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A new convergence theorem in a reflexive Banach space

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Abstract

In this paper, fixed points of an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense and a bifunction are investigated based on a monotone projection algorithm. A strong convergence theorem is established in a reflexive Banach space.

Keywords: Equilibrium problem point, quasi-ϕ-nonexpansive mapping, fixed point, projection, variational inequality.
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1. Introduction

Let C be a convex and closed subset of a Banach space E and let $B : C \times C \to \mathbb{R}$ be a bifunction. Consider the following equilibrium problem in the terminology of Blum and Oettli [4]; find $\bar{x} \in C$ such that $B(\bar{x}, y) \geq 0, \forall y \in C$. The equilibrium problem, which was introduced by Ky Fan [15] in 1972, has had a great impact and influence in the development of several branches of pure and applied sciences. The equilibrium problem, which includes variational inequality problems, variational inclusion problems, Nash equilibrium and game theory as special cases, has been shown that it provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization; see [3], [12]-[16], [19], and the references therein.

Fixed point methods recently have been used to study solutions of the equilibrium problem. Krasnoselskii-Mann iteration method, which is also known as a one-step iteration method, is an important method to study

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fixed points of nonlinear operators, in particular, nonexpansive operators. However, Krasnoselskii-Mann iteration only has the weak convergence for nonexpansive mappings; see [17] and the references therein. There are a lot of real world problems, including economics [20], image recovery [12], quantum physics [13], and control theory [16], which exist in infinite dimension spaces. In such problems, strong convergence or norm convergence is often much more desirable than the weak convergence. To obtain the strong convergence of the Krasnoselskii-Mann iteration, regularization techniques recently have been extensively investigated; see [7]-[11], [18], [21]-[23], [27]-[32] and the references therein.

In this paper, we suggest and analyze a monotone projection algorithm for the equilibrium problem and a fixed point problem. Strong convergence of the algorithm is obtained without the aid of compactness on a non-uniformly convex Banach space. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, convergence analysis of the algorithm is obtained on a non-uniformly convex Banach space. Some subresults are also provided as corollaries.

2. Preliminaries

Let E be a real Banach space and let E^* be the dual space of E. Let S^E be the unit sphere of E. Recall that E is said to be a strictly convex space iff $\|x+y\| < 2$ for all $x, y \in S^E$ and $x \neq y$. Recall that E is said to have a Gâteaux differentiable norm iff

$$\lim_{t\to 0} \frac{\|x\| - \|x+ty\|}{t}$$

exists for each $x, y \in S^E$. In this case, we also say that E is smooth. E is said to have a uniformly Gâteaux differentiable norm iff for each $y \in S^E$, the limit is attained uniformly for all $x \in S^E$. E is also said to have a uniformly Fréchet differentiable norm iff the above limit is attained uniformly for $x, y \in S^E$. In this case, we say that E is uniformly smooth.

Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{y \in E^* : \|x\|^2 = \langle x, y\rangle = \|y\|^2\}.$$

It is known,

- if E is uniformly smooth, then J is uniformly norm-to-norm continuous on every bounded subset of E;
- if E is a strictly convex Banach space, then J is strictly monotone;
- if E is a smooth Banach space, then J is single-valued and demicontinuous, i.e., continuous from the strong topology of E to the weak star topology of E;
- if E is a reflexive and strictly convex Banach space with a strictly convex dual E^* and $J^* : E^* \to E$ is the normalized duality mapping in E^*, then $J^{-1} = J^*$;
- if E is a smooth, strictly convex and reflexive Banach space, then J is single-valued, one-to-one and onto;
- if E is uniformly smooth, then it is smooth and reflexive.

It is also known that E^* is uniformly convex if and only if E is uniformly smooth.

From now on, we use $\rightharpoonup$ and $\to$ to stand for the weak convergence and strong convergence, respectively.

Recall that E is said to have the Kadec-Klee property iff $\lim_{n\to\infty} \|x_n - x\| = 0$ as $n \to \infty$, for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$ as $n \to \infty$.

