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Fixed and common fixed point results for cyclic mappings of Ω-distance

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Abstract

Jleli and Samet in [M. Jleli, B. Samet, Int. J. Anal., **2012** (2012), 7 pages] pointed out that some of fixed point theorems in G-metric spaces can be derived from classical metric spaces. In this paper, we utilize the concept of Ω-distance in sense of Saadati *et al.* [R. Saadati, S. M. Vaezpour, P. Vetro, B. E. Rhoades, Math. Comput. Modeling, **52** (2010), 797–801] to introduce new fixed point and common fixed point results for mappings of cyclic form, through the concept of G-metric space in sense of Mustafa and Sims [Z. Mustafa, B. Sims, J. Nonlinear Convex Anal., **7** (2006), 289–297]. We underline that the method of Jleli and Samet cannot be applied to our results.

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1. Introduction

In the past decade, Mustafa and Sims introduced a new generalization of the usual notion of metric space, which they named generalized metric space or simply G-metric space [15]. After that this notion was fructified by several scientists which proved valuable fixed point theorems in G-metric spaces; please, see Aydi *et al.* [3, 4]; Chandok *et al.* [6]; Chough *et al.* [8], Karapinar and Agarwal [13], Popa and Patriciu [18], Shatanawi and Postolache [24]. Jleli and Samet [11] and Samet *et al.* [20] proved that some of fixed

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point theorems in G-metric spaces can be obtained from usual metric spaces or from quasi metric spaces. Karapinar and Agarwal [13] proved that the approach of Jleli and Samet [11] and Samet *et al.* [20] cannot be applied if the contraction condition in the statement of the theorem is not reducible to two variables and they introduced and proved diverse interesting results in G-metric spaces.

In recent years, scientists studied many fixed point and common fixed point theorems for mappings of cyclic form in different metric space. In [1], Al-Thagafi and Shahzad refer to the existence of a best proximity point for a cyclic contraction map in a reflexive Banach space. In [2], Anurdha and P. Veeramani introduce the notion of proximal pointwise contraction, and state results on the best proximity point associated with a pair of weakly compact convex subsets of a Banach space. In [5], Bilgili and Karapinar proved the existence and uniqueness of some fixed points of certain cyclic mappings, by means of auxiliary functions in the context of G-metric spaces. Chandok and Postolache [7] introduce fixed point results for weakly Chatterjea-type cyclic contractions. In [9], Eldered and Veeramani introduce existence results for a best proximity point in the context of a cyclic mapping, and provide an algorithm for the determination of a best proximity point in the context of a uniformly convex Banach space. Karapinar and Erhan [14] introduce a class of cyclic contractions on partial metric spaces and give some results on fixed points in this framework. In [12], Karpagam and Agrawal utilize the notion of cyclic orbital Meir–Keeler contraction to prove sufficient conditions for the existence of fixed points and best proximity points. In [16], Păcurar and Rus develop a fixed point theory for cyclic ϕ-contractions, while in [17], some existence results of periodic points involving cyclic representations are introduced by Gabriela Petruşel. In [21], Shatanawi and Manro deal with some fixed point theorems for a mapping endowed with a cyclical generalized contractive condition defined by a pair of altering distance functions in complete partial metric spaces. In [25], Shatanawi and Postolache introduce the notion of a cyclic (ψ, A, B)-contraction for a pair of self-mappings and prove some common fixed point theorems for this class of mappings. In [26], a new class of mappings, called p-cyclic ψ-contractions, which contains ψ-cyclic contraction mappings as a subclass is introduced by Vetro.

In their distinguished research [19], Saadati *et al.* introduced the notion of Ω-distance associated with a G-metric and provided to academic community interesting results. In [10], Gholizadeh *et al.* state with the concept of Ω-distance on a complete, partially ordered G-metric space. In [22, 23] Shatanawi and Pitea used the notion of Ω-distance to prove some fixed and coupled fixed point theorems for nonlinear contractions.

