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Local conjugacy theorems for C^1 operators between Banach manifolds

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Abstract

In this paper, by the generalized inverse theory of bounded linear operators, the local conjugacy theorem for C^1 operators between Banach manifolds is established. According to this theorem, the conditions which can be used to make sure that a C^1 operator can be linearized are provided. Local conjugacy theorems for nonlinear Fredholm operators, nonlinear semi-Fredholm operators and finite rank operators are introduced.

Keywords: Conjugacy theorem, generalized inverse, linearization, Banach manifold.
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1. Introduction

Let X and Y be normed linear spaces, and $B(X,Y)$ be the space of all bounded linear operators from X to Y. For any $T \in B(X,Y)$, if the null space $\mathrm{N}(T)$ and the range space $\mathrm{R}(T)$ are topologically complemented in the space X and Y, respectively. Then there exists a linear inclined projector generalized inverse $T^+ \in B(Y,X)$ for T such that

$$TT^+T = T, \quad T^+TT^+ = T^+, \quad T^+T = I_X - P, \quad TT^+ = Q,$$

where I_X denotes the identity operator on X, P and Q are the continuous linear projectors form X and Y onto $\mathrm{N}(T)$ and $\mathrm{R}(T)$, respectively.

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There are many differences between the properties of inverse and the properties of generalized inverse. For example, the continuity. Let us consider bounded linear operators A and A_t, for all $t \geq 0$, and suppose that $A_t \to A$ as $t \to 0$. If A_t and A are invertible, then their inverses $A_t^{-1} \to A^{-1}$ as $t \to 0$. But their generalized inverses A_t^+ and A^+ do not have this property. In order to make the properties of generalized inverse clear, the perturbation, expression and existence of generalized inverse are widely discussed, see [2, 3, 4, 5, 6, 7, 8]. We apply the generalized inverse theory to discuss the local conjugacy problem.

It is well known that if x_0 is a submersion point, immersion point or sub-immersion point of the map f, then f is locally conjugate to the linearization $f'(x_0)$ near x_0, see [1, 13]. These conclusions are always said to be local linearization theorems or local conjugacy theorems. In [1], Berger mentioned a local conjugacy problem. He proved that f is a C^1 Fredholm operator of index p, then f is conjugate to a linear projection P in a sufficiently small neighborhood of x_0 under some certain conditions. He pointed out that a classical local conjugacy theorem of advanced calculus is unknown for C^1 operators, even if f is a Fredholm operator. A series of studies on local conjugacy problem in Banach spaces through generalized inverse theory were given in [9, 10]. Furthermore, in [12], the concept of local conjugation of C^1 mappings between Banach manifolds was introduced, and a rank theorem for nonlinear semi-Fredholm operators between Banach manifolds was established by Shi and Ma.

Our main goal is to generalize the results mentioned above, and provide the local conjugacy theorems for C^1 operators between Banach manifolds. The main theorem (Theorem 3.1) gives an answer to the question which Berger mentioned for C^1 operators. As the corollaries, this theorem deduces local conjugacy theorems for nonlinear Fredholm operators (Theorem 3.3), nonlinear semi-Fredholm operators (Corollary 3.4) and finite rank operators (Corollary 3.5). All these theorems provide the characteristics which can make sure that these operators could be locally linearized. These theorems deduce the results in [12] by Shi and Ma.

The structure of this paper is as follows. In Section 2, we recall some notions needed in the sequel. In Section 3, we set up a local conjugacy theorem for C^1 Fredholm operators between Banach manifolds, and as consequences, set up local conjugacy theorems for C^1 semi-Fredholm operators and finite rank operators.

All the symbols in this paper could be found in [11, 13].

2. Preliminaries

In this section we present the basic notions of Fredholm operator, generalized inverse and local conjugacy problem.

Let X and Y be Banach spaces, the mapping $f : X \to Y$ is said to be an *upper semi-Fredholm operator* at x if its null space $\mathrm{N}(f(x))$ is finite dimensional and its range $\mathrm{R}(f(x))$ is closed. f is said to be a *lower semi-Fredholm operator* at x if its range $\mathrm{R}(f(x))$ is closed and finite codimensional. f is said to be a *Fredholm operator* at x if it is both upper and lower semi-Fredholm operator.

