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Topological properties of L-partial pseudo-quasi-metric spaces

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Abstract

As an application of partial metrics and fuzzy set theory, the concept of L-partial pseudo-quasi-metric spaces is introduced and its topological properties are investigated. It is shown that L-partial pseudo-quasi-metrics are reasonable generalizations of partial pseudo-quasi-metrics and pointwise metrics in the sense of Shi. Also, it is proved that an L-partial pseudo-quasi-metric space can be endowed with an L-cotopology and a pointwise quasi-uniformity. Moreover, an L-partial pseudo-quasi-metric and its induced pointwise quasi-uniformity induce the same L-cotopology.

Keywords: Partial metric, L-partial metric, L-cotopology, pointwise metric.
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1. Introduction

Metric spaces play an important part in the research of general topology. Every metric space can be endowed with a uniformity and a topological structure. And there are close relations among metrics, uniformities and topological structures. In [10], Matthews introduced the notion of partial metrics, as a part of the study of denotational semantics of dataflow networks. In particular, he established a nice relationship between partial metric spaces and the so-called weightable quasi-metric spaces. In a partial metric space, the self-distance for any point need not be equal to zero. In fact, partial metrics are also generalizations of metrics. Partial metrics have received much attention for its applications in computer science [11, 14, 15]. Moreover, the existence of several connections between partial metrics and topological aspects of domain theory has been pointed [16, 17, 18, 23, 24].

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Since Zadeh introduced fuzzy set theory, there have been many different kinds of fuzzy metrics, including Erceg metric [2], KM metric [9], GV metric [3, 4, 5], pointwise metric [20] and (L, M)-fuzzy metric [21]. In the framework of fuzzy metric spaces, researchers usually discussed their topological properties. Erceg showed each Erceg metric space can be endowed with an L-topology and an L-uniformity in the sense of Hutton [7, 8]. Yue and Shi [28] proved each GV metric can induce a fuzzifying topology and a fuzzifying uniformity. Shi showed each pointwise pseudo-metric space can be equipped with a pointwise quasi-uniformity [19] and each (L, M)-fuzzy pseudo-quasi-metric space can be equipped with an (L, M)-fuzzy topology. In [12, 13], Pang and Shi proved that each (L, M)-fuzzy pseudo-quasi-metric (resp. pseudo-metric) space can be endowed with a pointwise (L, M)-fuzzy quasi-uniformity [25] (resp. uniformity [27]).

As an application of metrics or fuzzy set theory, the concept of fuzzy metrics is actually a combination of metrics and fuzzy sets. Inspired by this, Yue and Gu [26] proposed the concept of fuzzy partial metrics, which provided a new research direction for partial metrics. In fact, the notion of fuzzy partial metrics in the sense of Yue and Gu is a simultaneous generalization of KM metrics and partial metrics. As we all know, pointwise metrics in the sense of Shi generalized metrics in a different way from KM metrics. Based on this fact, we will provide a new method to generalize partial metrics in the the sense of pointwise metrics. Moreover, we will investigate its topological properties.

In this paper, we first generalize pointwise metrics in the sense of Shi and partial metrics in the sense of Matthews to a more general framework, so called L-partial metrics. Then we shall endow an L-partial pseudo-quasi-metric space with an L-cotopology and a pointwise quasi-uniformity, and show that an L-partial pseudo-quasi-metric and its induced pointwise quasi-uniformity induce the same L-cotopology.

2. Preliminaries

Throughout this paper, L denotes a completely distributive lattice. The smallest element and the largest element in L are denoted by $\perp$ and $\top$, respectively. For $a, b \in L$, we say that a is wedge below b in L, in symbols $a \prec b$, if for every subset $D \subseteq L$, $\bigvee D \geqslant b$ implies $d \geqslant a$ for some $d \in D$ [1]. A complete lattice L is completely distributive if and only if $b = \bigvee\{a \in L \mid a \prec b\}$ for each $b \in L$. An element a in L is called co-prime if $a \leqslant b \vee c$ implies $a \leqslant b$ or $a \leqslant c$ [6]. The set of non-zero co-prime elements in L is denoted by $J(L)$.

For a nonempty set X, L^X denotes the set of all L-subsets on X. L^X is also a completely distributive lattice when it inherits the structure of lattice L in a natural way, by defining $\vee$, $\wedge$, $\leqslant$ pointwisely. The set of non-zero co-prime elements in L^X is denoted by $J(L^X)$. Each member in $J(L^X)$ is also called a point. It is easy to see that $J(L^X)$ is exactly the set of all fuzzy points x_λ $(\lambda \in J(L))$. The smallest element and the largest element in L^X are denoted by $\underline{\perp}$ and $\underline{\top}$, respectively. Let (L^X, η) denote an L-cotopological space, i.e., η contains $\underline{\perp}$ and $\underline{\top}$, and is closed under finite unions and arbitrary intersections.