In this paper, we utilize the concept of Ω-distance in sense of Saadati *et al.* [19] to introduce new fixed and common fixed point results for mappings of cyclic form, through the concept of G-metric space in sense of Mustafa and Sims. We underline that the method of Jleli and Samet cannot be applied to our results.

2. Preliminaries

Now, we start by recalling the definition of a cyclic mapping; please, see for instance [16], [25].

Definition 2.1. Let A and B be two nonempty subsets of a space X. A mapping $T\colon A \cup B \to A \cup B$ is called cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$.

A more general notion than that of a cyclic mapping is that of pair of mappings with cyclic form.

Definition 2.2. Let A and B be two nonempty subsets of a space X. A pair of mappings $f, g\colon A\cup B \to A\cup B$ is said to have a cyclic form if $f(A) \subseteq B$ and $g(B) \subseteq A$.

The notion of a G-metric space was given by Mustafa and Sims as follows [15].

Definition 2.3 ([15])**.** Let X be a nonempty set, and let $G\colon X \times X \times X \to \mathbb{R}^+$ be a function satisfying:

(G1) $G(x, y, z) = 0$ if $x = y = z$;
(G2) $G(x, x, y) > 0$ for all $x, y \in X$, with $x \neq y$;
(G3) $G(x, y, y) \leq G(x, y, z)$ for all $x, y, z \in X$, with $y \neq z$;
(G4) $G(x, y, z) = G(p\{x, y, z\})$, for each permutation of $\{x, y, z\}$ (the symmetry);
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, $\forall\, x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or more specifically G-metric on X, and the pair (X, G) is called a G-metric space.

The notion of convergence and that of a Cauchy sequence in the setting of a G-metric space are given as follows:

Definition 2.4 ([15])**.** Let (X,G) be a G-metric space, and let (x_n) be a sequence of points of X. We say that (x_n) is G-convergent to x if for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq k$.

Definition 2.5 ([15])**.** Let (X,G) be a G-metric space, a sequence $(x_n) \subseteq X$ is said to be G-Cauchy if for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq k$.

Definition 2.6 ([15])**.** A G-metric space (X,G) is said to be G-complete or complete G-metric space if every G-Cauchy sequence in (X,G) is G-convergent in (X,G).

The definition of Ω-distance is given as follows; please, see Saadati *et al.* [19]:

Definition 2.7 ([19])**.** Let (X,G) be a G-metric space. Then a function $\Omega\colon X \times X \times X \to [0,\infty)$ is called an Ω-distance on X if the following conditions are satisfied:

(a) $\Omega(x,y,z) \leq \Omega(x,a,a) + \Omega(a,y,z), \quad \forall\, x,y,z,a \in X$,
(b) for any $x, y \in X, \Omega(x,y,\cdot), \Omega(x,\cdot,y)\colon X \to X$ are lower semi continuous,
(c) for each $\epsilon > 0$, there exists a $\delta > 0$ such that $\Omega(x,a,a) \leq \delta$ and $\Omega(a,y,z) \leq \delta$ imply $G(x,y,z) \leq \epsilon$.

Definition 2.8 ([19])**.** Let (X,G) be a G-metric space and Ω be an Ω-distance on X. Then we say that X is Ω-bounded if there exists $M \geq 0$ such that $\Omega(x,y,z) \leq M$, for all $x,y,x \in X$.

The following lemma plays a crucial role in the development of our results.

Lemma 2.9 ([19])**.** *Let X be a metric space with metric G and Ω be an Ω-distance on X. Let (x_n),(y_n) be sequences in X, (α_n),(β_n) be sequences in $[0,\infty)$ converging to zero and let $x,y,z,a \in X$. Then we have the following:*