Let $f : U \subset X \to Y$ be a $C^r (r \geq 1)$ mapping, $x_0 \in U$, $f'(x)$ denote the Fréchet differential of f at x. f is locally conjugate to $f'(x_0)$ near x_0 if and only if there exist two neighborhoods U_0 at x_0 and V_0 at 0, with locally diffeomorphisms $u : U_0 \to u(U_0)$ and $v : V_0 \to v(V_0)$, such that

$$u(x_0) = 0,\ u'(x_0) = I,\ v(0) = f(x_0),\ v'(0) = I,$$

and

$$f(x) = (v \circ f'(x_0) \circ u)(x)$$

for all $x \in U_0$.

In case of Banach manifolds, the local conjugacy of a mapping f could be induced directly by it's representation under some admissible charts, since the chart space is Banach spaces.

Definition 2.1 ([12])**.** Let M and N be C^r Banach manifolds, $r \geq 1$, $f : M \to N$ be a C^1 mapping. $f(x)$ is locally conjugate to $f'(x_0)$ near x_0, if under some admissible charts (U, φ) of M at x_0 and (V, ψ) of N at $y_0 = f(x_0)$, the representative $\overline{f} = \psi \circ f \circ \varphi^{-1}$ of f is locally conjugate to $\overline{f}'(x_\varphi^0)$ near $x_\varphi^0 = \varphi(x_0)$.

A bounded linear operator T_x is *locally fine* at $x_0 \in X$, if $T_0 = T_{x_0}$ has a generalized inverse T_0^+ and there exists a neighborhood U_0 at x_0 such that

$$\mathrm{R}(T_x) \cap \mathrm{N}(T_0^+) = \{0\}.$$

This induces the following concept of locally fine point of C^1 mappings in Banach spaces and Banach manifolds.

Let X and Y be two Banach spaces, $U \subset X$ is an open set, and $f : U \subset X \to Y$ be a C^1 mapping. The point $x \in U$ is called to be locally fine point of f if $f'(x)$ is locally fine at the point x.

Furthermore, assume that M, N are C^k Banach manifolds ($k \geq 1$), $f : M \to N$ be a C^1 mapping. A point $x_0 \in M$ is locally fine point of f, if and only if $x_\varphi^0 = \varphi(x_0)$ is a locally fine point of the representation $\overline{f}$ of f, where $\overline{f} = \psi \circ f \circ \varphi^{-1}$ with charts (U, φ) of M at x and (V, ψ) of N at $y = f(x)$. That is, there exists a bounded generalized inverse T_0^+ of $\overline{f}'(x_\varphi^0)$ such that

$$\mathrm{R}(\overline{f}'(x_\varphi)) \cap \mathrm{N}(T_0^+) = \{0\}$$

at the point $x_\varphi \in \varphi(U)$ which is close to the point x_φ^0.

By the following commutative diagram, it is easy to see that the concepts of locally fine point and locally conjugacy are independent of the choice of admissible charts (U_j, φ_j) and (V_j, ψ_j), where $j = 1, 2$.

$$\begin{tikzcd}
& TM_{x_0} \arrow[r, "f'(x_0)"'] & TN_{f(x_0)} \arrow[d, "v"'] & \\
X_{\varphi_1} \arrow[ur] \arrow[d, "\varphi_2\circ\varphi_1^{-1}"'] & x_0 \in U_1 \subset M \arrow[l, "\varphi_1"'] \arrow[u, "u"'] \arrow[r, "f"'] \arrow[d, "i|_{U_1\cap U_2}"'] & f(x_0) \in V_1 \subset N \arrow[r, "\psi_1"] \arrow[d, "i|_{V_1\cap V_2}"] & Y_{\psi_1} \arrow[ul] \arrow[d, "\psi_2\circ\psi_1^{-1}"] \\
X_{\varphi_2} \arrow[uur] & x_0 \in U_2 \subset M \arrow[l, "\varphi_2"'] \arrow[r, "f"] & f(x_0) \in V_2 \subset N \arrow[r, "\psi_2"] & Y_{\psi_2} \arrow[uul]
\end{tikzcd}$$

