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Computing center conditions for resonant infinity via integrating factor method

Yusen Wu[a,*], Feng Li[b]

[a]*School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, Henan, P. R. China*
[b]*School of Science, Linyi University, Linyi 276005, Shandong, P. R. China*

Abstract

In this literature, the calculation of generalized center conditions is addressed for resonant infinity of a polynomial vector field in $\mathbb{C}^2$. The technique is taking resonant infinity into elementary resonant origin by a homeomorphism. Afterwards, an algorithm to compute generalized singular point quantities is developed, which is a good approach to find the necessary conditions of generalized center for any rational resonance ratio. Finally, the necessary and sufficient conditions of generalized center for resonant infinity are obtained.

Keywords: Generalized complex center, resonant infinity, integrating factor method.
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1. Introduction

The classical *problem of center* is invariably restricted to the following polynomial real planar vector fields

$$\dot{x} = -y + P(x, y), \ \dot{y} = x + Q(x, y), \tag{1.1}$$

with $x, y, t \in \mathbb{R}$, P and Q are polynomials belonging to some natural class (e.g. of degree $\leq n$, homogeneous of degree n). One has to find conditions, on the coefficients of P and Q, under which a neighborhood of the origin is covered by periodic solution of the system (1.1).

The above problem was completely solved only in the following two general situations:

(i) When P and Q are homogeneous polynomials of degree 2 (by Dulac and Kapteyn);

(ii) When P and Q are homogeneous polynomials of degree 3 (by Sibirskiĭ [9, 10]).

*Corresponding author
Email addresses: `wuyusen621@126.com` (Yusen Wu), `lf0539@126.com` (Feng Li)

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The mentioned center problem is the subject of much work (see monographs [5, 13]), here we do not cite many more concrete literatures.

When we treat (1.1) as a system in the complex plane (with complex time), then after a simple change of variable it is equivalent to

$$\dot{x} = x + \cdots, \ \dot{y} = -y + \cdots, \tag{1.2}$$

i.e. to a $1:-1$ resonant saddle. The existence of a center is equivalent to the existence of a local analytic first integral of the form $H = xy + \cdots$ (Equivalent condition: absence of the resonant terms $(xy)^k(x\partial_x + y\partial_y)$ in the normal form).

On the other hand, to our notice, a natural generalization of the center problem is proposed in [14] to consider the case of a polynomial vector field in $\mathbb{C}^2$ with $p:-q$ resonant elementary singular point

$$\dot{x} = px + P(x,y), \ \dot{y} = -qy + Q(x,y), \tag{1.3}$$

with $p, q \in \mathbb{Z}^+$. The only way to get necessary conditions for a center is to compute the $p:-q$ resonant focus numbers, the analogues of the Poincaré-Lyapunov focus quantities. If p and q coprimes, then one can calculate the successive terms in the Taylor expansion of the supposed first integral and the $p:-q$ resonant focus numbers g_k are the coefficients of the obstacles to its existence:

$$H = x^q y^p + \cdots, \ \dot{H} = \sum g_k (x^q y^p)^{k+1}. \tag{1.4}$$

The $g_k's$ are polynomials in the coefficients of the system and can be calculated algorithmically.

Looking for conditions for the existence of a local analytic first integral $H = x^q y^p + \cdots$ (i.e. for the existence of a $p:-q$ resonant center) for system (1.3) has stimulated a great deal of effort from then on. Equivalent condition for the mentioned problem is the absence of the resonant terms $(x^q y^p)^k(px\partial_x + qy\partial_y)$ in the normal form. For the $1:-2$ resonant singular point the integrability problem is completely solved in [3, 14] where necessary and sufficient conditions (20 cases) are given. Lotka-Volterra systems of the form

$$\dot{x} = x + ax^2 + bxy, \ \dot{y} = -\lambda y + cxy + dy^2 \ (\lambda > 0) \tag{1.5}$$

is sufficiently general to give important information on the organization of strata in families of polynomial systems. One can find parameters such that system (1.5) is normalizable, normalizable but not integrable, integrable but not linearizable. Necessary and sufficient conditions for integrability and linearizability are already known in [2, 14] for the case $\lambda \in \mathbb{N}$, that is the $1:-n$ resonant cases. In [4], some sufficient conditions are given in the case of general λ. For the case $\lambda = \frac{p}{2}$ or $\frac{2}{p}, p \in \mathbb{N}^+$, necessary and sufficient conditions for integrable and linearizable systems are given. They have proven that, in the case $\lambda = \frac{p}{2}$ or $\frac{2}{p}, p \in \mathbb{N}^+$, if $a = d = 0, bc \neq 0$ then system (1.5) is integrable but not linearizable, and raised the question for general rational λ, other open problems are also suggested. In [6], some sufficient conditions for the systems (1.5) with $3:-q$ resonance were given, and the integrability of the particular cases of $3:-4$ and $3:-5$ resonances were investigated.