Let C be a convex and closed subset of E and let T be a mapping on C. Recall that a point p is said to be a fixed point of T iff $p = Tp$. T is said to be closed iff for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x'$

and $\lim_{n\to\infty} Tx_n = y'$, then $x' \in D(T)$ and $Tx' = y'$. Let D be a bounded subset of C. Recall that T is said to be uniformly asymptotically regular on C iff

$$\limsup_{n\to\infty} \sup_{x\in D}\{\|T^{n+1}x - T^n x\|\} = 0.$$

Next, we assume that E is a smooth Banach space which means J is single-valued. Study the functional

$$\phi(x,y) := \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle, \quad \forall x,y \in E.$$

Let C be a closed convex subset of a real Hilbert space H. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$, for all $y \in C$. The operator P_C is called the metric projection from H onto C. It is known that P_C is firmly nonexpansive. In [2], Alber studied a new mapping Π_C in a Banach space E which is an analogue of P_C, the metric projection, in Hilbert spaces. Recall that the generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of $\phi(x,y)$, which implies from the definition of ϕ that

$$\phi(x,y) - \phi(x,z) = \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x,y,z \in E, \tag{2.1}$$

and

$$(\|y\| + \|x\|)^2 \ge \phi(x,y) \ge (\|x\| - \|y\|)^2, \quad \forall x,y \in E. \tag{2.2}$$

Recall that T is said to be quasi-ϕ-nonexpansive [23] iff

$$Fix(T) \neq \emptyset, \phi(p,Tx) \le \phi(p,x), \quad \forall x \in C, \forall p \in Fix(T).$$

In the framework of Hilbert spaces, it reduces to the

$$Fix(T) \neq \emptyset, \|p - Tx\| \le \|p - x\|, \quad \forall x \in C, \forall p \in Fix(T).$$

T is said to be asymptotically quasi-ϕ-nonexpansive [24] iff there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$Fix(T) \neq \emptyset, \phi(p,T^n x) \le (\mu_n + 1)\phi(p,x), \quad \forall x \in C, \forall p \in Fix(T), \forall n \ge 1.$$

In the framework of Hilbert spaces, it reduces to the

$$Fix(T) \neq \emptyset, \|p - T^n x\| \le \sqrt{\mu_n + 1}\|p - x\|, \quad \forall x \in C, \forall p \in Fix(T).$$

Remark 2.1. The class of asymptotically quasi-ϕ-nonexpansive mappings and the class of quasi-ϕ-nonexpansive mappings are more desirable than the class of asymptotically relatively nonexpansive mappings and the class of relatively nonexpansive mappings; see [1] and [6] and the references therein.

T is said to be asymptotically quasi-ϕ-nonexpansive in the intermediate sense [25] iff $Fix(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{p\in Fix(T), x\in C} \big(\phi(p,T^n x) - \phi(p,x)\big) \le 0.$$

Putting $\xi_n = \max\{0, \sup_{p\in Fix(T),x\in C} \big(\phi(p,T^n x) - \phi(p,x)\big)\}$, we see $\xi_n \to 0$ as $n \to \infty$. It follows that

$$\phi(p,T^n x) \le \phi(p,x) + \xi_n, \quad \forall x \in C, \forall p \in Fix(T), \forall n \ge 1.$$

In the framework of Hilbert spaces, it reduces to the

$$\limsup_{n\to\infty} \sup_{p\in Fix(T), x\in C} \big(\|p - T^n x\| - |p - x\|\big) \le 0.$$

Remark 2.2. The class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense [25] is reduced to the class of asymptotically quasi-nonexpansive mappings in the intermediate sense, which was considered in [5] as a non-Lipschitz continuous mapping, in the framework of Hilbert spaces.

Let $B : C \times C \to \mathbb{R}$ be a bifunction. Recall that the following equilibrium problem in the terminology of Blum and Oettli [4]: find $\bar{x} \in C$ such that $B(\bar{x}, y) \geq 0$, $\forall y \in C$. We use $Sol(B)$ to denote the solution set of the equilibrium problem. That is, $Sol(B) = \{x \in C : B(x, y) \geq 0, \forall y \in C\}$. The following restrictions on bifunction B are essential in this paper.