Lemma 2.2 ([9])**.** *Suppose that $T_0 \in B(E, F)$ has a general inverse $T_0^+ \in B(F, E)$ and $\Lambda : B(E, F) \to B(E, \mathrm{R}(T_0) \times \mathrm{N}(T_0))$ is defined by*

$$(\Lambda T)x = (T_0 T_0^+ T x, (I_E - T_0^+ T_0)x), \quad x \in E,$$

then we have
(i) Λ is continuous,
(ii) there exists a neighborhood V_0 at T_0 such that

$$\mathrm{R}(T_0 T_0^+ T) = \mathrm{R}(T_0), \quad (\Lambda T) \in B^*(E, \mathrm{R}(T_0) \times \mathrm{N}(T_0)), \tag{2.1}$$

$$\mathrm{N}(T_0 T_0^+ T) = (\Lambda T)^{-1}(0, \mathrm{N}(T_0)), \quad T \in V_0, \tag{2.2}$$

where $B^*(E, \mathrm{R}(T_0) \times \mathrm{N}(T_0))$ denotes the set of all invertible operators in $B(E, \mathrm{R}(T_0) \times \mathrm{N}(T_0))$.

3. Main Results

In this section, we provide the local conjugacy theorem for C^1 operators between Banach manifolds. As the corollaries, a series of local conjugacy theorems for Fredholm operators, semi-Fredholm operators, and finite rank operators between Banach manifolds are established.

Theorem 3.1. *Let M, N be C^k Banach manifolds, $x_0 \in M$, $k \geq 1$. Suppose that $f : M \to N$ is C^1. Then $f(x)$ is locally conjugate to $f'(x_0)$ near x_0 if and only if under some admissible charts (U, φ) of M at x_0 and (V, ψ) of N at $y_0 = f(x_0)$, we have x_φ^0 is a locally fine point of $\overline{f}'(x_\varphi)$.*

Proof. **The sufficiency.**

The local fine property of $\overline{f}'(x_\varphi)$ at x_φ^0 implies that $\overline{f}'(x_\varphi^0)$ has a generalized inverse $(\overline{f}'(x_\varphi^0))^+$. Let

$$u(x_\varphi) = (\overline{f}'(x_\varphi^0))^+(\overline{f}(x_\varphi) - \overline{f}(x_\varphi^0)) + (I - (\overline{f}'(x_\varphi^0))^+\overline{f}'(x_\varphi^0))(x_\varphi - x_\varphi^0),$$

where $(\overline{f}'(x_\varphi^0))^+$ is the generalized inverse of $\overline{f}'(x_\varphi^0)$, $x_\varphi^0 = \varphi(x_0)$, I denotes the identity operator. Obviously, $u(x_\varphi^0) = 0$ and $u'(x_\varphi^0) = I$. By the inverse map theorem, there exists an open disk $D_r^{X_\varphi}(0)$ such that

$$u : u^{-1}(D_r^{X_\varphi}(0)) \to D_r^{X_\varphi}(0)$$

is a diffeomorphism. According to the continuity of $(\overline{f}'(x_\varphi^0))^+(\overline{f}(x_\varphi) - \overline{f}(x_\varphi^0))$ at x_φ^0, there exists an open disk $D_\rho^{X_\varphi}(x_\varphi^0) \subset u^{-1}(D_r^{X_\varphi}(0))$, such that

$$(\overline{f}'(x_\varphi^0))^+(\overline{f}(x_\varphi) - \overline{f}(x_\varphi^0)) \in D_r^{X_\varphi}(0) \tag{3.1}$$

for all $x_\varphi \in D_\rho^{X_\varphi}(x_\varphi^0)$. Then

$$u : D_\rho^{X_\varphi}(x_\varphi^0) \to u(D_\rho^{X_\varphi}(x_\varphi^0)) \text{ is a diffeomorphism,} \tag{3.2}$$

and

$$\begin{aligned}(\overline{f} \circ u^{-1})((\overline{f}'(x_\varphi^0))^+(\overline{f}(x_\varphi) - \overline{f}(x_\varphi^0)) + (I - (\overline{f}'(x_\varphi^0))^+\overline{f}'(x_\varphi^0))(x_\varphi - x_\varphi^0))\\ = (\overline{f} \circ u^{-1})((\overline{f}'(x_\varphi^0))^+(\overline{f}(x_\varphi) - \overline{f}(x_\varphi^0))).\end{aligned} \tag{3.3}$$