Y. Wu and C. Zhang ([11]) explored the problems of generalized center conditions and integrability of resonant infinity for the following complex polynomial differential system

$$\begin{aligned} \frac{dz}{dT} &= pz^{n+1}w^n + \sum_{\alpha+\beta=0}^{2n} a_{\alpha\beta} z^\alpha w^\beta, \\ \frac{dw}{dT} &= -qw^{n+1}z^n - \sum_{\alpha+\beta=0}^{2n} b_{\alpha\beta} w^\alpha z^\beta, \end{aligned} \tag{1.6}$$

where $z, w, T, a_{\alpha\beta}, b_{\alpha\beta} \in \mathbb{C}, p, q \in \mathbb{Z}^+, (p,q) = 1, n \in \mathbb{N}$. A new recursive algorithm for computing generalized singular point quantities at resonant singular point was derived. Compared with the above results, by using the method of integrating factor method, we develop a parallel recursive algorithm to the calculation of generalized singular point quantities at resonant infinity in this paper.

The organization of this paper is as follows. Sec. 2 is a section of generalities: we give the definitions of generalized singular point quantity, generalized complex center, algebraic equivalence, etc., which we will use in this paper. In Sec. 3 we state and prove the main result. In Sec. 4, as for the experimental part of our study, we specialize to a class of cubic systems and discuss the conditions under which resonant infinity can be a generalized complex center.

2. Generalized singular point quantity and integrability

First of all, we need to clarify the main notation as well as the definitions, lemmas and theorems.
Consider the complex polynomial differential system with the form

$$\begin{aligned} \tfrac{dz}{dT} &= pz + \sum_{\alpha+\beta=2}^{\infty} a_{\alpha\beta} z^{\alpha} w^{\beta} = Z(z,w), \\ \tfrac{dw}{dT} &= -qw - \sum_{\alpha+\beta=2}^{\infty} b_{\alpha\beta} w^{\alpha} z^{\beta} = -W(z,w). \end{aligned} \tag{2.1}$$

Lemma 2.1 ([1, 4])**.** *For system* (2.1)*, we can derive uniquely the following formal series*

$$\xi = z + \sum_{k+j=2}^{\infty} c_{kj} z^k w^j, \quad \eta = w + \sum_{k+j=2}^{\infty} d_{kj} w^k z^j, \tag{2.2}$$

where $p_0 = q_0 = 1,\ c_{k+1,k} = d_{k+1,k} = 0,\ k = 1, 2, \cdots$*, such that system* (2.1) *can be transformed into its normal form*

$$\frac{d\xi}{dT} = p\xi \sum_{i=0}^{\infty} p_i (\xi^q \eta^p)^i, \quad \frac{d\eta}{dT} = -q\eta \sum_{i=0}^{\infty} q_i (\xi^q \eta^p)^i. \tag{2.3}$$

Definition 2.2. For system (2.1), the quantity $\mu_k = p_k - q_k$ is called the generalized singular point quantity of order k of the origin. If $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0, \mu_k \neq 0$, then the origin is called a fine singular point of order k. If for all $k, \mu_k = 0$, then the origin is called a generalized complex center.

Remark 2.3. If system (2.1) is a real system, then "μ_k" defined in Definition 2.2 is "the saddle quantity of order k" defined in [14].

Lemma 2.4 ([12])**.** *The origin of system* (2.1) *is a generalized complex center if and only if system* (2.1) *has a regular first integral at the origin.*

Definition 2.5 ([7])**.** For system $(1.6)_{p=q=1}$ and any positive integer k, if there exist $\zeta_1, \zeta_2, \cdots, \zeta_{k-1}$ which are polynomials in $a_{\alpha\beta}, b_{\alpha\beta}$, such that

$$\mu_k + \zeta_1 \mu_1 + \zeta_2 \mu_2 + \cdots + \zeta_{k-1} \mu_{k-1} = \lambda_k, \tag{2.4}$$

we say that μ_k and λ_k are algebraic equivalence denoted by $\mu_k \sim \lambda_k$.