(R-1) $B(a, a) \equiv 0, \forall a \in C$;

(R-2) $B(b, a) + B(a, b) \leq 0, \forall a, b \in C$;

(R-3) $B(a, b) \geq \limsup_{t\downarrow 0} B(tc + (1 - t)a, b)$, $\forall a, b, c \in C$;

(R-4) $b \mapsto B(a, b)$ is convex and lower semi-continuous, $\forall a \in C$.

We remark here that B is said to be monotone iff $B(x, y) + B(y, x) \leq 0$ for all $x, y \in C$. $y \mapsto B(x, y)$ is convex iff $B(tx + (1-t)y, z) \leq tB(x, z) + (1-t)B(y, z)$ for all $x, y, z \in C$ and $t \in (0, 1)$. $y \mapsto B(x, y)$ is lower semi-continuous iff $B(x, y_n) \to B(x, y)$ whenever $y_n \to y$ as $n \to \infty$. It is known that the indicator function of an open set is lower semi-continuous.

In addition, we also need the following lemmas to prove our main results.

Lemma 2.3 ([4]). *Let E be a strictly convex, reflexive, and smooth Banach space and let C be a closed and convex subset of E. Let $x \in E$. Then $\langle y - x_0, Jx - Jx_0\rangle \leq 0$, $\forall y \in C$ if and only if $x_0 = \Pi_C x$ and*

$$\phi(y, \Pi_C x) \leq \phi(y, x) - \phi(\Pi_C x, x), \quad \forall y \in C.$$

Lemma 2.4 ([4], [23]). *Let E be a strictly convex, smooth, and reflexive Banach space and let C be a closed convex subset of E. Let B be a bifunction with restrictions (R-1), (R-2), (R-3) and (R-4). Let $x \in E$ and let $r > 0$. Then there exists $z \in C$ such that $rB(z, y) \leq \langle y - z, Jz - Jx\rangle$, $\forall y \in C$. Define a mapping $W^{B,r}$ by*

$$W^{B,r}x = \{z \in C : rB(z, y) + \langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C\}.$$

The following conclusions hold:

(1) *$W^{B,r}$ is single-valued quasi-ϕ-nonexpansive.*

(2) *$Sol(B) = Fix(W^{B,r})$ is closed and convex.*

Lemma 2.5 ([26]). *Let E be a uniformly convex Banach space and let r be a positive real number. Then there exists a convex, strictly increasing and continuous function $\lambda : [0, 2r] \to \mathbb{R}$ such that $\lambda(0) = 0$ and*

$$\|(1 - t)b + ta\|^2 + t(1 - t)\lambda(\|b - a\|) \leq t\|a\|^2 + (1 - t)\|b\|^2$$

for all $a, b \in S_r^E := \{a \in E : \|a\| \leq r\}$ and $t \in [0, 1]$.

3. Main results

Theorem 3.1. *Let E be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let C be a convex and closed subset of E and let B be a bifunction with conditions (R-1), (R-2), (R-3) and (R-4). Let T be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense on*

C. Assume that T is uniformly asymptotically regular and closed on C and $Fix(T) \cap Sol(B)$ is nonempty. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ r_n B(u_n, \mu) \geq \langle u_n - \mu, Ju_n - Jx_n \rangle, \forall \mu \in C_n, \\ y_n = J^{-1}\big((1-\alpha_n)Ju_n + \alpha_n JT^n x_n\big), \\ C_{n+1} = \{z \in C_n : \phi(z, x_n) + \xi_n \geq \phi(z, y_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\{\alpha_n\}$ is a real sequence in (0,1) such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\{r_n\} \subset [r, \infty)$ is a real sequence, where r is some positive real number and $\xi_n = \max\{\sup_{p\in Fix(T), x\in C} \big(\phi(p, T^n x) - \phi(p, x)\big), 0\}$. Then $\{x_n\}$ converges strongly to $\Pi_{Fix(T)\cap Sol(B)} x_1$.