Now we show (3.3). Let $y_1 = (\overline{f}'(x_\varphi^0))^+(\overline{f}(x_\varphi) - \overline{f}(x_\varphi^0))$ and $y_2 = y_1 + (I - (\overline{f}'(x_\varphi^0))^+\overline{f}'(x_\varphi^0))(x_\varphi - x_\varphi^0)$, so $y_2 = u(x_\varphi)$. Since (3.1) and (3.2), we see that for any $x_\varphi \in D_\rho^{X_\varphi}(x_\varphi^0)$, both y_1 and y_2 belong to $D_r^{X_\varphi}(0)$, so that

$$ty_1 + (1-t)y_2 = y_1 + (1-t)(I - (\overline{f}'(x_\varphi^0))^+\overline{f}'(x_\varphi^0))(x_\varphi - x_\varphi^0) \in D_r^{X_\varphi}(0)$$

for any $x_\varphi \in D_\rho^{X_\varphi}(x_\varphi^0)$ and each $t \in [0,1]$.

Consider that $\Phi(t) = (\overline{f} \circ u^{-1})(y_1 + (1-t)(I - (\overline{f}'(x_\varphi^0))^+\overline{f}'(x_\varphi^0))(x_\varphi - x_\varphi^0))$. By differentiation,

$$\frac{d}{dt}\Phi(t) = (\overline{f}' \circ u^{-1})(ty_1 + (1-t)y_2) \cdot ((u')^{-1} \circ u^{-1})(ty_1 + (1-t)y_2) \cdot ((\overline{f}'(x_\varphi^0))^+\overline{f}'(x_\varphi^0) - I).$$

Since that $\mathrm{R}((\overline{f}'(x_\varphi^0))^+\overline{f}'(x_\varphi^0) - I) = \mathrm{N}(\overline{f}'(x_\varphi^0))$, we obtain that $\frac{d}{dt}\Phi(t) = 0$, for all $t \in [0,1]$, then (3.3) holds.

In the following, we proceed to construct another diffeomorphism v which is required for locally conjugate. Let

$$v(y_\psi) = (\overline{f} \circ u^{-1} \circ (\overline{f}'(x_\varphi^0))^+)(y_\psi) + (I - \overline{f}'(x_\varphi^0)(\overline{f}'(x_\varphi^0))^+)y_\psi,$$

for all $y_\psi \in D_l^{Y_\psi}(0)$. Obviously, $v(0) = \overline{f}(x_\varphi^0)$ and

$$v'(0) = \overline{f}'(x_\varphi^0) \cdot (u^{-1})'(0) \cdot (\overline{f}'(x_\varphi^0))^+ + (I - \overline{f}'(x_\varphi^0)(\overline{f}'(x_\varphi^0))^+) = \overline{f}'(x_\varphi^0)(\overline{f}'(x_\varphi^0))^+ + (I - \overline{f}'(x_\varphi^0)(\overline{f}'(x_\varphi^0))^+) = I.$$

By the inverse map theorem we assume that there is an open disk $D_m^{Y_\psi}(0)$ with $0 < m < l$, such that

$$v : D_m^{Y_\psi}(0) \to v(D_m^{Y_\psi}(0))$$

is a diffeomorphism. Since that the boundedness of $\overline{f}'(x_\varphi^0)$, there is an open disk $D_q^{X_\varphi}(x_\varphi^0) \subset D_\rho^{X_\varphi}(x_\varphi^0)$ such that $\overline{f}'(x_\varphi^0)(x_\varphi) \in D_m^{Y_\psi}(0)$, for all $x \in u(D_q^{X_\varphi}(x_\varphi^0))$. Notice that (3.2) and (3.3) keep valid in $D_q^{X_\varphi}(x_\varphi^0) \subset D_\rho^{X_\varphi}(x_\varphi^0)$, it follows that

$$\begin{aligned}\overline{f}(x_\varphi) = (\overline{f} \circ u^{-1} \circ u)(x_\varphi) &= (\overline{f} \circ u^{-1})((\overline{f}'(x_\varphi^0))^+(\overline{f}(x_\varphi) - \overline{f}(x_\varphi^0)) + (I - (\overline{f}'(x_\varphi^0))^+\overline{f}'(x_\varphi^0))(x_\varphi - x_\varphi^0))\\ &= (\overline{f} \circ u^{-1})((\overline{f}'(x_\varphi^0))^+(\overline{f}(x_\varphi) - \overline{f}(x_\varphi^0)))\end{aligned} \tag{3.4}$$

