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On the Appell type λ-Changhee polynomials

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Abstract

In the paper, by virtue of the p-adic fermionic integral on $\mathbb{Z}_p$, the authors consider a λ-analogue of the Changhee polynomials and present some properties and identities of these polynomials.

Keywords: Identity, property, Appell type λ-Changhee polynomial, Changhee polynomial, degenerate Changhee polynomial.

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1. Introduction

Let p be a fixed odd prime number and let $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote respectively the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm is normally defined as $|p| = \frac{1}{p}$. Recently, degenerate Changhee polynomials $\mathrm{Ch}_{n,\lambda}(x)$ are defined [6, p. 296] by

$$\frac{2\lambda}{2\lambda+\ln(1+\lambda t)}\left[1+\frac{\ln(1+\lambda t)}{\lambda}\right]^x = \sum_{n=0}^{\infty}\mathrm{Ch}_{n,\lambda}(x)\frac{t^n}{n!}.$$

When $x = 0$, we call $\mathrm{Ch}_{n,\lambda} = \mathrm{Ch}_{n,\lambda}(0)$ the degenerate Changhee numbers.

It is common knowledge that the Euler polynomials $E_n(x)$ are given by

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}$$

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and that, when $x = 0$, we call $E_n = 2^n E_n\left(\frac{1}{2}\right)$ the Euler numbers.

Let $C(\mathbb{Z}_p)$ be the space of all continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ was defined [3, p. 134] by

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\,\mathrm{d}\,\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}\left(x + p^N\mathbb{Z}_p\right) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x.$$

From the above definition, we can derive

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0),$$

where $f_1(x) = f(x+1)$. Consequently, it follows from [2, p. 1256], [4, p. 994], and [5, p. 366] that

$$\int_{\mathbb{Z}_p} (1+t)^{x+y}\,\mathrm{d}\,\mu_{-1}(y) = \sum_{n=0}^{\infty} \mathrm{Ch}_n(x)\frac{t^n}{n!},$$

where $\mathrm{Ch}_n(x)$ are called the Changhee polynomials.

We note that the Euler polynomials $E_n(x)$ may also be represented by

$$\int_{\mathbb{Z}_p} e^{(x+y)t}\,\mathrm{d}\,\mu_{-1}(y) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.$$

The purpose of this paper is to construct a new type of polynomials, the Appell type λ-Changhee polynomials, and to investigate some properties and identities of these polynomials.

2. Appell type λ-Changhee polynomials

Assume that $\lambda, t \in \mathbb{C}_p$ such that

$$|\,\lambda t\,|_p < p^{-1/(p-1)} \quad \text{and} \quad (1+\lambda t)^{x/\lambda} = e^{x\ln(1+\lambda t)/\lambda}.$$

Now we define the Appell type λ-Changhee polynomials $\mathfrak{Ch}_n(x|\lambda)$ by

$$\int_{\mathbb{Z}_p} e^{y\ln(1+\lambda t)/\lambda + xt}\,\mathrm{d}\,\mu_{-1}(y) = \frac{2}{(1+\lambda t)^{1/\lambda}+1}e^{xt} = \sum_{n=0}^{\infty} \mathfrak{Ch}_n(x|\lambda)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, we call $\mathfrak{Ch}_n(\lambda) = \mathfrak{Ch}_n(0|\lambda)$ the λ-Changhee numbers. Note that $\mathfrak{Ch}_n(1) = \mathrm{Ch}_n$ for $n \geq 0$.

Theorem 2.1. *For $n \geq 0$, we have*

$$\mathfrak{Ch}_n(x|\lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathfrak{Ch}_m(\lambda)x^{n-m}.$$

Proof. From (2.1), we can derive

$$\sum_{n=0}^{\infty} \mathfrak{Ch}_n(x|\lambda)\frac{t^n}{n!} = \left[\sum_{n=0}^{\infty} \mathfrak{Ch}_n(\lambda)\right]\left(\sum_{l=0}^{\infty} \frac{x^l}{l!}t^l\right) = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n} \binom{n}{m} \mathfrak{Ch}_m(\lambda)x^{n-m}\right]\frac{t^n}{n!}.$$

Equating coefficients on the very ends of the above identity arrives at the required result. □

Theorem 2.2. *For $n \geq 0$, we have*

$$\mathfrak{Ch}_n(x|\lambda) = \sum_{k=0}^{n}\sum_{m=0}^{k} \binom{n}{k} \lambda^{k-m} E_m(0) S_1(k,m) x^{n-k},$$

where $S_1(n,m)$ is the Stirling number of the first kind.