Definition 2.1 ([22])**.** Let (L^X, η) be an L-cotopological space. A closed L-subset P is called a closed remote-neighborhood (or a closed R-nbd, in short) of $e \in J(L^X)$ if $e \not\leqslant P$. An L-subset Q is called a remote-neighborhood (or a R-nbd, in short) of $e \in J(L^X)$ if there is a closed R-nbd P of e such that $P \geqslant Q$. For each $e \in J(L^X)$, $\eta(e)$ $(\eta^-(e))$ denotes the set of all R-nbds (closed R-nbds) of e. $\xi \subseteq \eta(e)$ is called an R-nbd base of e if $\forall P \in \eta(e)$, $\exists Q \in \xi$ such that $Q \geqslant P$.

Definition 2.2 ([22])**.** (L^X, η) is said to be the first countable if each point has a countable R-nbd basis.

Definition 2.3 ([20])**.** A pointwise pseudo-quasi-metric (pq-metric, in short) on L^X is a map $\mathcal{M} : J(L^X) \times J(L^X) \longrightarrow [0, +\infty)$ satisfying: $\forall a, b, c \in J(L^X)$,

(PM1) $\mathcal{M}(a, a) = 0$;
(PM2) $\mathcal{M}(a, c) \leqslant \mathcal{M}(a, b) + \mathcal{M}(b, c)$;
(PM3) $\mathcal{M}(a, b) = \bigwedge_{c \prec b} \mathcal{M}(a, c)$;
(PM4) $a \leqslant b \Longrightarrow \mathcal{M}(a, c) \leqslant \mathcal{M}(b, c)$.

For a pointwise pq-metric $\mathcal{M}$ on L^X, the pair $(X, \mathcal{M})$ is called a pointwise pq-metric space.

Remark 2.4. In [20], the lattice L is required to be a completely distributive DeMorgan algebra, where $'$ is an order-reversing involution on L. Based on this, Shi defined the concept of a pointwise pseudo-metric on L^X, which is a map $\mathcal{M} : J(L^X) \times J(L^X) \longrightarrow [0, +\infty)$ satisfying (PM1)–(PM4) and

(PM5) Given $u, v \in J(L^X)$, $\bigwedge_{a \not\leqslant u'} \mathcal{M}(a, v) = \bigwedge_{b \not\leqslant v'} \mathcal{M}(b, u)$.

The pair $(X, \mathcal{M})$ is called a pointwise pseudo-metric space. It will be called a pointwise metric if $\mathcal{M}$ satisfies

(PM6) $\mathcal{M}(a, b) = 0$ if and only if $a \leqslant b$.

The pair $(X, \mathcal{M})$ is called a pointwise metric space.

Let $\mathcal{D}(L^X)$ be the set of all maps from $J(L^X)$ to L^X such that $a \not\leqslant d(a)$ for all $a \in J(L^X)$. For any $f, g \in \mathcal{D}(L^X)$, we define:
(1) $f \leqslant g$ if and only if $\forall a \in J(L^X), f(a) \leqslant g(a)$,
(2) $(f \vee g)(a) = f(a) \vee g(a)$,
(3) $(f \diamond g)(a) = \bigwedge\{f(b) \mid b \not\leqslant g(a)\}$.

Then we can prove that $f \vee g, f \diamond g \in \mathcal{D}(L^X), f \diamond g \leqslant f, f \diamond g \leqslant g$ and the operations "$\vee$" and "$\diamond$" satisfy the associate law.

Definition 2.5 ([19])**.** A pointwise quasi-uniformity on L^X is a nonempty subset $\mathscr{U}$ of $\mathcal{D}(L^X)$ satisfying:

(PU1) $f \in \mathscr{U}, g \in \mathcal{D}(L^X), g \leqslant f$ implies $g \in \mathscr{U}$;
(PU2) $f, g \in \mathscr{U}$ implies $f \vee g \in \mathscr{U}$;
(PU3) $f \in \mathscr{U}$ implies $\exists g \in \mathscr{U}$ such that $g \diamond g \geqslant f$.

For a pointwise quasi-uniformity $\mathscr{U}$ on L^X, the pair $(X, \mathscr{U})$ is called a pointwise quasi-uniform space.

Definition 2.6 ([10])**.** A partial metric on X is a map $p : X \times X \longrightarrow [0, +\infty)$ satisfying: $\forall x, y, z \in X$,

(P1) $p(x, x) \leqslant p(x, y)$,
(P2) $p(x, y) \leqslant p(x, z) + p(z, y) - p(z, z)$,
(P3) $p(x, y) = p(y, x)$,
(P4) $x = y$ iff $p(x, y) = p(x, x) = p(y, y)$.

For a partial metric p on X, the pair (X, p) is called a partial metric space. We call p a partial pseudo-quasi-metric (pseudo-metric) if it satisfies (P1)–(P2) ((P1)–(P3)).

3. L-partial pseudo-quasi-metrics

In this section, we introduce the concept of L-partial pseudo-quasi-metrics and show that it is a generalization of both pointwise pq-metrics and partial pq-metrics.