(1) *If $\Omega(y,x_n,x_n) \leq \alpha_n$ and $\Omega(x_n,y,z) \leq \beta_n$ for $n \in \mathbb{N}$, then $G(y,y,z) < \epsilon$ and hence $y = z$;*
(2) *If $\Omega(y_n,x_n,x_n) \leq \alpha_n$ and $\Omega(x_n,y_m,z) \leq \beta_n$ for any $m > n \in \mathbb{N}$, then $G(y_n,y_m,z) \to 0$ and hence $y_n \to z$;*
(3) *If $\Omega(x_n,x_m,x_l) \leq \alpha_n$ for any $m,n,l \in \mathbb{N}$ with $n \leq m \leq l$, then (x_n) is a G-Cauchy sequence;*
(4) *If $\Omega(x_n,a,a) \leq \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a G-Cauchy sequence.*

3. Main results

In this section, we introduce some results for mappings of cyclic form in the notion of Ω-distance, in the setting of Saadati *et al.* [19].

Theorem 3.1. *Let (X,G) be a complete G-metric space and Ω be an Ω-distance on X such that X is Ω-bounded. Let A and B be nonempty, closed subsets of X, $A \cap B \neq \emptyset$ with $X = A \cup B$. Let $\phi\colon [0,\infty) \to [0,\infty)$ be a non-decreasing function such that $\sum\limits_{n=1}^{\infty} (\phi^n t)^{\frac{1}{2}} < \infty, \forall t > 0$. Suppose that the pair (f,g), $f,g\colon A \cup B \to A \cup B$ has a cyclic form, and that the following conditions are satisfied*

$$\Omega(fx,gy,gz) \leq \phi\Omega(x,y,z), \quad \forall x \in A, \quad \textit{and} \quad \forall y,z \in B, \tag{3.1}$$

$$\Omega(fx,gy,fz) \leq \phi\Omega(x,y,z), \quad \forall x,z \in A, \quad \textit{and} \quad \forall y \in B, \tag{3.2}$$

$$\Omega(gx,fy,fz) \leq \phi\Omega(x,y,z), \quad \forall y,z \in A, \quad \textit{and} \quad \forall x \in B, \tag{3.3}$$

and

$$\Omega(gx,fy,gz) \leq \phi\Omega(x,y,z), \quad \forall y \in A, \quad \textit{and} \quad \forall x,z \in B. \tag{3.4}$$

Also, assume that if $fu \neq u$ or $gu \neq u$, then

$$\inf\{\Omega(fx,gfx,u) : x \in X\} > 0.$$

If f or g is continuous, then f and g have a unique common fixed point in $A \cap B$.

Proof. Let $x_0 \in A$. Since $f(A) \subseteq B$, then $fx_0 = x_1 \in B$. Also, since $g(B) \subseteq A$, then $gx_1 = x_2 \in A$. Continuing this process we obtain a sequence (x_n) in X such that $fx_{2n} = x_{2n+1}$, $x_{2n} \in A$ and $gx_{2n+1} = x_{2n+2}$, $x_{2n+1} \in B$, $n \in \mathbb{N}$.
Let $s \in \mathbb{N}$. If s is even, then by (3.1) we get

$$\Omega(x_{2n+1}, x_{2n+2}, x_{2n+s}) = \Omega(fx_{2n}, gx_{2n+1}, gx_{2n+s-1}) \leq \phi\Omega(x_{2n}, x_{2n+1}, x_{2n+s-1}). \tag{3.5}$$

If s is odd, then by (3.2) we get

$$\Omega(x_{2n+1}, x_{2n+2}, x_{2n+s}) = \Omega(fx_{2n}, gx_{2n+1}, fx_{2n+s-1}) \leq \phi\Omega(x_{2n}, x_{2n+1}, x_{2n+s-1}). \tag{3.6}$$

From (3.5) and (3.6) we have

$$\Omega(x_{2n+1}, x_{2n+2}, x_{2n+s}) \leq \phi\Omega(x_{2n}, x_{2n+1}, x_{2n+s-1}), \quad \forall\, n, s \in \mathbb{N}. \tag{3.7}$$

Again, if s is even, then by (3.3) we get

$$\Omega(x_{2n}, x_{2n+1}, x_{2n+s-1}) = \Omega(gx_{2n-1}, fx_{2n}, fx_{2n+s-2}) \leq \phi\Omega(x_{2n-1}, x_{2n}, x_{2n+s-2}). \tag{3.8}$$