Lemma 2.6 ([8])**.** *For system* (2.1)*, we can derive successively the following formal series*

$$M(z,w) = \sum_{\alpha+\beta=p+q-2}^{\infty} c_{\alpha\beta} z^{\alpha} w^{\beta} = z^{q-1} w^{p-1} + h.o.t., \tag{2.5}$$

where $c_{kq,kp} = 0, k = 1, 2, 3, \cdots$*, h.o.t. stands for high order terms, such that*

$$\frac{\partial(MZ)}{\partial z} - \frac{\partial(MW)}{\partial w} = z^{q-1} w^{p-1} \sum_{m=1}^{\infty} (m+1) \lambda_m (z^q w^p)^m. \tag{2.6}$$

If $\lambda_1 = \lambda_2 = \cdots = \lambda_{m-1} = 0, \lambda_m \neq 0$*, then* $\mu_1 = \mu_2 = \cdots = \mu_{m-1} = 0, \mu_m \neq 0$*, and* $\lambda_m \sim pq\mu_m,\ m = 1, 2, \cdots$*, where " $\sim$ " is the symbol of algebraic equivalence.*

Theorem 2.7 ([11])**.** *Infinity of system* (1.6) *is a generalized complex center if and only if there exist a non-zero real number s and a first integral*

$$G(z,w)=(z^{-q}w^{-p})^s\sum_{k=0}^{\infty}\frac{g_{(2n+1)k}(z,w)}{(zw)^{(n+1)k}}. \tag{2.7}$$

3. The algorithm

By means of transformation

$$z=\frac{z_1}{(z_1w_1)^{n+1}},\ w=\frac{w_1}{(z_1w_1)^{n+1}},\ dT=(2n+1)(z_1w_1)^{n(2n+1)}dT_1 \tag{3.1}$$

and renaming (z_1,w_1,T_1) by (z,w,T), system (1.6) is brought to

$$\begin{aligned}\tfrac{dz}{dT}&=p^*z+\sum_{\alpha+\beta=0}^{2n}[na_{\alpha\beta}+(n+1)b_{\beta+1,\alpha-1}]z^{\alpha+1}w^{\beta+1}(zw)^{(2n-\alpha-\beta)(n+1)}=\widetilde{Z}(z,w),\\ \tfrac{dw}{dT}&=-q^*w-\sum_{\alpha+\beta=0}^{2n}[nb_{\alpha\beta}+(n+1)a_{\beta+1,\alpha-1}]w^{\alpha+1}z^{\beta+1}(zw)^{(2n-\alpha-\beta)(n+1)}=-\widetilde{W}(z,w),\end{aligned} \tag{3.2}$$

where

$$p^*=np+(n+1)q,\quad q^*=nq+(n+1)p. \tag{3.3}$$

Accordingly, infinity of system (1.6) becomes the origin of system (3.2). Note that transformation (3.1) is a homeomorphism, thus the study of infinity of system (1.6) is equivalent to the study of the origin of system (3.2). The origin is an elementary $p^*:-q^*$ resonant singular point of system (3.2).

Remark 3.1. For system (3.2), the functions on the right hand side have the following peculiarities:

(i) There exist two complex straight line solutions $z=0$ and $w=0$.

(ii) The degree of every monomial higher than one is $(2n+1-\alpha-\beta)(2n+1)+1,\alpha+\beta=0,1,\cdots,2n$.

From Lemma 2.6, we have

Theorem 3.2. *For system* (3.2)*, we can derive successively the following formal series*

$$\widetilde{M}(z,w)=\sum_{\alpha+\beta=p^*+q^*-2}^{\infty}\widetilde{c}_{\alpha\beta}z^\alpha w^\beta=z^{q^*-1}w^{p^*-1}+h.o.t., \tag{3.4}$$

where $\widetilde{c}_{kq^*,kp^*}=0,k=1,2,3,\cdots,$ *such that*

$$\frac{\partial(\widetilde{M}\widetilde{Z})}{\partial z}-\frac{\partial(\widetilde{M}\widetilde{W})}{\partial w}=z^{q^*-1}w^{p^*-1}\sum_{m=1}^{\infty}(m+1)\widetilde{\lambda}_m(z^{q^*}w^{p^*})^m, \tag{3.5}$$