Definition 3.1. An L-partial pseudo-quasi-metric (pq-metric, in short) on L^X is a map $\mathcal{P} : J(L^X) \times J(L^X) \longrightarrow [0, +\infty)$ satisfying: $\forall a, b, c \in J(L^X)$,

(LPM1) $\mathcal{P}(a, a) \leqslant \mathcal{P}(a, b)$;
(LPM2) $\mathcal{P}(a, b) \leqslant \mathcal{P}(a, c) + \mathcal{P}(c, b) - \mathcal{P}(c, c)$;
(LPM3) $\mathcal{P}(a, b) = \bigwedge_{c \prec b} \mathcal{P}(a, c)$;
(LPM4) $a \leqslant b \Longrightarrow \mathcal{P}(a, c) - \mathcal{P}(a, a) \leqslant \mathcal{P}(b, c) - \mathcal{P}(b, b)$.

For an L-partial pq-metric $\mathcal{P}$ on L^X, the pair $(X, \mathcal{P})$ is called an L-partial pq-metric space.

If $'$ is an order-reversing involution on L, then we can define L-partial pseudo-metrics and L-partial metrics.

Definition 3.2. An L-partial pq-metric on L^X is called an L-partial pseudo-metric (p-metric, in short) if it satisfies

(LPM5) Given $u, v \in J(L^X)$, $\bigwedge_{a \not\leqslant u'} \mathcal{P}(a, v) = \bigwedge_{b \not\leqslant v'} \mathcal{P}(b, u)$.

It will be called an L-partial metric if it satisfies

(LPM6) $\mathcal{P}(a, a) = \mathcal{P}(a, b) = \mathcal{P}(b, b)$ if and only if $a \leqslant b$.

Remark 3.3.

(1) If $\mathcal{P}(a, a) = 0$ for all $a \in J(L^X)$, then this L-partial (pq-, p-) metric is a pointwise (pq-, p-) metric.
(2) If $L = \{0, 1\}$, then the axioms (LPM1), (LPM2), (LPM5) and (LPM6) are equivalent to the following axioms, respectively.
  (P1) $\mathcal{P}(x, x) \leqslant \mathcal{P}(x, y)$,
  (P2) $\mathcal{P}(x, y) \leqslant \mathcal{P}(x, z) + \mathcal{P}(z, y) - \mathcal{P}(z, z)$,
  (P3) $\mathcal{P}(x, y) = \mathcal{P}(y, x)$,
  (P4) $x = y$ iff $\mathcal{P}(x, y) = \mathcal{P}(x, x) = \mathcal{P}(y, y)$.

Hence, a $\{0, 1\}$-partial (pq-, p-) metric is a partial (pq-, p-) metric. By (1) and (2), we know L-partial metrics are generalizations of both pointwise metrics and partial metrics.

Example 3.4. Let $X = \mathbb{R}^+$, the set of non-negative real numbers and let $L = I = [0, 1]$, the unit interval. Then $J(L^X) = \{x_\alpha \mid x > 0,\ \alpha \in (0, 1]\}$. Define $\mathcal{P} : J(I^X) \times J(I^X) \longrightarrow [0, +\infty)$ as follows:

$$\forall x_\alpha,\ y_\beta \in J(I^X),\ \mathcal{P}(x_\alpha, y_\beta) = \max\{x, y\}.$$

We prove that $\mathcal{P}$ is an I-partial pq-metric on I^X. It suffices to verify that $\mathcal{P}$ satisfies (LPM1)–(LPM4). In fact,

(LPM1) If $x_\alpha,\ y_\beta \in J(I^X)$, then $\mathcal{P}(x_\alpha, x_\alpha) = x \leqslant \max\{x, y\} = \mathcal{P}(x_\alpha, y_\beta)$.
(LPM2) Take $x_\alpha,\ y_\beta,\ z_\gamma \in J(I^X)$. Then

$$\begin{aligned}\mathcal{P}(x_\alpha, y_\beta) + \mathcal{P}(y_\beta, z_\gamma) - \mathcal{P}(y_\beta, y_\beta) &= \max\{x, y\} + \max\{y, z\} - y \\ &\geqslant \max\{x, z\} \\ &= \mathcal{P}(x_\alpha, z_\gamma).\end{aligned}$$

(LPM3) Take $x_\alpha,\ y_\beta,\ z_\gamma \in J(I^X)$. Then $z_\gamma \prec y_\beta$ is equivalent to $z = y$ and $\gamma < \beta$. It therefore follows that

$$\begin{aligned}\bigwedge_{z_\gamma \prec y_\beta} \mathcal{P}(x_\alpha, z_\gamma) &= \bigwedge_{\gamma < \beta} \mathcal{P}(x_\alpha, y_\gamma) \\ &= \bigwedge_{\gamma < \beta} \max\{x, y\} \\ &= \max\{x, y\} = \mathcal{P}(x_\alpha, y_\beta).\end{aligned}$$

(LPM4) Take $x_\alpha,\ y_\beta,\ z_\gamma \in J(I^X)$ with $x_\alpha \leqslant y_\beta$. Then $x = y$ and $\alpha \leqslant \beta$. Hence we have

$$\begin{aligned}\mathcal{P}(x_\alpha, z_\gamma) - \mathcal{P}(x_\alpha, x_\alpha) &= \max\{x, z\} - x \\ &= \max\{y, z\} - y \\ &= \mathcal{P}(y_\beta, z_\gamma) - \mathcal{P}(y_\beta, y_\beta).\end{aligned}$$

In the classical case, we know that each partial pq-metric can induce a pq-metric. In the following theorem, it is shown that each L-partial pq-metric can also induce a pointwise pq-metric.