Also, if s is odd, then by (3.4) we get

$$\Omega(x_{2n}, x_{2n+1}, x_{2n+s-1}) = \Omega(gx_{2n-1}, fx_{2n}, gx_{2n+s-2}) \leq \phi\Omega(x_{2n-1}, x_{2n}, x_{2n+s-2}). \tag{3.9}$$

From (3.8) and (3.9) we have

$$\Omega(x_{2n}, x_{2n+1}, x_{2n+s-1}) \leq \phi\Omega(x_{2n-1}, x_{2n}, x_{2n+s-2}), \quad \forall n, s \in \mathbb{N}. \tag{3.10}$$

Thus from (3.7) and (3.10), we have

$$\Omega(x_n, x_{n+1}, x_{n+s-1}) \leq \phi\Omega(x_{n-1}, x_n, x_{n+s-2}), \quad \forall\, n, s \in \mathbb{N}. \tag{3.11}$$

Hence inequality (3.11) becomes

$$\Omega(x_n, x_{n+1}, x_{n+s-1}) \leq \phi^n\Omega(x_0, x_1, x_{s-1}).$$

Therefore for any $l, m, n \in \mathbb{N}$ with $l > m > n$, $l = m + t$, $m = n + k$ we have

$$\begin{aligned}\Omega(x_n, x_m, x_l) &\leq \Omega(x_n, x_{n+1}, x_{n+1}) + \Omega(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + \Omega(x_{m-1}, x_m, x_l)\\ &\leq \phi^n\Omega(x_0, x_1, x_1) + \phi^{n+1}\Omega(x_0, x_1, x_1) + \cdots + \phi^{m-1}\Omega(x_0, x_1, x_{t+1}).\end{aligned}$$

Since X is Ω-bounded then there exists $M \geq 0$ such that $\Omega(x, y, z) \leq M$, $\forall\, x, y, z \in X$. Hence

$$\begin{aligned}\Omega(x_n, x_m, x_l) &\leq \phi^n(M) + \phi^{n+1}(M) + \cdots + \phi^{m-1}(M)\\ &\leq (\phi^{n-1}(M))^{\frac{1}{2}} \left(\sum_{i=1}^{m-n} (\phi^i(M))^{\frac{1}{2}}\right)\\ &\leq (\phi^{n-1}(M))^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} (\phi^i(M))^{\frac{1}{2}}\right).\end{aligned}$$

Since $\sum\limits_{i=1}^{\infty} (\phi^i(M))^{\frac{1}{2}} \leq \infty$, we have

$$\lim_{l,m,n\to\infty} \Omega(x_n, x_m, x_l) = 0.$$

By Lemma 2.9, (x_n) is a G-Cauchy sequence. So there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Since (x_n) is G-convergent to u, then every subsequence of (x_n) is also G-convergent to u. So that the subsequences $(x_{2n+1}) = (fx_{2n})$ and $(x_{2n+2}) = (gx_{2n+1})$ are G-convergent to u.

Without loss of generality, we assume that f is continuous. So

$$\lim_{n\to\infty} fx_{2n} = fu.$$

Since

$$\lim_{n\to\infty} x_{2n+1} = u,$$

then by uniqueness of the limit we have $fu = u$. By the lower semi-continuity of Ω, we get

$$\Omega(x_n, x_m, u) \le \liminf_{p\to\infty} \Omega(x_n, x_m, x_p) \le \epsilon,\ \forall m \ge n.$$

Now, suppose that $gu \ne u$, then we get

$$0 < \inf\{\Omega(fx, gfx, u) : x \in X\} \le \inf\{\Omega(x_n, x_{n+1}, u) : n\,\text{odd}\} \le \epsilon$$

for every $\epsilon > 0$ which is a contradiction. Therefore $fu = gu = u$.