for all $x_\varphi \in D_q^{X_\varphi}(x_\varphi^0)$. Since that v is a diffeomorphism and $D_m^{Y_\psi}(0) \subset D_l^{Y_\psi}(0)$, $(v \circ \overline{f}'(x_\varphi^0) \circ u)(x_\varphi)$ is determined for every $x_\varphi \in D_q^{X_\varphi}(x_\varphi^0)$. We have

$$\begin{aligned}(v \circ \overline{f}'(x_\varphi^0) \circ u)(x_\varphi) &= (v \circ \overline{f}'(x_\varphi^0))((\overline{f}'(x_\varphi^0))^+(\overline{f}(x_\varphi) - \overline{f}(x_\varphi^0)) + (I - (\overline{f}'(x_\varphi^0))^+\overline{f}'(x_\varphi^0))(x_\varphi - x_\varphi^0)) \\ &= v(\overline{f}'(x_\varphi^0)(\overline{f}'(x_\varphi^0))^+(\overline{f}(x_\varphi) - \overline{f}(x_\varphi^0))) \\ &= (\overline{f} \circ u^{-1})((\overline{f}'(x_\varphi^0))^+\overline{f}'(x_\varphi^0)(\overline{f}'(x_\varphi^0))^+(\overline{f}(x_\varphi) - \overline{f}(x_\varphi^0))) \\ &\quad + (I - \overline{f}'(x_\varphi^0)(\overline{f}'(x_\varphi^0))^+)(\overline{f}'(x_\varphi^0)(\overline{f}'(x_\varphi^0))^+(\overline{f}(x_\varphi) - \overline{f}(x_\varphi^0))) \\ &= (\overline{f} \circ u^{-1})((\overline{f}'(x_\varphi^0))^+(\overline{f}(x_\varphi) - \overline{f}(x_\varphi^0)) \end{aligned} \tag{3.5}$$

for any $x_\varphi \in D_q^{X_\varphi}(x_\varphi^0)$. Combining (3.4) and (3.5) together, we have

$$\overline{f}(x_\varphi) = (v \circ \overline{f}'(x_\varphi^0) \circ\ u)(x_\varphi)$$

for all $x_\varphi \in D_q^{X_\varphi}(x_\varphi^0)$, where $u : D_q^{X_\varphi}(x_\varphi^0) \to u(D_q^{X_\varphi}(x_\varphi^0))$ and $v : D_m^{Y_\psi}(0) \to v(D_m^{Y_\psi}(0))$ are both diffeomorphisms.

The necessity.

If $\overline{f}(x_\varphi) = (v \circ \overline{f}'(x_\varphi^0) \circ\ u)(x_\varphi)$, for each x_φ near x_φ^0. Let

$$G(x_\varphi) = (u')^{-1}(x_\varphi)(\overline{f}'(x_\varphi^0))(v')^{-1}(\overline{f}'(x_\varphi^0) \circ\ u(x_\varphi)).$$

Then

$$G(x_\varphi)\overline{f}'(x_\varphi)G(x_\varphi) = G(x_\varphi) \ \text{ and } \ \overline{f}'(x_\varphi)G(x_\varphi)\overline{f}'(x_\varphi) = \overline{f}'(x_\varphi).$$

So $G(x_\varphi)$ is a bounded generalized inverse of $\overline{f}'(x_\varphi)$, for each x_φ near x_φ^0, and $\lim_{x_\varphi \to x_\varphi^0} G(x_\varphi) = G(x_\varphi^0)$. That is to say, the generalized inverse $G(x_\varphi)$ is continuous as $x_\varphi \to x_\varphi^0$, and then x_φ^0 is a locally fine point of $\overline{f}'(x_\varphi)$. □

Definition 3.2. Let M and N be C^k Banach manifolds, $k \geq 1$. The mapping $f : U_0 \subset M \to N$ is called a Fredholm operator at x, if and only if the linearization $f'(x)$ is a Fredholm operator from tangent space TM_x to tangent space $TN_{f(x)}$.