Theorem 3.5. *Let $\mathcal{P}$ be an L-partial pq-metric on L^X and define $\mathcal{M}_{\mathcal{P}} : J(L^X) \times J(L^X) \longrightarrow [0, +\infty)$ as follows:*

$$\forall a, b \in J(L^X),\ \mathcal{M}_{\mathcal{P}}(a, b) = \mathcal{P}(a, b) - \mathcal{P}(a, a).$$

Then $\mathcal{M}_{\mathcal{P}}$ is a pointwise pq-metric on L^X.

Proof. It suffices to prove that $\mathcal{M}_{\mathcal{P}}$ satisfies (PM1)–(PM4).

(PM1) $\mathcal{M}_{\mathcal{P}}(a, a) = \mathcal{P}(a, a) - \mathcal{P}(a, a) = 0$.

(PM2) Take any $a, b, c \in J(L^X)$. Then

$$\begin{aligned}\mathcal{M}_{\mathcal{P}}(a, b) + \mathcal{M}_{\mathcal{P}}(b, c) &= \mathcal{P}(a, b) - \mathcal{P}(a, a) + \mathcal{P}(b, c) - \mathcal{P}(b, b)\\ &= (\mathcal{P}(a, b) + \mathcal{P}(b, c) - \mathcal{P}(b, b)) - \mathcal{P}(a, a)\\ &\geqslant \mathcal{P}(a, c) - \mathcal{P}(a, a) \quad \text{(by (LPM2))}\\ &= \mathcal{M}_{\mathcal{P}}(a, c).\end{aligned}$$

(PM3) Take any $a, b \in J(L^X)$. Then

$$\begin{aligned}\mathcal{M}_{\mathcal{P}}(a, b) &= \mathcal{P}(a, b) - \mathcal{P}(a, a)\\ &= \bigwedge_{c \prec b} \mathcal{P}(a, c) - \mathcal{P}(a, a) \quad \text{(by (LPM3))}\\ &= \bigwedge_{c \prec b} (\mathcal{P}(a, c) - \mathcal{P}(a, a))\\ &= \bigwedge_{c \prec b} \mathcal{M}_{\mathcal{P}}(a, c).\end{aligned}$$

(PM4) Take any $a, b \in J(L^X)$ with $a \leqslant b$. Then

$$\mathcal{M}_{\mathcal{P}}(a, c) = \mathcal{P}(a, c) - \mathcal{P}(a, a) \leqslant \mathcal{P}(b, c) - \mathcal{P}(b, b) = \mathcal{M}_{\mathcal{P}}(b, c).$$

This proves that $\mathcal{M}_{\mathcal{P}}$ is a pointwise pq-metric on L^X. □

4. *L*-cotopologies induced by *L*-partial pq-metrics

In this section, we demonstrate that an L-partial pseudo-quasi-metric can induce an L-closure operator and further it can induce an L-cotopology. Moreover, this induced L-cotopology has a countable R-nbd basis, so it is the first countable.

Definition 4.1. Let $\mathcal{P}$ be an L-partial pq-metric on L^X and define $R_t : J(L^X) \longrightarrow L^X$ for each $t > 0$ as follows:

$$\forall a \in J(L^X),\quad R_t(a) = \bigvee \{b \in J(L^X) \mid \mathcal{P}(a, b) - \mathcal{P}(a, a) \geqslant t\}.$$

Then $\{R_t \mid t \in (0, +\infty)\}$ is called the family of remote-neighborhood maps (R-nbd maps, in short) of $\mathcal{P}$.

Lemma 4.2. *If $\mathcal{P}$ is an L-partial pq-metric on L^X, then for $a, b, c \in J(L^X)$, we have*

$$b \leqslant c \Longrightarrow \mathcal{P}(a, c) \leqslant \mathcal{P}(a, b).$$

Proof. By (LPM3), it follows that

$$\mathcal{P}(a, c) = \bigwedge_{e \prec c} \mathcal{P}(a, e) \leqslant \bigwedge_{e \prec b} \mathcal{P}(a, e) = \mathcal{P}(a, b).$$

□

Lemma 4.3. *If $\mathcal{P}$ is an L-partial pq-metric on L^X with the family of R-nbd maps $\{R_t \mid t \in (0,+\infty)\}$, then for $a, b \in J(L^X)$, we have*
$$b \leqslant R_t(a) \Longleftrightarrow \mathcal{P}(a,b) - \mathcal{P}(a,a) \geqslant t.$$

Proof. It suffices to show that $b \leqslant R_t(a) \Longrightarrow \mathcal{P}(a,b) - \mathcal{P}(a,a) \geqslant t$. Take $c \in J(L^X)$ with $c \prec b \leqslant R_t(a)$. Then there exists $e \in J(L^X)$ such that $c \leqslant e$ and $\mathcal{P}(a,e) - \mathcal{P}(a,a) \geqslant t$. By Lemma 4.2, it follows that $\mathcal{P}(a,c) - \mathcal{P}(a,a) \geqslant t$. By the arbitrariness of c, we obtain
$$\mathcal{P}(a,b) - \mathcal{P}(a,a) = \bigwedge_{c \prec b} \mathcal{P}(a,c) - \mathcal{P}(a,a) = \bigwedge_{c \prec b} (\mathcal{P}(a,c) - \mathcal{P}(a,a)) \geqslant t,$$
as desired. □