Proof. Let $e, f \in Fix(T)$, and $g = se + (1-s)f$, where $s \in (0,1)$. Note that $\phi(e, T^n g) \leq \phi(e, g) + \xi_n$, and $\phi(f, T^n g) \leq \phi(f, g) + \xi_n$. In view of (2.1), we obtain that

$$\phi(e, T^n g) = \phi(e, g) + 2\langle e - g, Jg - JT^n g\rangle + \phi(g, T^n g),$$

and

$$\phi(f, T^n g) = \phi(f, g) + 2\langle f - g, Jg - JT^n g\rangle + \phi(g, T^n g).$$

It follows that

$$\phi(g, T^n g) \leq 2\langle g - e, Jg - J(T^n g)\rangle + \xi_n, \tag{3.1}$$

and

$$\phi(g, T^n g) \leq 2\langle g - f, Jg - J(T^n g)\rangle + \xi_n. \tag{3.2}$$

Multiplying $1-s$ and s on the both sides of (3.1) and (3.2), respectively yields that $\phi(g, T^n g) \leq \xi_n$. Hence, one has $\lim_{n\to\infty} \|T^n g\| = \|g\|$. Since E is uniformly smooth, we have E^* is uniformly convex. Hence, E^* is reflexive, we may, without loss of generality, assume that $J(T^n g) \rightharpoonup v^* \in E^*$. In view of the reflexivity of E, we have $J(E) = E^*$. This shows that there exists an element $v \in E$ such that $Jv = v^*$. It follows that

$$\phi(g, T^n g) = \|g\|^2 - 2\langle g, J(T^n g)\rangle + \|J(T^n g)\|^2.$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above, we obtain that

$$\begin{aligned} 0 &\geq \|g\|^2 - 2\langle g, v^*\rangle + \|v^*\|^2 \\ &= \|g\|^2 - 2\langle g, Jv\rangle + \|Jv\|^2 \\ &= \|g\|^2 - 2\langle g, Jv\rangle + \|v\|^2 \\ &= \phi(g, v) \\ &\geq 0. \end{aligned}$$

This implies that $g = v$, that is, $Jg = v^*$. It follows that

$$J(T^n g) \rightharpoonup Jg \in E^*.$$

Using the Kadec-Klee property of E^*, we obtain

$$\lim_{n\to\infty} \|J(T^n g) - Jg\| = 0.$$

Since J^{-1} is a demicontinuous mapping, we see that $T^n g \rightharpoonup g$. By virtue of the Kadec-Klee property of E, we see $T^n g \to g$ as $n \to \infty$. Hence, $TT^n g = T^{n+1}g \to g$, as $n \to \infty$. In view of the closedness of T, we obtain that $g \in Fix(T)$. This shows that $Fix(T)$ is convex. Using Lemma 2.4, we find that $Sol(B)$ is convex and closed. This proves that $Sol(B) \cap Fix(T)$ is convex and closed. So, $Proj_{Sol(B)\cap Fix(T)}x$ is well defined, for any element x in E.

Next, we prove that C_n is convex and closed. It is obvious that $C_1 = C$ is convex and closed. Assume that C_m is convex and closed for some $m \geq 1$. Let $p_1, p_2 \in C_{m+1}$. It follows that

$$p = sp_1 + (1-s)p_2 \in C_m,$$

where $s \in (0,1)$. Notice that

$$\phi(p_1, y_m) - \phi(p_1, x_m) \leq \xi_m,$$

and

$$\phi(p_2, y_m) - \phi(p_2, x_m) \leq \xi_m.$$

Hence, one has

$$2\langle p_1, Jx_m - Jy_m\rangle - \|x_m\|^2 \leq \xi_m - \|u_m\|^2,$$

and

$$2\langle p_2, Jx_m - Jy_m\rangle - \|x_m\|^2 \leq \xi_m - \|u_m\|^2.$$

Using the above two inequalities, one has

$$\phi(p, x_m) + \xi_m \geq \phi(z, y_m).$$

This shows that C_{m+1} is closed and convex. Hence, C_n is a convex and closed set. This proves that $\Pi_{C_{n+1}}x_1$ is well defined.