The index of f is defined to be

$$\mathrm{ind}(f'(x)) = \mathrm{dimN}(f'(x)) - \mathrm{codimR}(f'(x)).$$

Theorem 3.3. *Let M, N be C^k Banach manifolds, $k \geq 1$, $U(x_0)$ is an open set containing point $x_0 \in X$, $f : U(x_0) \subset M \to N$ be a C^1 Fredholm operator. Then f is conjugate to $f'(x_0)$ in the neighborhood U_0 of x_0 if and only if under some admissible chart (U, φ) of M at x_0 and admissible chart (V, ψ) of N at $f(x_0)$, we have*

$$\mathrm{dimN}(\overline{f}'(x_\varphi)) = \mathrm{dimN}(\overline{f}'(x_\varphi^0))$$

or

$$\mathrm{codimR}(\overline{f}'(x_\varphi)) = \mathrm{codimR}(\overline{f}'(x_\varphi^0)).$$

Proof. Let $f : U(x_0) \subset M \to N$ be a C^1 Fredholm operator, we first show that x_φ^0 is a locally fine point of $\overline{f}'(x)$ if and only if under some admissible chart (U, φ) of M at x_0 and so admissible chart (V, ψ) of N at $f(x_0)$, we have

$$\mathrm{dimN}(\overline{f}'(x_\varphi)) = \mathrm{dimN}(\overline{f}'(x_\varphi^0))$$

or

$$\mathrm{codimR}(\overline{f}'(x_\varphi)) = \mathrm{codimR}(\overline{f}'(x_\varphi^0)).$$

The sufficiency.

By the definition of Fredholm operators, we know that $\dim N(\overline{f}'(x_\varphi))$ and $\operatorname{codim} R(\overline{f}'(x_\varphi))$ are both finite for all $x_\varphi \in \varphi(U)$. Therefore, the chart space X_φ and Y_ψ have the following direct sum decompositions

$$X_\varphi = N(\overline{f}'(x_\varphi)) \oplus N(\overline{f}'(x_\varphi))^-$$

and

$$Y_\psi = R(\overline{f}'(x_\varphi)) \oplus R(\overline{f}'(x_\varphi))^-$$

for all $x_\varphi \in \varphi(U)$.

Clearly,

$$\overline{f}'(x_\varphi)\mid_{N(\overline{f}'(x_\varphi))^-} : N(\overline{f}'(x_\varphi))^- \to R(\overline{f}'(x_\varphi))$$

is onto and one-to-one, which gives a generalized inverse $\overline{f}'(x_\varphi)^+$ of $\overline{f}'(x_\varphi)$ as follows:

$$(\overline{f}'(x_\varphi))^+h = \begin{cases} \left(\overline{f}'(x_\varphi)\mid_{N(\overline{f}'(x_\varphi))^-}\right)^{-1}h, & h \in R(\overline{f}'(x_\varphi)) \\ 0, & h \in R(\overline{f}'(x_\varphi))^- \end{cases}.$$

Thus, by Lemma 2.2, there is a neighborhood at x_0 contained in U, written as U_0, in which (2.1) and (2.2) are satisfied.

If $\dim N(\overline{f}'(x_\varphi)) = \dim N(\overline{f}'(x^0_\varphi)) < \infty$ is constant, for all $x_\varphi \in \varphi(U_0)$, then it follows from (2.2) and

$$N(\overline{f}'(x^0_\varphi)(\overline{f}'(x^0_\varphi))^+\overline{f}'(x_\varphi)) \supset N(\overline{f}'(x_\varphi)),$$

that

$$N(\overline{f}'(x^0_\varphi)(\overline{f}'(x^0_\varphi))^+\overline{f}'(x_\varphi)) = N(\overline{f}'(x_\varphi)).$$

Note that

$$N(\overline{f}'(x^0_\varphi)(\overline{f}'(x^0_\varphi))^+\overline{f}'(x_\varphi)) = N(\overline{f}'(x_\varphi)) + \{u \in X_\varphi : \overline{f}'(x_\varphi)[u] = h \in R(\overline{f}'(x^0_\varphi))^-\},$$

where

$$[u] \in X_\varphi / N(\overline{f}'(x_\varphi)),\ \overline{f}'(x_\varphi)[u] = \overline{f}'(x_\varphi)u,$$

and

$$\begin{aligned} R(\overline{f}'(x^0_\varphi))^- &= R(I_{Y_\psi} - \overline{f}'(x^0_\varphi)(\overline{f}'(x^0_\varphi))^+) \\ &= N((\overline{f}'(x^0_\varphi))^+), \end{aligned}$$

then

$$R(\overline{f}'(x_\varphi)) \cap R(\overline{f}'(x^0_\varphi))^- = R(\overline{f}'(x_\varphi)) \cap N((\overline{f}'(x^0_\varphi))^+) = \{0\}.$$