Theorem 4.4. *If $\mathcal{P}$ is an L-partial pq-metric on L^X with the family of R-nbd maps $\{R_t \mid t \in (0,+\infty)\}$, then the following statements hold:*

(LPR1) $\forall a \in J(L^X),\ \bigwedge_{t>0} R_t(a) = \perp$,
(LPR2) $\forall a \in J(L^X),\ \forall t > 0,\ a \nleqslant R_t(a)$,
(LPR3) $\forall t, s > 0,\ R_t \diamond R_s \geqslant R_{t+s}$,
(LPR4) $\forall a \in J(L^X),\ R_t(a) = \bigwedge_{s<t} R_s(a)$,
(LPR5) $\forall t > 0,\ R_t$ *is order-preserving.*

Proof. (LPR1) Suppose that $\bigwedge_{t>0} R_t(a) \neq \perp$ for some $a \in J(L^X)$. Then there exists $b \in J(L^X)$ such that $b \leqslant \bigwedge_{t>0} R_t(a) \neq \perp$. It follows that $b \leqslant R_t(a)$ for all $t > 0$. By Lemma 4.3, we have $\mathcal{P}(a,b) - \mathcal{P}(a,a) \geqslant t$ for all $t > 0$, which is contradict to $\mathcal{P}(a,b) \in [0,+\infty)$.

(LPR2) Since $\mathcal{P}(a,a) - \mathcal{P}(a,a) = 0 \ngeqslant t$ for all $t > 0$, by Lemma 4.3, it follows that $a \nleqslant R_t(a)$ for all $t > 0$.

(LPR3) Let $a, c \in J(L^X)$ with $c \nleqslant R_t \diamond R_s(a)$. By the definition of $R_t \diamond R_s$, there exists $b \in J(L^X)$ such that $b \nleqslant R_s(a)$ and $c \nleqslant R_t(b)$. By Lemma 4.3, we have $\mathcal{P}(a,b) - \mathcal{P}(a,a) < s$ and $\mathcal{P}(b,c) - \mathcal{P}(b,b) < t$. Hence, with (LPM2), it follows that
$$\mathcal{P}(a,c) - \mathcal{P}(a,a) \leqslant \mathcal{P}(a,b) + \mathcal{P}(b,c) - \mathcal{P}(b,b) - \mathcal{P}(a,a) < t + s.$$

This implies $c \nleqslant R_{t+s}(a)$. By the arbitrariness of a and c, we have $R_t \diamond R_s \geqslant R_{t+s}$.

(LPR4) Take any $a, b \in J(L^X)$. Then
$$\begin{aligned} b \leqslant R_t(a) &\Longleftrightarrow \mathcal{P}(a,b) - \mathcal{P}(a,a) \geqslant t \\ &\Longleftrightarrow \forall s < t,\ \mathcal{P}(a,b) - \mathcal{P}(a,a) \geqslant s \\ &\Longleftrightarrow \forall s < t,\ b \leqslant R_s(a) \\ &\Longleftrightarrow b \leqslant \bigwedge_{s<t} R_s(a). \end{aligned}$$

(LPR5) Take any $a, b \in J(L^X)$ with $a \leqslant b$. Then by Lemma 4.3 and with (LPM4), we have
$$\begin{aligned} c \leqslant R_t(a) &\Longleftrightarrow \mathcal{P}(a,c) - \mathcal{P}(a,a) \geqslant t \\ &\Longrightarrow \mathcal{P}(b,c) - \mathcal{P}(b,b) \geqslant t \\ &\Longleftrightarrow c \leqslant R_t(b). \end{aligned}$$

This shows $R_t(a) \leqslant R_t(b)$, as desired. □

Lemma 4.5. *Let $\mathcal{P}$ be an L-partial pq-metric on L^X and define $c_{\mathcal{P}} : L^X \longrightarrow L^X$ by*
$$\forall A \in L^X,\ c_{\mathcal{P}}(A) = \bigvee \left\{ a \in J(L^X) \mid \bigwedge_{c \leqslant A} \mathcal{P}(a,c) = \mathcal{P}(a,a) \right\}.$$

Then the following statements hold:

(1) $c_{\mathcal{P}}(A) = \bigvee\{a \in J(L^X) \mid \forall t > 0,\ A \nleqslant R_t(a)\}$,
(2) $\forall a \in J(L^X)$, $a \leqslant c_{\mathcal{P}}(A) \Longleftrightarrow \forall t > 0,\ A \nleqslant R_t(a)$.

Proof. (1) Put $P = \left\{a \in J(L^X) \mid \bigwedge_{c\leqslant A} \mathcal{P}(a, c) = \mathcal{P}(a, a)\right\}$ and $Q = \{a \in J(L^X) \mid \forall t > 0,\ A \nleqslant R_t(a)\}$. We need only prove $P = Q$.