Next, we prove $Sol(B) \cap Fix(T) \subset C_n$. Note that $Sol(B) \cap Fix(T) \subset C_1 = C$ is clear. Suppose that $Sol(B) \cap Fix(T) \subset C_m$ for some positive integer m. For any $w \in Sol(B) \cap Fix(T) \subset C_m$, we see that

$$\begin{aligned}
\phi(w, y_m) &= \|(1-\alpha_m)Ju_m + \alpha_m JT^m x_m\|^2 + \|w\|^2 \\
&\quad - 2\langle w, (1-\alpha_m)Ju_m + \alpha_m JT^m x_m\rangle \\
&\leq (1-\alpha_m)\|u_m\|^2 + \alpha_m\|T^m x_m\|^2 + \|w\|^2 \\
&\quad - 2(1-\alpha_m)\langle w, Ju_m\rangle - 2\alpha_m\langle w, JT^m x_m\rangle \\
&= (1-\alpha_m)\phi(w, u_m) + \alpha_m\phi(w, T^m x_m) \\
&\leq \phi(w, x_m) + \alpha_m\xi_m,
\end{aligned}$$

where

$$\xi_m = \max\{\sup_{p\in Fix(T), x\in C}\big(\phi(p, T^m x) - \phi(p, x)\big), 0\}.$$

This shows that $w \in C_{m+1}$. This implies that $Sol(B) \cap Fix(T) \subset C_n$. Using Lemma 2.3, one has $\langle z - x_n, Jx_1 - Jx_n\rangle \leq 0$, for all $z \in C_n$. It follows that

$$\langle w - x_n, Jx_1 - Jx_n\rangle \leq 0, \forall w \in Sol(B) \cap Fix(T) \subset C_n.$$

Using Lemma 2.3 yields that

$$\phi(x_n, x_1) \leq \phi(\Pi_{Fix(T)\cap Sol(B)}x_1, x_1),$$

which implies that $\{\phi(x_n, x_1)\}$ and $\{x_n\}$ are bounded. Since E is a reflexive Banach space, we may assume that $x_n \rightharpoonup \bar{x} \in C_n$. Therefore, one has

$$\phi(x_n, x_1) \leq \phi(\bar{x}, x_1).$$

This implies that

$$\phi(\bar{x}, x_1) \le \liminf_{n\to\infty}(\|x_n\|^2 + \|x_1\|^2 - 2\langle x_n, Jx_1\rangle) = \liminf_{n\to\infty} \phi(x_n, x_1) \le \phi(\bar{x}, x_1).$$

It follows that

$$\lim_{n\to\infty} \phi(x_n, x_1) = \phi(\bar{x}, x_1).$$

Hence, we have $\lim_{n\to\infty} \|x_n\| = \|\bar{x}\|$. Using the Kadec-Klee of the spaces, one obtains that sequence $\{x_n\}$ converges strongly to $\bar{x}$ as $n \to \infty$. It follows that

$$\phi(x_{n+1}, x_n) \le \phi(x_{n+1}, x_1) - \phi(x_n, x_1).$$

Therefore, we have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0.$$

Since $x_{n+1} \in C_{n+1}$, one sees that

$$0 \le \phi(x_{n+1}, y_n) \le \phi(x_{n+1}, x_n) + \xi_n.$$

It follows that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

Hence, one has

$$\lim_{n\to\infty} (\|y_n\| - \|x_{n+1}\|) = 0.$$

This implies that

$$\lim_{n\to\infty} \|Jy_n\| = \lim_{n\to\infty} \|y_n\| = \|\bar{x}\| = \|J\bar{x}\|.$$

This implies that $\{Jy_n\}$ is bounded. Assume that $\{Jy_n\}$ converges weakly to $y^* \in E^*$. In view of the reflexivity of E, we see $J(E) = E^*$. This shows that there exists an element $y \in E$ such that $Jy = y^*$. It follows that

$$\phi(x_{n+1}, y_n) + 2\langle x_{n+1}, Jy_n\rangle = \|x_{n+1}\|^2 + \|Jy_n\|^2.$$

Taking $\liminf_{n\to\infty}$, one has

$$\begin{aligned} 0 &\ge \|\bar{x}\|^2 - 2\langle \bar{x}, y^*\rangle + \|y^*\|^2 \\ &= \|\bar{x}\|^2 + \|Jy\|^2 - 2\langle \bar{x}, Jy\rangle \\ &= \phi(\bar{x}, y) \\ &\ge 0. \end{aligned}$$