If $\operatorname{codim} R(\overline{f}'(x_\varphi)) = \operatorname{codim} R(\overline{f}'(x^0_\varphi)) < \infty$ is constant, for all $x_\varphi \in \varphi(U_0)$, then

$$R(\overline{f}'(x^0_\varphi)(\overline{f}'(x^0_\varphi))^+\overline{f}'(x_\varphi)) = R(\overline{f}'(x^0_\varphi)),$$

so

$$Y_\psi = R(\overline{f}'(x^0_\varphi)) \oplus R(\overline{f}'(x^0_\varphi))^- = R(\overline{f}'(x_\varphi)) + R(\overline{f}'(x^0_\varphi))^- = R(\overline{f}'(x_\varphi)) \oplus \mathrm{R},$$

where R satisfies $R(\overline{f}'(x^0_\varphi))^- = \mathrm{R} \oplus (R(\overline{f}'(x_\varphi)) \cap R(\overline{f}'(x^0_\varphi))^-)$. Hence

$$R(\overline{f}'(x_\varphi)) \cap R(\overline{f}'(x^0_\varphi))^- = \{0\}$$

and

$$\mathrm{R}(\overline{f}'(x_\varphi)) \cap \mathrm{N}((\overline{f}'(x_\varphi^0))^+) = \{0\}.$$

Then we can draw the conclusion that x_φ^0 is a locally fine point of $\overline{f}'(x_\varphi)$.

The necessity.

Assume that x_φ^0 is a locally fine point of $\overline{f}'(x_\varphi)$ to find out that

$$\dim\mathrm{N}(\overline{f}'(x_\varphi)) = \dim\mathrm{N}(\overline{f}'(x_\varphi^0))$$

or

$$\mathrm{codim}\mathrm{R}(\overline{f}'(x_\varphi)) = \mathrm{codim}\mathrm{R}(\overline{f}'(x_\varphi^0)),$$

under some admissible chart (U, φ) of M at x_0 and admissible chart (V, ψ) of N at $f(x_0)$.

If x_φ^0 is a locally fine point of $\overline{f}'(x_\varphi)$, then under some admissible charts (U, φ) of M at x_0 and (V, ψ) of N at $f(x_0)$,

$$\mathrm{R}(\overline{f}'(x_\varphi)) \cap \mathrm{N}((\overline{f}'(x_\varphi^0))^+) = \{0\},$$

then the following four equalities hold in some neighborhood of x_φ^0,

$$(I - (\overline{f}'(x_\varphi^0))^+\overline{f}'(x_\varphi^0))\mathrm{N}(\overline{f}'(x_\varphi)) = \mathrm{N}(\overline{f}'(x_\varphi^0)),$$

$$(I - (\overline{f}'(x_\varphi))^+\overline{f}'(x_\varphi))\mathrm{N}(\overline{f}'(x_\varphi^0)) = \mathrm{N}(\overline{f}'(x_\varphi)),$$

$$(I - \overline{f}'(x_\varphi^0)(\overline{f}'(x_\varphi^0))^+)\mathrm{N}((\overline{f}'(x_\varphi))^+) = \mathrm{N}((\overline{f}'(x_\varphi^0))^+),$$

$$(I - \overline{f}'(x_\varphi)(\overline{f}'(x_\varphi))^+)\mathrm{N}((\overline{f}'(x_\varphi^0))^+) = \mathrm{N}((\overline{f}'(x_\varphi))^+).$$

Now the necessity of theorem follows.

By Theorem 3.1, we have f is locally conjugate to $f'(x_0)$ if and only if

$$\dim\mathrm{N}(\overline{f}'(x_\varphi)) = \dim\mathrm{N}(\overline{f}'(x_\varphi^0))$$

or

$$\mathrm{codim}\mathrm{R}(\overline{f}'(x_\varphi)) = \mathrm{codim}\mathrm{R}(\overline{f}'(x_\varphi^0)).$$

The proof is completed. □

A bounded linear operator has a generalized inverse if and only if it is a double splitting operator. Then the following Corollary 3.4 holds.