To check $P \supseteq Q$, take $a \notin P$, which means that $\bigwedge_{c\leqslant A} \mathcal{P}(a, c) \neq \mathcal{P}(a, a)$. By (LPM1), it follows that $t \triangleq \bigwedge_{c\leqslant A}(\mathcal{P}(a, c) - \mathcal{P}(a, a)) > 0$. Then for each $c \leqslant A$, $\mathcal{P}(a, c) - \mathcal{P}(a, a) \geqslant t$. By Lemma 4.3, we obtain $c \leqslant R_t(a)$ for each $c \leqslant A$. By the arbitrariness of c, it follows that $A \leqslant R_t(a)$. This shows $a \notin Q$.

To check $P \subseteq Q$, take $a \notin Q$. This means that there exists $t > 0$ such that $A \leqslant R_t(a)$. For each $c \leqslant A$, it follows that $c \leqslant R_t(a)$. By Lemma 4.3, we have $\mathcal{P}(a, c) - \mathcal{P}(a, a) \geqslant t$. By the arbitrariness of c, we obtain

$$\bigwedge_{c\leqslant A} \mathcal{P}(a, c) - \mathcal{P}(a, a) = \bigwedge_{c\leqslant A} (\mathcal{P}(a, c) - \mathcal{P}(a, a)) \geqslant t.$$

This shows $a \notin P$. As a result, we obtain that $P = Q$, as desired.

(2) The sufficiency is obvious. Next we prove the necessity. Take any $t > 0$. With (LPR2), we have $\bigvee_{b\prec a} b = a \nleqslant R_{\frac{t}{2}}(a)$. Then there exists $b \in J(L^X)$ such that $b \prec a$ and $b \nleqslant R_{\frac{t}{2}}(a)$. Further, it follows from $a \leqslant c_{\mathcal{P}}(A) = \bigvee\{c \in J(L^X) \mid \forall s > 0,\ A \nleqslant R_s(c)\}$ that there exists $e \in J(L^X)$ such that $b \leqslant e$ and $\forall s > 0, A \nleqslant R_s(e)$. Put $s = \frac{t}{2}$. By (LPR5), we have $A \nleqslant R_{\frac{t}{2}}(e) \geqslant R_{\frac{t}{2}}(b)$. Hence, $A \nleqslant R_{\frac{t}{2}}(b)$ and $b \nleqslant R_{\frac{t}{2}}(a)$. It therefore follows that $A \nleqslant R_{\frac{t}{2}} \diamond R_{\frac{t}{2}}(a) \geqslant R_t(a)$. This proves $A \nleqslant R_t(a)$. □

Theorem 4.6. *If $\mathcal{P}$ is an L-partial pq-metric on L^X, then $c_{\mathcal{P}}$ is an L-closure operator. Moreover, let $\eta_{\mathcal{P}}$ denote the L-cotopology induced by $c_{\mathcal{P}}$.*

Proof. (LC1) $c_{\mathcal{P}}(\underline{\perp}) = \bigvee \emptyset = \underline{\perp}$, $c_{\mathcal{P}}(\underline{\top}) = \bigvee\{a \in J(L^X)\} = \underline{\top}$.

(LC2) $A \leqslant c_{\mathcal{P}}(A)$. Take any $a \in J(L^X)$ with $a \nleqslant c_{\mathcal{P}}(A)$. By Lemma 4.5, it follows that there exists $t > 0$ such that $A \leqslant R_t(a)$. Since $a \nleqslant R_t(a)$, we obtain $a \nleqslant A$. By the arbitrariness of a, we obtain $A \leqslant c_{\mathcal{P}}(A)$.

(LC3) $c_{\mathcal{P}}(A \vee B) = c_{\mathcal{P}}(A) \vee c_{\mathcal{P}}(B)$. Take any $a \in J(L^X)$ such $a \nleqslant c_{\mathcal{P}}(A) \vee c_{\mathcal{P}}(B)$. Then $a \nleqslant c_{\mathcal{P}}(A)$ and $a \nleqslant c_{\mathcal{P}}(B)$. By Lemma 4.5, there exist $t,\ s > 0$ such that $A \leqslant R_t(a)$ and $B \leqslant R_s(a)$. Let $r = \min\{t, s\}$. By (LPR4), it follows that $A \vee B \leqslant R_t(a) \vee R_s(a) = R_r(a)$. This implies that $a \nleqslant c_{\mathcal{P}}(A \vee B)$. By the arbitrariness of a, we obtain $c_{\mathcal{P}}(A \vee B) \leqslant c_{\mathcal{P}}(A) \vee c_{\mathcal{P}}(B)$. The inverse inequality is obvious. Therefore, $c_{\mathcal{P}}(A \vee B) = c_{\mathcal{P}}(A) \vee c_{\mathcal{P}}(B)$.