and $\widetilde{\lambda}_m\sim p^*q^*\mu_m,m=1,2,\cdots.$

The coefficients $\widetilde{c}_{\alpha\beta}$ *and* $\widetilde{\lambda}_m$ *above are determined as follows: for* $\forall(\alpha,\beta)$*, when* $p^*(\alpha+1)=q^*(\beta+1)$*,* $\widetilde{c}_{\alpha\beta}$ *are arbitrary; when* $p^*(\alpha+1)\neq q^*(\beta+1)$*,*

$$\begin{aligned}\widetilde{c}_{\alpha\beta}=&\frac{1}{q^*(\beta+1)-p^*(\alpha+1)}\sum_{k+j=1}^{2n+1}\{[n\alpha-(n+1)\beta-1]a_{k,j-1}-[n\beta-(n+1)\alpha-1]b_{j,k-1}\}\\ &\times\widetilde{c}_{\alpha+nk+(n+1)j-(2n+1)(n+1),\beta+nj+(n+1)k-(2n+1)(n+1)}.\end{aligned} \tag{3.6}$$

For any positive integer m,

$$\widetilde{\lambda}_m=(2n+1)\sum_{k+j=1}^{2n+1}(pb_{j,k-1}-qa_{k,j-1})\widetilde{c}_{q^*(m+1)+nk+(n+1)j-(2n^2+3n+2),p^*(m+1)+nj+(n+1)k-(2n^2+3n+2)}. \tag{3.7}$$

In expressions (3.6) *and* (3.7), *for* $p^*+q^*-2 \le \alpha+\beta \le p^*+q^*+2n-2$, *we have already let*

$$\widetilde{c}_{\alpha\beta} = \begin{cases} 1, & \alpha = q^*-1, \beta = p^*-1, \\ 0, & for\ other\ (\alpha,\beta) \end{cases} \tag{3.8}$$

and if $\alpha < 0$ *or* $\beta < 0$, *let* $a_{\alpha\beta} = b_{\alpha\beta} = \widetilde{c}_{\alpha\beta} = 0$.

Proof. System (3.2) can also be expressed as

$$\begin{aligned} \frac{dz}{dT} &= p^*z + \sum_{k+j=1}^{2n+1} [na_{k,j-1} + (n+1)b_{j,k-1}] z^{k+1+(2n+1-k-j)(n+1)} w^{j+(2n+1-k-j)(n+1)} = \widetilde{Z}(z,w), \\ \frac{dw}{dT} &= -q^*w - \sum_{k+j=1}^{2n+1} [nb_{j,k-1} + (n+1)a_{k,j-1}] z^{k+(2n+1-k-j)(n+1)} w^{j+1+(2n+1-k-j)(n+1)} = -\widetilde{W}(z,w). \end{aligned} \tag{3.9}$$

Taking partial derivative of $\widetilde{M}\widetilde{Z}$ and $\widetilde{M}\widetilde{W}$ with respect to z and w, we have