Corollary 3.4. *Let M and N be C^k Banach manifolds, $k \geq 1$, $f : U(x_0) \subset M \to N$ be a C^1 semi-Fredholm operator, if $f'(x_0)$ has a generalized inverse $T_0^+ \in B(TN_{f(x)}, TM_x)$, then f is locally conjugate to $f'(x_0)$ in a neighborhood U_0 of x_0 if and only if under some admissible chart (U, φ) of M at x_0, and (V, ψ) of N at $f(x_0)$, we have*

$$\dim\mathrm{N}(\overline{f}'(x_\varphi)) = \dim\mathrm{N}(\overline{f}'(x_\varphi^0))$$

or

$$\mathrm{codim}\mathrm{R}(\overline{f}'(x_\varphi)) = \mathrm{codim}\mathrm{R}(\overline{f}'(x_\varphi^0)).$$

Corollary 3.5. *Let M and N be C^k Banach manifolds, $k \geq 1$, $f : U \subset M \to N$ be a C^1 mapping, $x_0 \in U$. If $\dim\mathrm{R}(f'(x))$ is finite constant in the neighborhood U_0 of x_0, then f is locally conjugate to $f'(x_0)$.*

Proof. Notice that $\dim\mathrm{R}(\overline{f}'(x_\varphi)) = \dim\mathrm{R}(\overline{f}'(x_\varphi^0))$ is finite, for all $x_\varphi \in \varphi(U)$, since $\dim\mathrm{R}(f'(x))$ is finite constant. So $\mathrm{codim}\mathrm{N}(\overline{f}'(x_\varphi)) = \mathrm{codim}\mathrm{N}(\overline{f}'(x_\varphi^0))$ and $\dim\mathrm{R}(\overline{f}'(x_\varphi)) = \dim\mathrm{R}(\overline{f}'(x_\varphi^0))$ are all finite. Hence, the chart spaces X_φ and Y_ψ have the direct sum decompositions $X_\varphi = \mathrm{N}(\overline{f}'(x_\varphi)) \oplus \mathrm{N}(\overline{f}'(x_\varphi))^-$ and $Y_\psi = \mathrm{R}(\overline{f}'(x_\varphi)) \oplus \mathrm{R}(\overline{f}'(x_\varphi))^-$, for all $x_\varphi \in \varphi(U)$. Then, $\overline{f}'|_{\mathrm{N}(\overline{f}'(x_\varphi))^-} : \mathrm{N}(\overline{f}'(x_\varphi))^- \to \mathrm{R}(\overline{f}'(x_\varphi))$ is a bijective mapping, further more, there exists a generalized inverse

$$(\overline{f}'(x_\varphi))^+z = \begin{cases} (\overline{f}'(x_\varphi)\mid_{\mathrm{N}(\overline{f}'(x_\varphi))^-})^{-1}z, & z \in \mathrm{R}(\overline{f}'(x_\varphi)) \\ 0, & z \in \mathrm{R}(\overline{f}'(x_\varphi))^- \end{cases}.$$

Thus, by Lemma 2.2, there exists a neighborhood at x_0 contained in U, written as U_0, in which (2.1) is satisfied. Then $\mathrm{R}(\overline{f}'(x^0_\varphi)(\overline{f}'(x^0_\varphi))^+(\overline{f}'(x_\varphi))) = \mathrm{R}(\overline{f}'(x^0_\varphi))$. So the chart space

$$Y_\psi = \mathrm{R}(\overline{f}'(x^0_\varphi)) \oplus \mathrm{R}(\overline{f}'(x^0_\varphi))^- = \mathrm{R}(\overline{f}'(x_\varphi)) + \mathrm{R}(\overline{f}'(x^0_\varphi))^- = \mathrm{R}(\overline{f}'(x_\varphi)) \oplus \mathrm{R},$$

where R satisfies that $\mathrm{R}(\overline{f}'(x^0_\varphi))^- = \mathrm{R} \oplus (\mathrm{R}(\overline{f}'(x_\varphi)) \cap \mathrm{R}(\overline{f}'(x^0_\varphi))^-)$. According to this,

$$\mathrm{R}(\overline{f}'(x_\varphi)) \cap \mathrm{R}(\overline{f}'(x^0_\varphi))^- = \{0\},$$

thus, x^0_φ is a locally fine point of $\overline{f}'(x_\varphi)$ and f is locally conjugate to $f'(x_0)$. □

As a direct conclusion of Corollary 3.5, the following result holds.

Corollary 3.6. *Let M and N be C^k Banach manifolds, $k \geq 1$, $f : U \subset M \to N$ be a C^1 finite rank mapping. Then f is locally conjugate to $f'(x_0)$.*

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