(LC4) $c_{\mathcal{P}}(c_{\mathcal{P}}(A)) = c_{\mathcal{P}}(A)$. By (LC2), $c_{\mathcal{P}}(c_{\mathcal{P}}(A)) \geqslant c_{\mathcal{P}}(A)$. To check $c_{\mathcal{P}}(c_{\mathcal{P}}(A)) \leqslant c_{\mathcal{P}}(A)$, let $a \in J(L^X)$ with $a \leqslant c_{\mathcal{P}}(c_{\mathcal{P}}(A))$, by Lemma 4.5, $c_{\mathcal{P}}(A) \nleqslant R_{\frac{t}{2}}(a)$ for all $t > 0$. Then there exists $b \in J(L^X)$ such that $b \leqslant c_{\mathcal{P}}(A)$ and $b \nleqslant R_{\frac{t}{2}}(a)$. It follow from $b \leqslant c_{\mathcal{P}}(A)$ that $A \nleqslant R_{\frac{t}{2}}(b)$. With $b \nleqslant R_{\frac{t}{2}}(a)$, we conclude that $A \nleqslant R_{\frac{t}{2}} \diamond R_{\frac{t}{2}}(a) \geqslant R_t(a)$. By Lemma 4.5, we have $a \leqslant c_{\mathcal{P}}(A)$. This proves that $c_{\mathcal{P}}(c_{\mathcal{P}}(A)) \leqslant c_{\mathcal{P}}(A)$, as desired.

As a consequence, $c_{\mathcal{P}}$ is an L-closure operator. Its induced L-cotopology, denoted by $\eta_{\mathcal{P}}$, has the following form $\eta_{\mathcal{P}} = \{A \in L^X \mid A = c_{\mathcal{P}}(A)\}$. □

Theorem 4.7. *If $\mathcal{P}$ is an L-partial pq-metric on L^X with the family of R-nbd maps $\{R_t \mid t \in (0, +\infty)\}$, then for each $e \in J(L^X)$, $\{R_t(e) \mid t \in (0, +\infty)\}$ is an R-nbd base at e in $\eta_{\mathcal{P}}$.*

Proof. We first prove $\{R_t(e) \mid t \in (0, +\infty)\} \subseteq \eta_{\mathcal{P}}$, i.e., $c_{\mathcal{P}}(R_t(e)) = R_t(e)$ for all $t > 0$. Suppose that $c_{\mathcal{P}}(R_t(e)) \nleqslant R_t(e)$, then there exists $a \in J(L^X)$ such that $a \leqslant c_{\mathcal{P}}(R_t(e))$ and $a \nleqslant R_t(e)$. Since $R_t(e) = \bigwedge_{s<t} R_s(e)$, there exists $s < t$ such that $a \nleqslant R_s(e)$. Put $r = t - s$. By Lemma 4.5 and with $a \leqslant c_{\mathcal{P}}(R_t(e))$, we have $R_t(e) \nleqslant R_r(a)$. Then it follows from $R_t(e) \nleqslant R_r(a)$ and $a \nleqslant R_s(e)$ that $R_t(e) \nleqslant R_r \diamond R_s(e) \geqslant R_{r+s}(e) = R_t(e)$, which a contradiction. This implies $c_{\mathcal{P}}(R_t(e)) \leqslant R_t(e)$. The inverse inequality is obvious. Hence, we obtain $c_{\mathcal{P}}(R_t(e)) = R_t(e)$. Further, by (LPR2), $e \nleqslant R_t(e)$ for all $t > 0$. Therefore, by Definition 2.1, we obtain $\{R_t(e) \mid t \in (0, +\infty)\} \subseteq \eta^-_{\mathcal{P}}(e) \subseteq \eta_{\mathcal{P}}(e)$.

Next we show $\{R_t(e) \mid t \in (0, +\infty)\}$ is an R-nbd base at e in $\eta_{\mathcal{P}}$. Take any $M \in \eta_{\mathcal{P}}(e)$. Then there exists $N \in \eta^-_{\mathcal{P}}(e)$ such that $e \nleqslant N \geqslant M$. Since $N \in \eta_{\mathcal{P}}$, we have $e \nleqslant N = c_{\mathcal{P}}(N)$. By Lemma 4.5, there exists $s > 0$ such that $R_s(e) \geqslant N \geqslant M$, as desired. □

By Theorem 4.7, the following result is obvious.

Corollary 4.8. *If $\mathcal{P}$ is an L-partial pq-metric on L^X, then $(L^X, \eta_{\mathcal{P}})$ is the first countable.*

5. Pointwise quasi-uniformities induced by L-partial pq-metrics

In this section, we show that an L-partial pq-metric can induce a pointwise quasi-uniformity. Further, we show that an L-partial pq-metric and its induced pointwise quasi-uniformity induce the same L-cotopology.

Theorem 5.1. *If $\mathcal{P}$ is an L-partial pq-metric on L^X with the family of R-nbd maps $\{R_t \mid t \in (0, +\infty)\}$, then $\{R_t \mid t \in (0, +\infty)\}$ is a base for a pointwise quasi-uniformity, which is said to be induced by the L-partial pq-metric $\mathcal{P}$.*

Proof. Define $\mathscr{U}_{\mathcal{P}} \subseteq \mathcal{D}(L^X)$ as follows:

$$\mathscr{U}_{\mathcal{P}} = \{f \in \mathcal{D}(L^X) \mid \exists t > 0, \ s.t. \ f \leqslant R_t\}.$$

Next we prove $\mathscr{U}_{\mathcal{P}}$ is a pointwise quasi-uniformity on L^X.