Since $(x_{2n}) \subseteq A$ and A is closed, we have $u \in A$. Also, since $(x_{2n+1}) \subseteq B$ and B is closed, we have $u \in B$. Hence u is a common fixed point of f and g in $A \cap B$.

To prove the uniqueness of u assume that there exists $v \in X$ such that $fv = gv = v$. Then by (3.1) we have

$$\Omega(u, v, v) = \Omega(fu, gv, gv) \le \phi\Omega(u, v, v) < \Omega(u, v, v),$$

a contradiction. Thus $\Omega(u, v, v) = 0$. Also,

$$\Omega(v, u, v) = \Omega(fv, gu, gv) \le \phi\Omega(v, u, v) < \Omega(v, u, v),$$

a contradiction. Thus $\Omega(v, u, v) = 0$. According to the definition of an Ω-distance, we get that $G(u, u, v) = 0$; that is, $u = v$. □

By choosing $A = B = X$ in Theorem 3.1, we get the following result:

Corollary 3.2. *Let (X, G) be a complete G-metric space and Ω be an Ω-distance on X such that X is Ω-bounded. Let $\phi\colon [0,\infty) \to [0,\infty)$ be a non-decreasing function such that $\sum_{n=1}^{\infty} (\phi^n t)^{\frac{1}{2}} < \infty$, $\forall t > 0$. Suppose that $f, g : X \to X$ be two mappings such that the following condition holds:*

$$\max\{\Omega(fx, gy, gz), \Omega(fx, gy, fz), \Omega(gx, fy, fz), \Omega(gx, fy, gz)\} \le \phi\Omega(x, y, z)$$

holds $\forall x, y, z \in X$. Moreover, assume that if $fu \ne u$ or $gu \ne u$, then

$$\inf\{\Omega(fx, gfx, u) : x \in X\} > 0.$$

If f or g is continuous, then f and g have a unique common fixed point.

By defining $\phi\colon [0,+\infty) \to [0,+\infty)$ via $\phi(t) = \alpha t$, where $\alpha \in [0, 1)$ we have the following result:

Corollary 3.3. *Let (X, G) be a complete G-metric space and Ω be an Ω-distance on X such that X is Ω-bounded. Let A and B be two nonempty closed subsets of X, $A \cap B \ne \emptyset$ with $X = A \cup B$. Suppose that pair (f, g), $f, g\colon A \cup B \to A \cup B$ has a cyclic form, and there exists $\alpha \in [0, 1)$ such that the following conditions hold true*

$$\Omega(fx, gy, gz) \le \alpha\Omega(x, y, z), \quad \forall x \in A, \quad \textit{and}\ \ \forall y, z \in B,$$

$$\Omega(fx, gy, fz) \le \alpha\Omega(x, y, z), \quad \forall x, z \in A, \quad and \ \ \forall y \in B,$$
$$\Omega(gx, fy, fz) \le \alpha\Omega(x, y, z), \quad \forall y, z \in A, \quad and \ \ \forall x \in B,$$

and

$$\Omega(gx, fy, gz) \le \alpha\Omega(x, y, z), \quad \forall y \in A, \quad and \ \ \forall x, z \in B.$$

Also, assume that if $fu \ne u$ or $gu \ne u$, then

$$\{\Omega(fx, gfx, u) : x \in X\} > 0.$$

If f or g is continuous, then f and g have a unique common fixed point in $A \cap B$.

By choosing $A = B = X$ in Corollary 3.3, we get the following result:

Corollary 3.4. *Let (X, G) be a complete G-metric space and Ω be an Ω-distance on X such that X is Ω-bounded. Let $\phi\colon [0, \infty) \to [0, \infty)$ be a non-decreasing function such that $\sum_{n=1}^{\infty} (\phi^n t)^{\frac{1}{2}} < \infty, \quad \forall t > 0$. Suppose that $f, g\colon X \to X$ be two mappings such that the following condition holds:*

$$\max\{\Omega(fx, gy, gz), \Omega(fx, gy, fz), \Omega(gx, fy, fz), \Omega(gx, fy, gz)\} \le \alpha\Omega(x, y, z)$$

$\forall x, y, z \in X$. Moreover, assume that if $fu \ne u$ or $gu \ne u$, then

$$\inf\{\Omega(fx, gfx, u) : x \in X\} > 0.$$

If f or g is continuous, then f and g have a unique common fixed point.