That is, $\bar{x} = y$, which in turn implies that $J\bar{x} = y^*$. Hence, $Jy_n \rightharpoonup J\bar{x} \in E^*$. Using the Kadec-Klee property again, we obtain $\lim_{n\to\infty} Jy_n = J\bar{x}$. Since J^{-1} is demi-continuous and E has the Kadec-Klee property, one gets $y_n \to \bar{x}$, as $n \to \infty$. It follows that

$$\lim_{n\to\infty} \big(\phi(z, x_n) - \phi(z, y_n)\big) = 0, \quad \forall z \in Fix(T) \cap Sol(B). \tag{3.3}$$

By using Lemma 2.5, one has

$$\begin{aligned} \phi(z, y_n) &\le (1-\alpha_n)\|u_n\|^2 + \alpha_n\|T^n x_n\|^2 + \|z\|^2 \\ &\quad - 2(1-\alpha_n)\langle z, Ju_n\rangle - 2\alpha_n\langle z, JT^n x_n\rangle - \alpha_n(1-\alpha_n)\lambda(\|Ju_n - JT^n x_n\|) \\ &= (1-\alpha_n)\phi(z, u_n) - \alpha_n(1-\alpha_n)\lambda(\|Ju_n - JT^n x_n\|) + \alpha_n\phi(z, T^n x_n) \\ &\le \phi(z, x_n) - \alpha_n(1-\alpha_n)\lambda(\|Ju_n - JT^n x_n\|) + \alpha_n\xi_n, \end{aligned}$$

where

$$\xi_m = \max\{\sup_{p\in Fix(T), x\in C} \big(\phi(p, T^m x) - \phi(p, x)\big), 0\}.$$

Using (3.3) and Lemma 2.5, one has

$$\lim_{j\to\infty} \||Ju_n - JT^n x_n\| = 0.$$

It follows that $JT^n x_n \to J\bar{x}$. Since $J^{-1} : E^* \to E$ is demi-continuous, one has $T^n x_n \rightharpoonup \bar{x}$. This further implies $\|T^n x_n\| \to \|\bar{x}\|$ as $n \to \infty$. Since E has the Kadec-Klee property, one has

$$\lim_{n\to\infty} \||\bar{x} - T^n x_n\| = 0.$$

Since T is also uniformly asymptotically regular, one has $\lim_{n\to\infty} \|\bar{x} - T^{n+1} x_n\| = 0$. That is, $T(T^n x_n) \to \bar{x}$. Using the closedness of T, we find $T\bar{x} = \bar{x}$. This proves $\bar{x} \in Fix(T)$.

Next, we show that $\bar{x} \in Sol(B)$. Since B is monotone, we find that $B(\mu, \bar{x}) \le 0$. For $0 < t < 1$, define

$$\mu^t = (1 - t)\bar{x} + t\mu.$$

This implies that $0 \ge B(\mu^t, \bar{x})$. Hence, we have

$$0 = B(\mu^t, \mu^t) \le tB(\mu^t, \mu).$$

It follows that $B(\bar{x}, \mu) \ge 0$, $\forall \mu \in C$. This implies that $\bar{x} \in Sol(B)$.

Finally, we prove $\bar{x} = \Pi_{Sol(B)\cap Fix(T)} x_1$. Note the fact $\langle x_n - z, Jx_1 - Jx_n\rangle \ge 0$, $\forall z \in Sol(B) \cap Fix(T)$. It follows that

$$\langle \bar{x} - z, Jx_1 - J\bar{x}\rangle \ge 0, \quad \forall z \in Fix(T) \cap Sol(B).$$

Using Lemma 2.3, we find that that $\bar{x} = \Pi_{Fix(T)\cap Sol(B)} x_1$. This completes the proof. □

Remark 3.2. Theorem 3.1, which mainly improves the corresponding results in [18], [23], [31], [32], unify the recent results of the monotone projection algorithms. The nonlinear mapping in Theorem 3.1 is more generalized than the corresponding results in the literature which only involve (asymptotically) quasi-ϕ-nonexpansive mappings. The sequence $\{u_n\}$ is generated shrinkingly instead of always finding in set C and it also deserves mentioning that there is no bounded restriction imposed on the common solution set $Fix(T) \cap Sol(B)$.