Proof. From Theorem 2.1, it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,x}\mathfrak{Ch}_n(x|\lambda) &= \sum_{m=1}^{n}\binom{n}{m}\mathfrak{Ch}_m(\lambda)(n-m)x^{n-m-1} = n\sum_{m=1}^{n}\binom{n-1}{m-1}\mathfrak{Ch}_m(\lambda)x^{n-m-1}\\ &= n\sum_{m=0}^{n-1}\binom{n-1}{m}\mathfrak{Ch}_{n-m-1}(\lambda)x^{m} = n\mathfrak{Ch}_{n-1}(x|\lambda).\end{aligned}$$

This means that $\mathfrak{Ch}_n(x|\lambda)$ is an Appel sequence. Furthermore, we observe that

$$\begin{aligned}\int_{\mathbb{Z}_p} e^{y\ln(1+\lambda t)/\lambda+xt}\,\mathrm{d}\,\mu_{-1}(y) &= \left(\sum_{m=0}^{\infty}\lambda^{-m}\int_{\mathbb{Z}_p} y^m\mu_{-1}(y)\frac{1}{m!}(\ln(1+\lambda t))^m\right)\left(\sum_{l=0}^{\infty}\frac{x^l}{l!}t^l\right)\\ &= \left(\sum_{m=0}^{\infty}\lambda^{-m}E_m(0)\sum_{k=m}^{\infty}S_1(k,m)\frac{\lambda^k t^k}{k!}\right)\left(\sum_{l=0}^{\infty}\frac{x^l}{l!}t^l\right)\\ &= \left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}\lambda^{k-m}E_m(0)S_1(k,m)\right)\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty}\frac{x^l}{l!}t^l\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^{k-m}E_m(0)S_1(k,m)x^{n-k}\right)\frac{t^n}{n!}.\end{aligned}$$

Combining this with (2.1) yields the required identity. □

Theorem 2.3. *For $n\ge 0$, we have*

$$\sum_{m=0}^{n}\mathfrak{Ch}_m(x|\lambda)\lambda^{n-m}S_2(n,m) = \sum_{m=0}^{n}\binom{n}{m}B_m\left(\frac{x}{\lambda}\right)\lambda^m E_{n-m}(0),$$

where $S_2(m,n)$ is the Stirling number of the second kind.

Proof. By replacing t by $\frac{e^{\lambda t}-1}{\lambda}$ in (2.1), we obtain

$$\begin{aligned}\frac{2}{e^t+1}e^{x(e^{\lambda t}-1)/\lambda} &= \sum_{m=0}^{\infty}\mathfrak{Ch}_m(x|\lambda)\frac{1}{m!}\left(\frac{e^{\lambda t}-1}{\lambda}\right)^m\\ &= \sum_{m=0}^{\infty}\mathfrak{Ch}_m(x|\lambda)\frac{1}{\lambda^m}\sum_{n=m}^{\infty}S_2(n,m)\frac{\lambda^n t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\mathfrak{Ch}_m(x|\lambda)\lambda^{n-m}S_2(n,m)\right]\frac{t^n}{n!}.\end{aligned}\tag{2.2}$$

Recall from [1, p. 265] that the Bell polynomials $B_n(x)$ are generated by

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

Therefore, we acquire that

$$\frac{2}{e^t+1}e^{x(e^{\lambda t}-1)/\lambda} = \left(\sum_{l=0}^{\infty}\frac{E_l(0)}{l!}t^l\right)\left(\sum_{m=0}^{\infty}B_m\left(\frac{x}{\lambda}\right)\frac{\lambda^m t^m}{m!}\right) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}B_m\left(\frac{x}{\lambda}\right)\lambda^m E_{n-m}(0)\right)\frac{t^n}{n!}.$$

Comparing this with (2.2) leads to the required identity. □

For $r \in \mathbb{N}$, define the higher order λ-Changhee polynomials $\mathfrak{Ch}_n^{(r)}(x)$ by

$$\int \cdots \int_{\mathbb{Z}_p^r} e^{(x_1+\cdots+x_r)\ln(1+\lambda t)/\lambda + xt} \,\mathrm{d}\,\mu_{-1}(x_1)\cdots \mathrm{d}\,\mu_{-1}(x_r) = \left[\frac{2}{(1+\lambda t)^{1/\lambda}+1}\right]^r e^{xt} = \sum_{n=0}^{\infty} \mathfrak{Ch}_n^{(r)}(x|\lambda)\frac{t^n}{n!}. \quad (2.3)$$

When $x = 0$, we call $\mathfrak{Ch}_n^{(r)}(\lambda) = \mathfrak{Ch}_n^{(r)}(0|\lambda)$ the higher order λ-Changhee numbers.