It is worth mentioning that the condition: If $fu \ne u$ or $gu \ne u$, then

$$\{\Omega(fx, gfx, u) : x \in X\} > 0$$

in Theorem 3.1 can be dropped if g is replaced by f. Hence, we have the following result:

Theorem 3.5. *Let (X, G) be a complete G-metric space and Ω be an Ω-distance on X such that X is Ω-bounded. Let A and B be two nonempty closed subsets of X, $A \cap B \ne \emptyset$ with $X = A \cup B$. Let $\phi\colon [0, \infty) \to [0, \infty)$ be a non-decreasing function such that $\sum_{n=1}^{\infty} (\phi^n t)^{\frac{1}{2}} < \infty$, $\forall t > 0$. Suppose that $f\colon A \cup B \to A \cup B$ is a cyclic mapping. Also, suppose that the following conditions hold:*

$$\Omega(fx, fy, fz) \le \phi\Omega(x, y, z), \quad \forall x \in A, \quad and \ \ \forall y, z \in B,$$
$$\Omega(fx, fy, fz) \le \phi\Omega(x, y, z), \quad \forall x, z \in A, \quad and \ \ \forall y \in B,$$
$$\Omega(fx, fy, fz) \le \phi\Omega(x, y, z), \quad \forall y, z \in A, \quad and \ \ \forall x \in B,$$

and

$$\Omega(fx, fy, fz) \le \phi\Omega(x, y, z), \quad \forall y \in A, \quad and \ \ \forall x, z \in B.$$

If f is continuous, then f has a unique fixed point in $A \cap B$.

Proof. Following the proof of Theorem 3.1 word by word we can deduce the proof of this theorem. □

By choosing $A = B = X$ in Theorem 3.5, we get the following result:

Corollary 3.6. *Let (X, G) be a complete G-metric space and Ω be an Ω-distance on X such that X is Ω-bounded. Let $\phi\colon [0, \infty) \to [0, \infty)$ be a non-decreasing function such that $\sum_{n=1}^{\infty} (\phi^n t)^{\frac{1}{2}} < \infty$, $\forall t > 0$. Let $f\colon X \to X$ is a mapping such that the following condition holds:*

$$\max\{\Omega(fx, fy, fz), \Omega(fx, fy, fz), \Omega(fx, fy, fz), \Omega(fx, fy, fz)\} \le \phi\Omega(x, y, z)$$

$\forall x, y, z \in X$. If f is continuous, then f has a unique fixed point.

By defining $\phi\colon [0,+\infty) \to [0,+\infty)$ by the formula $\phi(t) = \alpha t$, where $\alpha \in [0,1)$, we have the following result:

Corollary 3.7. *Let (X,G) be a complete G-metric space and Ω be an Ω-distance on X such that X is Ω-bounded. Let A and B be two nonempty closed subsets of X, $A\cap B \neq \emptyset$ with $X = A\cup B$. Suppose that $f\colon A\cup B \to A\cup B$ is a cyclic mappings. Assume there exists $\alpha \in [0,1)$ such that the following conditions hold true*

$$\begin{aligned}
\Omega(fx,fy,fz) &\le \alpha\Omega(x,y,z), \quad \forall x\in A, \quad \textit{and} \quad \forall y,z\in B,\\
\Omega(fx,fy,fz) &\le \alpha\Omega(x,y,z), \quad \forall x,z\in A, \quad \textit{and} \quad \forall y\in B,\\
\Omega(fx,fy,fz) &\le \alpha\Omega(x,y,z), \quad \forall y,z\in A, \quad \textit{and} \quad \forall x\in B,
\end{aligned}$$

and

$$\Omega(fx,fy,fz) \le \alpha\Omega(x,y,z), \quad \forall y\in A, \quad \textit{and} \quad \forall x,z\in B.$$

If f is continuous, then f has a unique fixed point in $A\cap B$.