If T is a quadi-ϕ-nonexpansive mapping, we have the following result.

Corollary 3.3. *Let E be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let C be a convex and closed subset of E and let B be a bifunction with conditions (R-1), (R-2), (R-3) and (R-4). Let T be an asymptotically quasi-ϕ-nonexpansive mapping on C. Assume that T is closed on C and $Fix(T) \cap Sol(B)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ r_n B(u_n, \mu) \ge \langle u_n - \mu, Ju_n - Jx_n\rangle, \forall \mu \in C_n, \\ y_n = J^{-1}\big((1 - \alpha_n)Ju_n + \alpha_n JTx_n\big), \\ C_{n+1} = \{z \in C_n : \phi(z, x_n) \ge \phi(z, y_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\{\alpha_n\}$ is a real sequence in (0,1) such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\{r_n\} \subset [r, \infty)$ is a real sequence, where r is some positive real number. Then $\{x_n\}$ converges strongly to $\Pi_{Fix(T)\cap Sol(B)} x_1$.

From Theorem 3.1, the following results are not hard to derive.

Corollary 3.4. *Let E be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let C be a convex and closed subset of E and let B be a bifunction with conditions (R-1), (R-2), (R-3) and (R-4) such that $Sol(B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ r_n B(u_n, \mu) \geq \langle u_n - \mu, Ju_n - Jx_n \rangle, \forall \mu \in C_n, \\ y_n = J^{-1}\big((1-\alpha_n)Ju_n + \alpha_n Jx_n\big), \\ C_{n+1} = \{z \in C_n : \phi(z, x_n) \geq \phi(z, y_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\{\alpha_n\}$ is a real sequence in (0,1) such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\{r_n\} \subset [r, \infty)$ is a real sequence, where r is some positive real number. Then $\{x_n\}$ converges strongly to $\Pi_{Sol(B)} x_1$.

Corollary 3.5. *Let E be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let C be a convex and closed subset of E. Let T be an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense on C. Assume that T is uniformly asymptotically regular and closed on C and $Fix(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}\big((1-\alpha_n)Jx_n + \alpha_n JT^n x_n\big), \\ C_{n+1} = \{z \in C_n : \phi(z, x_n) + \xi_n \geq \phi(z, y_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\xi_n = \max\{\sup_{p\in Fix(T), x\in C} \big(\phi(p, T^n x) - \phi(p, x)\big), 0\}$, $\{\alpha_n\}$ is a real sequence in (0,1) such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then $\{x_n\}$ converges strongly to $\Pi_{Fix(T)\cap Sol(B)} x_1$.

In the framework of Hilbert spaces, one has $\sqrt{\phi(x,y)} = \|x - y\|$, $\forall x, y \in E$. The generalized projection is reduced to the metric projection and the class of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense is reduced to the class of asymptotically quasi-nonexpansive mappings in the intermediate sense. Using Theorem 3.1, we also have the following results.

Corollary 3.6. *Let E be a Hilbert space. Let C be a convex and closed subset of E and let B be a bifunction with conditions (R-1), (R-2), (R-3) and (R-4). Let T be an asymptotically quasi-nonexpansive mapping in the intermediate sense on C. Assume that T is uniformly asymptotically regular and closed on C and $Fix(T) \cap Sol(B)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ r_n B(u_n, \mu) \geq \langle u_n - \mu, u_n - x_n \rangle, \forall \mu \in C_n, \\ y_n = (1-\alpha_n)Ju_n + \alpha_n JT^n x_n, \\ C_{n+1} = \{z \in C_n : \|x_n - z\|^2 + \xi_n \geq \|z - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \end{cases}$$

where $\{\alpha_n\}$ is a real sequence in (0,1) such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\{r_n\} \subset [r, \infty)$ is a real sequence, where r is some positive real number and $\xi_n = \max\{\sup_{p\in Fix(T), x\in C} \big(\|p - T^n x\|^2 - \|p - x\|^2\big), 0\}$. Then $\{x_n\}$ converges strongly to $\Pi_{Fix(T)\cap Sol(B)} x_1$.

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