$$\begin{aligned} &\frac{\partial(\widetilde{M}\widetilde{Z})}{\partial z} - \frac{\partial(\widetilde{M}\widetilde{W})}{\partial w} = (\frac{\partial \widetilde{M}}{\partial z}\widetilde{Z} - \frac{\partial \widetilde{M}}{\partial w}\widetilde{W}) + (\frac{\partial \widetilde{Z}}{\partial z} - \frac{\partial \widetilde{W}}{\partial w})\widetilde{M} \\ &= \sum_{\alpha+\beta=p^*+q^*-2}^{\infty} \alpha \widetilde{c}_{\alpha\beta} z^{\alpha-1} w^{\beta} \{p^*z + \sum_{k+j=1}^{2n+1} [na_{k,j-1} + (n+1)b_{j,k-1}] \\ &\quad \times z^{k+1+(2n+1-k-j)(n+1)} w^{j+(2n+1-k-j)(n+1)}\} \\ &\quad - \sum_{\alpha+\beta=p^*+q^*-2}^{\infty} \beta \widetilde{c}_{\alpha\beta} z^{\alpha} w^{\beta-1} \{q^*w + \sum_{k+j=1}^{2n+1} [nb_{j,k-1} + (n+1)a_{k,j-1}] \\ &\quad \times z^{k+(2n+1-k-j)(n+1)} w^{j+1+(2n+1-k-j)(n+1)}\} \\ &\quad + \sum_{\alpha+\beta=p^*+q^*-2}^{\infty} \widetilde{c}_{\alpha\beta} z^{\alpha} w^{\beta} \{(p^*-q^*) + \sum_{k+j=1}^{2n+1} \{[k+1+(2n+1-k-j)(n+1)][na_{k,j-1} + (n+1)b_{j,k-1}] \\ &\quad - [j+1+(2n+1-k-j)(n+1)][nb_{j,k-1} + (n+1)a_{k,j-1}]\} z^{k+(2n+1-k-j)(n+1)} w^{j+(2n+1-k-j)(n+1)}\} \\ &= \sum_{\alpha+\beta=p^*+q^*-2}^{\infty} [p^*(\alpha+1) - q^*(\beta+1)] \widetilde{c}_{\alpha\beta} z^{\alpha} w^{\beta} \\ &\quad + \sum_{\alpha+\beta=p^*+q^*-2}^{\infty} \sum_{k+j=1}^{2n+1} \{\alpha[na_{k,j-1} + (n+1)b_{j,k-1}] - \beta[nb_{j,k-1} + (n+1)a_{k,j-1}]\} \\ &\quad \times \widetilde{c}_{\alpha\beta} z^{\alpha+k+(2n+1-k-j)(n+1)} w^{\beta+j+(2n+1-k-j)(n+1)} \\ &\quad + \sum_{\alpha+\beta=p^*+q^*-2}^{\infty} \sum_{k+j=1}^{2n+1} \{[k+1+(2n+1-k-j)(n+1)][na_{k,j-1} + (n+1)b_{j,k-1}] \\ &\quad - [j+1+(2n+1-k-j)(n+1)][nb_{j,k-1} + (n+1)a_{k,j-1}]\} \\ &\quad \times \widetilde{c}_{\alpha\beta} z^{\alpha+k+(2n+1-k-j)(n+1)} w^{\beta+j+(2n+1-k-j)(n+1)} \\ &= \sum_{\alpha+\beta=p^*+q^*-2}^{\infty} [p^*(\alpha+1) - q^*(\beta+1)] \widetilde{c}_{\alpha\beta} z^{\alpha} w^{\beta} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\alpha+\beta=p^*+q^*-2}^{\infty}\ \sum_{k+j=1}^{2n+1}\{[n\alpha-(n+1)\beta+(2n+1)k-(2n^2+3n+2)]a_{k,j-1}\\
&-[n\beta-(n+1)\alpha+(2n+1)j-(2n^2+3n+2)]b_{j,k-1}\}\\
&\times\widetilde{c}_{\alpha\beta}z^{\alpha+k+(2n+1-k-j)(n+1)}w^{\beta+j+(2n+1-k-j)(n+1)}.
\end{aligned}
$$

For $p^*+q^*-2\le\alpha+\beta\le p^*+q^*+2n-2$, let $\widetilde{c}_{\alpha\beta}$ be a piecewise constant function, such that

$$
\widetilde{c}_{\alpha\beta}=\begin{cases}1, & \alpha=q^*-1,\beta=p^*-1,\\ 0, & for\ other\ (\alpha,\beta).\end{cases} \tag{3.10}
$$

So

$$
\sum_{\alpha+\beta=p^*+q^*-2}^{\infty}[p^*(\alpha+1)-q^*(\beta+1)]\widetilde{c}_{\alpha\beta}z^{\alpha}w^{\beta}=\sum_{\alpha+\beta=p^*+q^*+2n-1}^{\infty}[p^*(\alpha+1)-q^*(\beta+1)]\widetilde{c}_{\alpha\beta}z^{\alpha}w^{\beta}. \tag{3.11}
$$