(PU1) If $f \in \mathscr{U}_{\mathcal{P}}$, $g \in \mathcal{D}(L^X)$ and $g \leqslant f$, then there exists $s > 0$ such that $g \leqslant f \leqslant R_s$. This shows $g \in \mathscr{U}_{\mathcal{P}}$.

(PU2) If $f \in \mathscr{U}_{\mathcal{P}}$ and $g \in \mathscr{U}_{\mathcal{P}}$. Then there exist $t_1, \ t_2 > 0$ such that $f \leqslant R_{t_1}$ and $g \leqslant R_{t_2}$. Put $t = \min\{t_1, t_2\}$. By (LPR4), we obtain $f \vee g \leqslant R_{t_1} \vee R_{t_2} = R_t$. This means $f \vee g \in \mathscr{U}_{\mathcal{P}}$.

(PU3) If $f \in \mathscr{U}_{\mathcal{P}}$, then there exists $s > 0$ such that $f \leqslant R_s$. Let $g = R_{\frac{s}{2}}$. Then $g \diamond g = R_{\frac{s}{2}} \diamond R_{\frac{s}{2}} \geqslant R_s \geqslant f$.

As a result, $\mathscr{U}_{\mathcal{P}}$ is a pointwise quasi-uniformity on L^X, and $\{R_t \mid t \in (0, +\infty)\}$ is one of its bases. □

Corollary 5.2. *If $\mathcal{P}$ is an L-partial pq-metric on L^X, then $\mathscr{U}_{\mathcal{P}}$ defined by*

$$\mathscr{U}_{\mathcal{P}} = \{f \in \mathcal{D}(L^X) \mid \exists t > 0, \ s.t. \ f \leqslant R_t\},$$

is a pointwise quasi-uniformity on L^X, which is said to be induced by the L-partial pq-metric $\mathcal{P}$.

Lemma 5.3 ([19])**.** *Let $(L^X, \mathscr{U})$ is a pointwise quasi-uniform space and define $c_{\mathscr{U}} : L^X \longrightarrow L^X$ as follows:*

$$c_{\mathscr{U}}(A) = \bigvee \{a \in J(L^X) \mid \forall f \in \mathscr{U}, \ A \nleq f(a)\}.$$

Then the following statements hold:

(1) $\forall a \in J(L^X), \ a \leqslant c_{\mathscr{U}}(A) \Longleftrightarrow \forall f \in \mathscr{U}, \ A \nleq f(a)$,

(2) *$c_{\mathscr{U}}$ is an L-closure operator. Hence, it may induce an L-cotopology, denoted by $\eta_{\mathscr{U}}$.*

Theorem 5.4. *If $\mathcal{P}$ is an L-partial pq-metric on L^X, then $c_{\mathscr{U}_{\mathcal{P}}} = c_{\mathcal{P}}$.*

Proof. Take any $A \in L^X$ and $a \in J(L^X)$. Then

$$\begin{aligned} a \leqslant c_{\mathscr{U}_{\mathcal{P}}}(A) &\Longleftrightarrow \forall f \in \mathscr{U}_{\mathcal{P}}, A \nleq f(a) \quad \text{(by Lemma 5.3)} \\ &\Longleftrightarrow \text{If } t > 0 \text{ and } f \leqslant R_t, \text{then } A \nleq f(a) \\ &\Longleftrightarrow \forall t > 0, A \nleq R_t(a) \\ &\Longleftrightarrow a \leqslant c_{\mathcal{P}}(A). \quad \text{(by Lemma 4.5)} \end{aligned}$$

This implies $c_{\mathscr{U}_{\mathcal{P}}}(A) = c_{\mathcal{P}}(A)$ for all $A \in L^X$, as desired. □

Corollary 5.5. *If $\mathcal{P}$ is an L-partial pq-metric on L^X, then $\eta_{\mathscr{U}_{\mathcal{P}}} = \eta_{\mathcal{P}}$.*

6. Conclusions

In this paper, we combine partial metrics and fuzzy set theory, and introduced the concept of L-partial (pq-,p-) metrics. This new concept generalized partial (pq-,p-) metrics to the fuzzy case in a different way from fuzzy partial metrics in the sense of Yue and Gu. L-partial pq-metrics also possessed nice topological properties. Each L-partial pq-metric can induce an L-cotopology and a pointwise quasi-uniformity, and an L-partial pq-metric and its induced pointwise quasi-uniformity induce the same L-cotopology.

As we all know, L-partial metrics can be considered as the combinations of partial metrics and pointwise metrics, and fuzzy partial metrics in the sense of Yue and Gu can be considered as the combinations of partial metrics and KM metrics. While (L, M)-fuzzy metrics in the sense of Shi are generalizations of both pointwise metrics and KM metrics. This motivates us to generalize partial metrics to the (L, M)-fuzzy case. In the future, we will consider proposing the concept of (L, M)-fuzzy partial metrics and investigate its topological properties.

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