By choosing $A = B = X$ in Corollary 3.7 we have the following result:

Corollary 3.8. *Let (X,G) be a complete G-metric space and Ω be an Ω-distance on X such that X is Ω-bounded. Let $f\colon X\to X$ be a mapping such that there exists $\alpha\in[0,1)$ with*

$$\max\{\Omega(fx,fy,fz),\Omega(fx,fy,fz),\Omega(fx,fy,fz),\Omega(fx,fy,fz)\} \le \alpha\Omega(x,y,z)$$

holds for all $x,y,z\in X$. If f is continuous, then f has a unique fixed point in $A\cap B$.

We introduce the following example to support our main result.

Example 3.9. Consider $X = [-1,1]$ and define

$$G\colon X\times X\times X \to [0,\infty), \quad G(x,y,z) = |x-y| + |y-z| + |x-z|.$$

Also, define

$$\Omega\colon X\times X\times X \to [0,\infty), \quad \Omega(x,y,z) = |x-y| + |x-z|.$$

Let $A = [-1,0]$, $B = [0,1]$ and $f,g\colon A\cup B \to A\cup B$ given by the formulae $fx = -\frac{1}{4}x$, $gx = -\frac{1}{2}x$, respectively. Consider $\phi\colon [0,\infty)\to[0,\infty)$ such that $\phi t = \frac{1}{2}t$. Then

(1) (X,G) is a complete G-metric space,
(2) Ω is an Ω-distance on X, and X is Ω-bounded,
(3) A and B are closed subsets of X with respect to the topology induced by G,
(4) f and g are continuous,
(5) $f(A)\subseteq B$, and $g(B)\subseteq A$,
(6) ϕ is a non decreasing function such that $\sum_{n=1}^{\infty}(\phi^n t)^{\frac{1}{2}} < \infty$, $\forall t>0$,
(7) The following inequalities hold:

$$\begin{aligned}
\Omega(fx,gy,gz) &\le \phi\Omega(x,y,z), \quad \forall x\in A \text{ and } \forall y,z\in B,\\
\Omega(fx,gy,fz) &\le \phi\Omega(x,y,z), \quad \forall x,z\in A \text{ and } \forall y\in B,\\
\Omega(gx,fy,fz) &\le \phi\Omega(x,y,z), \quad \forall y,z\in A \text{ and } \forall x\in B,
\end{aligned}$$

and

$$\Omega(gx,fy,gz) \le \phi\Omega(x,y,z), \quad \forall y\in A \text{ and } \forall x,z\in B.$$

(8) If $fu\neq u$ or $gu\neq u$, then

$$\Omega(fx,gfx,u) : x\in X\} > 0.$$

Proof. The proof of (1), (2), (3), (4) and (5) are clear.

To prove (6), remark that ϕ is a nondecreasing function.

Now let $t > 0$. Then $\sum_{n=1}^{\infty}(\phi^n t)^{\frac{1}{2}} = \sum_{n=1}^{\infty}\left(\left(\frac{1}{2}\right)^n t\right)^{\frac{1}{2}} = t^{\frac{1}{2}}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{\frac{n}{2}}$. Since $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{\frac{n}{2}}$ is a geometric series with base $(\frac{1}{2})^{\frac{1}{2}} < 1$, therefore $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{\frac{n}{2}} < \infty$, and so $t^{\frac{1}{2}}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{\frac{n}{2}} < \infty$.