By calling the new variables

$$
\alpha'=\alpha+k+(2n+1-k-j)(n+1),\ \beta'=\beta+j+(2n+1-k-j)(n+1), \tag{3.12}
$$

we have

$$
\alpha'+\beta'=\alpha+\beta+(2n+2-k-j)(2n+1)\ge p^*+q^*+2n-1. \tag{3.13}
$$

We preserve the notation (α,β) for (α',β'), then

$$
\begin{aligned}
&\sum_{\alpha+\beta=p^*+q^*-2}^{\infty}\ \sum_{k+j=1}^{2n+1}\big\{[n\alpha-(n+1)\beta+(2n+1)k-(2n^2+3n+2)]a_{k,j-1}\\
&\qquad-[n\beta-(n+1)\alpha+(2n+1)j-(2n^2+3n+2)]b_{j,k-1}\big\}\\
&\qquad\times\widetilde{c}_{\alpha\beta}z^{\alpha+k+(2n+1-k-j)(n+1)}w^{\beta+j+(2n+1-k-j)(n+1)}\\
&=\sum_{\alpha+\beta=p^*+q^*+2n-1}^{\infty}\ \sum_{k+j=1}^{2n+1}\{[n\alpha-(n+1)\beta-1]a_{k,j-1}-[n\beta-(n+1)\alpha-1]b_{j,k-1}\}\\
&\qquad\times\widetilde{c}_{\alpha-k-(2n+1-k-j)(n+1),\beta-j-(2n+1-k-j)(n+1)}z^{\alpha}w^{\beta}.
\end{aligned} \tag{3.14}
$$

Furthermore,

$$
\begin{aligned}
\frac{\partial(\widetilde{M}\widetilde{Z})}{\partial z}-\frac{\partial(\widetilde{M}\widetilde{W})}{\partial w}=&\sum_{\alpha+\beta=p^*+q^*+2n-1}^{\infty}\{[p^*(\alpha+1)-q^*(\beta+1)]\widetilde{c}_{\alpha\beta}\\
&+\sum_{k+j=1}^{2n+1}\{[n\alpha-(n+1)\beta-1]a_{k,j-1}-[n\beta-(n+1)\alpha-1]b_{j,k-1}\}\\
&\times\widetilde{c}_{\alpha+nk+(n+1)j-(2n+1)(n+1),\beta+nj+(n+1)k-(2n+1)(n+1)}\}z^{\alpha}w^{\beta}.
\end{aligned} \tag{3.15}
$$

Denote that

$$
\begin{aligned}
\nabla_{\alpha\beta}=&\sum_{k+j=1}^{2n+1}\{[n\alpha-(n+1)\beta-1]a_{k,j-1}-[n\beta-(n+1)\alpha-1]b_{j,k-1}\}\\
&\times\widetilde{c}_{\alpha+nk+(n+1)j-(2n+1)(n+1),\beta+nj+(n+1)k-(2n+1)(n+1)}.
\end{aligned} \tag{3.16}
$$

When $p^*(\alpha+1)-q^*(\beta+1)\neq 0$, let $[p^*(\alpha+1)-q^*(\beta+1)]\widetilde{c}_{\alpha\beta}+\nabla_{\alpha\beta}=0$, from expressions (3.15) and (3.16), one can obtain $\widetilde{c}_{\alpha\beta}=\frac{\nabla_{\alpha\beta}}{q^*(\beta+1)-p^*(\alpha+1)}$, namely, formula (3.6). When $p^*(\alpha+1)-q^*(\beta+1)=0$, comparing (3.5) with (3.15), one obtains $(m+1)\widetilde{\lambda}_m=\nabla_{q^*(m+1)-1,p^*(m+1)-1}$, namely, formula (3.7). □

This theorem gives a recurrent way to compute the generalized singular point quantities at resonant infinity of system (1.6) in terms of its coefficients.

4. An illustrative example

In this section, we present a class of concrete systems of the form (1.6) to illustrate the validity of the theoretical results obtained in the previous section.

Consider the cubic complex polynomial differential system as follows:

$$\begin{aligned}\frac{dz}{dT}&=a_{20}z^2+a_{11}zw+z^2w,\\ \frac{dw}{dT}&=-b_{20}w^2-b_{11}wz-2w^2z.\end{aligned}\tag{4.1}$$