Theorem 2.4. *For $n \geq 1$, we have*

$$\mathfrak{Ch}_n^{(r)}(x|\lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathfrak{Ch}_m^{(r)}(\lambda) x^{n-m} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}\,x}\mathfrak{Ch}_n^{(r)}(x|\lambda) = n\mathfrak{Ch}_{n-1}^{(r)}(x|\lambda).$$

Proof. This follows from the observation that

$$\sum_{n=0}^{\infty} \mathfrak{Ch}_n^{(r)}(x|\lambda)\frac{t^n}{n!} = \left[\frac{2}{(1+\lambda t)^{1/\lambda}+1}\right]^r e^{xt} = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\binom{n}{m}\mathfrak{Ch}_m^{(r)}(\lambda)x^{n-m}\right]\frac{t^n}{n!}.$$ □

Recall from [8, p. 12] that the higher order Euler polynomials $E_n^{(r)}(x)$ may be represented by

$$\int \cdots \int_{\mathbb{Z}_p^r} e^{(x_1+\cdots+x_r+x)t} \,\mathrm{d}\,\mu_{-1}(x_1)\cdots \mathrm{d}\,\mu_{-1}(x_r)\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}.$$

When $x = 0$, we call $E_n^{(r)}(0)$ the higher order modified Euler numbers.

Theorem 2.5. *For $n \geq 0$, we have*

$$\mathfrak{Ch}_n^{(r)}(x|\lambda) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^{k-m}E_m^{(r)}(0)S_1(k,m)x^{n-k}.$$

Proof. We observe that

$$\begin{aligned}
&\int \cdots \int_{\mathbb{Z}_p^r} e^{(x_1+\cdots+x_r)\ln(1+\lambda t)/\lambda + xt} \,\mathrm{d}\,\mu_{-1}(x_1)\cdots \mathrm{d}\,\mu_{-1}(x_r) \\
&\quad = \left(\sum_{m=0}^{\infty}\int \cdots \int_{\mathbb{Z}_p^r}(x_1+\cdots+x_r)^m \,\mathrm{d}\,\mu_{-1}(x_1)\cdots \mathrm{d}\,\mu_{-1}(x_r)\frac{[\ln(1+\lambda t)]^m}{m!\lambda^m}\right)\sum_{l=0}^{\infty}\frac{x^l}{l!}t^l \\
&\quad = \left(\sum_{m=0}^{\infty}\lambda^{-m}E_m^{(r)}(0)\sum_{k=m}^{\infty}S_1(k,m)\frac{\lambda^k t^k}{k!}\right)\left(\sum_{l=0}^{\infty}\frac{x^l}{l!}t^l\right) \\
&\quad = \left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}\lambda^{k-m}E_m^{(r)}(0)S_1(k,m)\right)\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty}\frac{x^l}{l!}t^l\right) \\
&\quad = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}\lambda^{k-m}E_m^{(r)}(0)S_1(k,m)x^{n-k}\right)\frac{t^n}{n!}.
\end{aligned}$$

Combination of this identity with (2.3) results in the required identity. □

Theorem 2.6. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n}\lambda^{n-m}\mathfrak{Ch}_m^{(r)}(x|\lambda)S_2(n,m) = \sum_{m=0}^{n}B_m\left(\frac{x}{\lambda}\right)\lambda^m E_{n-m}^{(r)}(0).$$

Proof. Substituting $\frac{e^{\lambda t}-1}{\lambda}$ for t in (2.3) gives

$$\left(\frac{2}{e^t+1}\right)^r e^{x(e^{\lambda t}-1)/\lambda} = \sum_{m=0}^{\infty} \mathfrak{Ch}_m^{(r)}(x|\lambda)\frac{1}{\lambda^m}\frac{1}{m!}\left(e^{\lambda t}-1\right)^m$$

$$= \sum_{m=0}^{\infty} \mathfrak{Ch}_m^{(r)}(x|\lambda)\frac{1}{\lambda^m}\sum_{n=m}^{\infty} S_2(n,m)\frac{\lambda^n t^n}{n!} = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n} \lambda^{n-m}\mathfrak{Ch}_m^{(r)}(x|\lambda)S_2(n,m)\right]\frac{t^n}{n!}.$$

On the other hand,

$$\left(\frac{2}{e^t+1}\right)^r e^{x(e^{\lambda t}-1)/\lambda} = \left[\sum_{l=0}^{\infty} E_l^{(r)}(0)\frac{t^l}{l!}\right]\left[\sum_{m=0}^{\infty} B_m\left(\frac{x}{\lambda}\right)\frac{\lambda^m t^m}{m!}\right] = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n} B_m\left(\frac{x}{\lambda}\right)\lambda^m E_{n-m}^{(r)}(0)\right]\frac{t^n}{n!}.$$

The required result thus follows. □

Remark 2.7. This paper is a slightly modified version of the preprint [7].

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