinate formula for volume and the Minkowski integral inequality, it follows that

$$\begin{aligned} V(\Phi_p^*(K_1 \#_p K_2))^{-p/n} &= \Big(\frac{1}{n} \int_{S^{n-1}} (h(\Phi_p(K_1 \#_p K_2), u)^p)^{-n/p} dS(u)\Big)^{-p/n} \\ &= n^{p/n} \| h(\Phi_p(K_1, u))^p + h(\Phi_p(K_2, u))^p \|_{-n/p} \\ &\geq n^{p/n} \| h(\Phi_p(K_1, u))^p \|_{-n/p} + n^{p/n} \| h(\Phi_p(K_2, u))^p \|_{-n/p} \\ &= V(\Phi_p^* K_1)^{-p/n} + V(\Phi_p^* K_2)^{-p/n}, \end{aligned}$$

with equality, if and only if $\Phi_p K_1$ and $\Phi_p K_2$ are dilates.

Therefore we have proved inequality (1.2). □

Proof of Theorem 1.5. Let $K_1, K_2 \in \mathcal{S}_0^n$ and $0 < p < n$. From Lemma 3.1 and the L_p-Minkowski inequality, for any $M \in \mathcal{S}_0^n$, it follows that

$$\begin{aligned} \widetilde{V}_p(M, \Psi_p(K_1 \widetilde{+}_{n-p} K_2)) &= \widetilde{V}_p(M, \Psi_p K_1 \widetilde{+}_p \Psi_p K_2) \\ &= \widetilde{V}_p(M, \Psi_p K_1) + \widetilde{V}_p(M, \Psi_p K_2) \\ &\leq V(M)^{(n-p)/n} (V(\Psi_p K_1)^{p/n} + V(\Psi_p K_2)^{p/n}), \end{aligned}$$

with equality, if and only if M, $\Psi_p K_1$ and $\Psi_p K_2$ are dilates.

By taking $M = \Psi_p(K_1 \widetilde{+}_{n-p} K_2)$, we get

$$V(\Psi_p(K_1 \widetilde{+}_{n-p} K_2))^{p/n} \leq V(\Psi_p K_1)^{p/n} + V(\Psi_p K_2)^{p/n},$$

with equality, if and only if $\Psi_p K_1$ and $\Psi_p K_2$ are dilates.

Therefore we have proved inequality (1.3).

If $p < 0$ or $p > n$, then we get

$$V(\Psi_p(K_1 \widetilde{+}_{n-p} K_2))^{p/n} \geq V(\Psi_p K_1)^{p/n} + V(\Psi_p K_2)^{p/n},$$

with equality, if and only if $\Psi_p K_1$ and $\Psi_p K_2$ are dilates. The inequality (1.4) is proved. □

Since the L_p projection body operator Π_p is an L_p Blaschke-Minkowski homomorphism, we get the following inequalities which were established by Lu and Leng in [11].

Corollary 3.2 ([11]). *Let $\Pi_p : \mathcal{K}_s^n \to \mathcal{K}_s^n$ be the L_p projection body operator. If $K_1, K_2 \in \mathcal{K}_s^n$ and $n \neq p \geq 1$, then*

$$V(\Pi_p(K_1 \#_p K_2))^{p/n} \geq V(\Pi_p K_1)^{p/n} + V(\Pi_p K_2)^{p/n}, \tag{3.1}$$

$$V(\Pi_p^*(K_1 \#_p K_2))^{-p/n} \geq V(\Pi_p^* K_1)^{-p/n} + V(\Pi_p^* K_2)^{-p/n}, \tag{3.2}$$

with equality in (3.1) *and* (3.2)*, if and only if $\Pi_p K_1$ and $\Pi_p K_2$ are dilates.*

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