Theorem 4.1. *Consider system* (4.1)*, the following assertions hold.*

(i) Computing with the recursive formulae in Theorem 3.1 *of* [11]*, we summarize the first three generalized singular point quantities at resonant infinity as follows:*

$$\begin{aligned}\lambda_1&=-\frac{3}{2}b_{20}(2a_{20}a_{11}+a_{11}b_{11}-b_{20}b_{11}),\\ \lambda_2&=\frac{1}{4}b_{20}(2a_{11}a_{20}+a_{11}b_{11}-b_{11}b_{20})(46a_{11}^2a_{20}+42a_{11}^2b_{11}+99a_{11}a_{20}b_{20}-108a_{11}b_{11}b_{20}\\ &\quad-67a_{20}b_{20}^2+48b_{11}b_{20}^2),\\ \lambda_3&=-\frac{1}{960}b_{20}(2a_{11}a_{20}+a_{11}b_{11}-b_{11}b_{20})(861216a_{11}^4a_{20}^2-617136a_{11}^4a_{20}b_{11}+339120a_{11}^4b_{11}^2\\ &\quad-1553120a_{11}^3a_{20}^2b_{20}+2384780a_{11}^3a_{20}b_{11}b_{20}-1236870a_{11}^3b_{11}^2b_{20}+2403420a_{11}^2a_{20}^2b_{20}^2\\ &\quad-3937820a_{11}^2a_{20}b_{11}b_{20}^2+1677465a_{11}^2b_{11}^2b_{20}^2-1728280a_{11}a_{20}^2b_{20}^3+2567800a_{11}a_{20}b_{11}b_{20}^3\\ &\quad-958230a_{11}b_{11}^2b_{20}^3+406524a_{20}^2b_{20}^4-563224a_{20}b_{11}b_{20}^4+194715b_{11}^2b_{20}^4).\end{aligned}\tag{4.2}$$

(ii) Computing with the recursive formulae (3.6) *and* (3.7)*, we summarize the first three generalized singular point quantities at resonant infinity as follows:*

$$\begin{aligned}\widetilde{\lambda}_1&=-\frac{3}{2}b_{20}(2a_{20}a_{11}+a_{11}b_{11}-b_{20}b_{11}),\\ \widetilde{\lambda}_2&=-\frac{1}{4}b_{20}(2a_{11}a_{20}+a_{11}b_{11}-b_{11}b_{20})(46a_{11}^2a_{20}+54a_{11}^2b_{11}+87a_{11}a_{20}b_{20}-132a_{11}b_{11}b_{20}\\ &\quad-43a_{20}b_{20}^2+48b_{11}b_{20}^2),\\ \widetilde{\lambda}_3&=-\frac{1}{960}b_{20}(2a_{11}a_{20}+a_{11}b_{11}-b_{11}b_{20})(861216a_{11}^4a_{20}^2-1020336a_{11}^4a_{20}b_{11}+646920a_{11}^4b_{11}^2\\ &\quad-1149920a_{11}^3a_{20}^2b_{20}+2842880a_{11}^3a_{20}b_{11}b_{20}-1955970a_{11}^3b_{11}^2b_{20}+1637520a_{11}^2a_{20}^2b_{20}^2\\ &\quad-3941120a_{11}^2a_{20}b_{11}b_{20}^2+2303865a_{11}^2b_{11}^2b_{20}^2-1005880a_{11}a_{20}^2b_{20}^3+2153500a_{11}a_{20}b_{11}b_{20}^3\\ &\quad-1130130a_{11}b_{11}^2b_{20}^3+194424a_{20}^2b_{20}^4-391324a_{20}b_{11}b_{20}^4+194715b_{11}^2b_{20}^4).\end{aligned}\tag{4.3}$$

On the basis of Theorem 4.1, it is easy to check that the following equalities are satisfied:

$$\begin{aligned}\lambda_1&=\widetilde{\lambda}_1,\\ \lambda_2&=\widetilde{\lambda}_2+2(2b_{20}-a_{11})(a_{20}b_{20}-a_{11}b_{11})\widetilde{\lambda}_1,\\ \lambda_3&=\widetilde{\lambda}_3+\frac{5}{24}(a_{20}b_{20}-a_{11}b_{11})(-1344a_{11}^3a_{20}+1026a_{11}^3b_{11}+2553a_{11}^2a_{20}b_{20}-2397a_{11}^2b_{11}b_{20}\\ &\quad-2408a_{11}a_{20}b_{20}^2+2088a_{11}b_{11}b_{20}^2+707a_{20}b_{20}^3-573b_{11}b_{20}^3)\widetilde{\lambda}_1,\end{aligned}\tag{4.4}$$

which suggest that $\lambda_m \sim \widetilde{\lambda}_m, m = 1, 2, 3$. And so we can obtain the same integrable conditions at resonant infinity as those in Theorem 5.2 of [11].

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