(7) Let $x \in A$ and $y, z \in B$. Then

$$\begin{aligned}\Omega(fx, gy, gz) &= \Omega\Big(-\frac{1}{4}x, -\frac{1}{2}y, -\frac{1}{2}z\Big) = \Big|-\frac{1}{4}x + \frac{1}{2}y\Big| + \Big|-\frac{1}{4}x + \frac{1}{2}z\Big| \\ &\le \Big|-\frac{1}{2}x + \frac{1}{2}y\Big| + \Big|-\frac{1}{2}x + \frac{1}{2}z\Big| \le \frac{1}{2}(|x-y| + |x-z|) \\ &\le \frac{1}{2}\Omega(x, y, z).\end{aligned}$$

Again, let $x, z \in A$ and $y \in B$. Then

$$\begin{aligned}\Omega(fx, gy, fz) &= \Omega\Big(-\frac{1}{4}x, -\frac{1}{2}y, -\frac{1}{4}z\Big) = \Big|-\frac{1}{4}x - -\frac{1}{2}y\Big| + \Big|-\frac{1}{4}x - -\frac{1}{4}z\Big| \\ &\le \Big|-\frac{1}{2}x + \frac{1}{2}y\Big| + \Big|-\frac{1}{4}x + \frac{1}{4}z\Big| \le \frac{1}{2}|x-y| + \frac{1}{4}|x-z| \\ &\le \frac{1}{2}(|x-y| + |x-z|) \le \frac{1}{2}\Omega(x, y, z).\end{aligned}$$

Also, let $x \in B$ and $y, z \in A$. Then

$$\begin{aligned}\Omega(gx, fy, fz) &= \Omega\Big(-\frac{1}{2}x, -\frac{1}{4}y, -\frac{1}{4}z\Big) = \Big|-\frac{1}{2}x - \frac{1}{4}y\Big| + \Big|-\frac{1}{2}x - \frac{1}{4}z\Big| \\ &\le \Big|-\frac{1}{2}x + \frac{1}{2}y\Big| + \Big|-\frac{1}{2}x + \frac{1}{2}z\Big| \le \frac{1}{2}|x-y| + \frac{1}{2}|x-z| \\ &\le \frac{1}{2}(|x-y| + |x-z|) \le \frac{1}{2}\Omega(x, y, z).\end{aligned}$$

Finally, let $x, z \in B$ and $y \in A$. Then

$$\begin{aligned}\Omega(gx, fy, gz) &= \Omega\Big(-\frac{1}{2}x, -\frac{1}{4}y, -\frac{1}{2}z\Big) = \Big|-\frac{1}{2}x - \frac{1}{4}y\Big| + \Big|-\frac{1}{2}x - \frac{1}{2}z\Big| \\ &\le \Big|-\frac{1}{2}x + \frac{1}{2}y\Big| + \Big|-\frac{1}{2}x + \frac{1}{2}z\Big| \le \frac{1}{2}|x-y| + \frac{1}{2}|x-z| \\ &\le \frac{1}{2}(|x-y| + |x-z|) \le \frac{1}{2}\Omega(x, y, z).\end{aligned}$$

(8) If $fu \ne gu$, then $u \ne 0$. Therefore

$$\begin{aligned}\inf\{\Omega(fx, gfx, u) : x \in X\} &= \inf\Big\{\Omega\Big(-\frac{1}{4}x, \frac{1}{8}x, u\Big) : x \in X\Big\} \\ &= \inf\Big\{\Big|-\frac{1}{4}x - \frac{1}{8}x\Big| + \Big|-\frac{1}{4}x - u\Big| : x \in X\Big\} \\ &= \inf\Big\{\frac{3}{8}|x| + \Big|\frac{1}{4}x + u\Big| : x \in X\Big\} \\ &= |u| > 0.\end{aligned}$$

Thus all hypotheses of Theorem 3.1 hold true. Hence f and g have a common fixed point in $A \cap B$. Here the common fixed point of f and g in $A \cap B$ is 0. $\square$

4. Conclusion

In their research [11], Jleli and Samet pointed out that some of fixed point theorems in G-metric spaces can be derived from classical metric spaces. In this paper, we took advantage of the concept of Ω-distance in sense of Saadati *et al.* [19] to introduce new fixed point and common fixed point results for mappings of cyclic form, through the concept of G-metric space in sense of Mustafa and Sims [15]. We underline that the method of Jleli and Samet cannot be applied to our results.

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