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Some geometric properties of generalized modular sequence spaces defined by Zweier operator

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Abstract

In this paper, the main purpose is to define generalized Cesàro sequence spaces by using the Zweier operator and to investigate the property (H) and uniform Opial property in the spaces when they are equipped with the Luxemburg norm.

Keywords: Generalized modular sequence spaces, Cesàro sequence spaces, property (H), uniform Opial property, Zweier operator.
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1. Introduction

There are many mathematicians who are interested in studying geometric properties of Banach spaces, because the geometric properties were identified as important characteristics and properties of the Banach spaces. For example, if Banach spaces have some geometric properties such as uniform rotund, P- convexity, Q- convexity, Banach-Saks property then they are reflexive spaces. The investigations of metric geometry of Banach spaces, date back to 1913, when Radon [17] introduced Kadec-Klee property (sometimes called the Radon-Riesz property, or property (H)) and, later Riesz [18, 19] who showed that the classical L_p-spaces, $1 < p < \infty$, have the Kadec-Klee property. Although the space $L_1[0, 1]$ (with Lebesgue measure) fails to have the Kadec-Klee property. In 1936, Clarkson [2] introduced the notion of the uniform convexity property (UC) or the uniform rotun property (UR) of Banach spaces, and it was shown that L_p with $1 < p < \infty$ are examples of such space. In 1967, Opial [14] introduced a new property which is called Opial property and

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proved that the sequence space $l_p(1 < p < \infty)$ have this property but $L_p[0,\pi](p \neq 2, 1 < p < \infty)$ do not have it. In 1980, Huff [6] introduced the nearly uniform convexity for Banach spaces and he also proved that every nearly uniformly convex Banach space is reflexive and it has the uniform Kadec-Klee property(UKK). In 1991, Kutzarova [8] defined and studied k-nearly uniformly Banach spaces. In 1992, Prus [16] introduced the notion of uniform Opial property. Recently, many mathematicians are also interested of geometric properties in sequence spaces. Some example of the geometry of sequence spaces and their generalizations have been extensively studied in [1, 4, 7, 15, 20, 22, 23, 24, 25].

The main purpose of this paper is to define generalized Cesàro sequence spaces for a bounded sequence of positive real numbers $p = p_k \geq 1$ with a sequence (q_n) of positive real numbers by using the Zweier operator. Also, we investigate the property (H) and Uniform Opial property equipped with the Luxemburg norm.

2. Preliminaries and Notation

Let l^0 be the space of all real sequences. For $1 \leq p < \infty$, the Cesàro sequence space (ces_p, for short) of Shue[22] is defined by

$$ces_p = \{x \in l^0 : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=0}^{n} |x(i)| \right)^p < \infty\}.$$

It is very useful in the theory of matrix operators and others(see [9, 12]).

In 1997, Bilgin [1] defined the sequences spaces $C(s,p)$ when $s \geq 0$ as follow:

$$ces(p,s) = \left\{ x \in l^0 : \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r k^{-s} |x(i)| \right)^{p_r} < \infty \right\}, \tag{2.1}$$

where $\sum_r$ denotes a sum over the range $2^r \leq k < 2^{r+1}$. If $s = 0$, then the spaces become to the spaces

$$ces(p) = \left\{ x \in l^0 : \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_r |x(i)| \right)^{p_r} < \infty \right\}, \tag{2.2}$$

which has been investigated by Lim [10, 11]. In 2005, Mursaleen [13] defined the Cesàro sequence space $ces[(p),(q)]$ with (q_n) is a sequence of positive real numbers and real bounded sequence (p_n) with $\inf p_r > 0$ by

$$ces[(p),(q)] = \{x \in l^0 : \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_k |x(i)| \right)^{p_r} < \infty\},$$

where $Q_{2^r} = q_{2^r} + q_{2^r+1} + q_{2^r+2} + \cdots + q_{2^{r+1}-1}$. If $q_n = 1$ for all $n \geq 1$, then $ces[(p),(q)]$ reduces to $ces(p)$.

The Z-transform of a sequence $x = x_k$ is defined by $(Zx)_n = y_n = \gamma x_n + (1-\gamma)x_{n-1}$ by using the Zweier operator

$$Z = (z_{nk}) = \begin{cases} \gamma & ; k = n, \\ 1-\gamma & ; k = n-1, \\ 0 & ; \text{otherwise} \end{cases}$$

for all $n, k \geq 1$ and scalar $\gamma \in \mathbb{F} \setminus \{0\}$, where $\mathbb{F}$ is the field of all complex or real numbers. The Zweier operator was studied by Şengönül and Kayaduman [21]. In 2013, Et et al. [5] used the Zweier operator define the new modular sequence spaces $\mathcal{Z}_\sigma(s,p)$ as follow:

$$\mathcal{Z}_\sigma(s,p) = \left\{ x \in l^0 : \sigma(\lambda x) < \infty \text{ for some } \lambda > 0 \right\},$$

where

$$\sigma(x) = \sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_{r} k^{-s}|\alpha x_k + (1-\alpha)x_{k-1}|\right)^{p_r}$$

and $s \geq 0$. This spaces is equipped with the Luxemburg norm

$$\|x\| = \inf\{\lambda > 0 : \sigma(\tfrac{x}{\lambda}) \leq 1\}.$$

Now, we define the generalized modular sequence space $C(\mathcal{Z}; p, q)$ for $p = (p_k)$ bounded sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$ and a sequence (q_n) of positive real numbers by

$$C(\mathcal{Z}; p, q) = \left\{x \in l^0 : \varrho(\lambda x) < \infty \ \text{ for some } \ \lambda > 0\right\}, \tag{2.3}$$

where

$$\varrho(x) = \sum_{r=0}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_{r} q_k|\gamma x_k + (1-\gamma)x_{k-1}|\right)^{p_r}$$

and the spaces is equipped with the Luxemburg norm

$$\|x\| = \inf\{\tau > 0 : \varrho(\tfrac{x}{\tau}) \leq 1\},$$

where $Q_{2^r} = q_{2^r} + q_{2^r+1} + q_{2^r+2} + \cdots + q_{2^{r+1}-1}$ and $\sum_r$ denotes a sum over the range $2^r \leq k < 2^{r+1}$. If we take $\gamma = 1$, then the spaces $C(\mathcal{Z}; p, q)$ become to $ces[(p),(q)]$. Also, If we take $\gamma = 1$ and $q_k = 1$ for all $k \geq 1$, then the spaces $C(\mathcal{Z}; p, q)$ become to $ces(p)$ studied by Lim [10, 11].

Let $(X, \|\cdot\|)$ be a real Banach space and let $B(X)$(resp., $S(X)$) be the closed unit ball (resp., the unit sphere) of X. A point $x \in S(X)$ is an $H-point$ of $B(X)$ if for any sequence (x_n) in X such that $\|x_n\| \to 1$ as $n \to \infty$, the week convergence of (x_n) to x implies that $\|x_n - x\| \to 0$ as $n \to \infty$. If every point of $S(X)$ is an $H-point$ of $B(X)$, then X is said to have the property (H). A Banach space X is said to have the Opial property (see [14]) if every sequence $\{x_n\}$ weakly convergent to x_0 satisfies

$$\liminf_{n\to\infty} \| x_n - x_0 \| \leq \liminf_{n\to\infty} \| x_n - x \|$$

for every $x \in X$. A Banach space X is said to have the uniform Opial property (see [16]), if for each $\varepsilon > 0$ there exists $\tau > 0$ such that for any weakly null sequence (x_n) in $S(X)$ and $x \in X$ with $\| x \| > \varepsilon$ there holds

$$1 + \tau \leq \liminf_{n\to\infty} \| x_n - x \|.$$

For example, the space in [4, 15] have the uniform Opial property.

Throughout this paper, for $x \in l^0$, $i \in \mathbb{N}$, we denote

$$\begin{aligned} e_i &= (\overbrace{0, 0, ..., 0}^{i-1}, 1, 0, 0, 0, ...), \\ x\,|_i &= (x(1), x(2), x(3), ..., x(i), 0, 0, 0, ...), \\ x\,|_{\mathbb{N}-i} &= (0, 0, 0, ..., x(i+1), x(i+2), ...). \end{aligned}$$

In addition, we recall the following inequalities:

$$|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k}) \tag{2.4}$$

and

$$|a_k + b_k|^{t_k} \leq |a_k|^{t_k} + |b_k|^{t_k}, \tag{2.5}$$

where $t_k = \frac{p_k}{M}$, $C = \max\{1, 2^{M-1}\}$, $M = \sup_k p_k$ for all $k \geq 1$.

Next, we start with a brief recollection of basic concepts and facts in modular spaces. For a real vector space X, a function $\rho : X \to [0, \infty]$ is called a *modular* if it satisfies the following conditions:

(i) $\rho(x) = 0$ if and only if $x = 0$;
(ii) $\rho(\alpha x) = \rho(x)$ for all scalar α with $|\alpha| = 1$;
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

The modular ρ is called convex if

(iv) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$, for all $x, y \in X$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

For a modular ρ on X, the space

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0^+\}$$

is called the *modular space*.

A sequence (x_n) in X_ρ is called *modular convergent* to $x \in X_\rho$ if there exists a $\lambda > 0$ such that $\rho(\lambda(x_n - x)) \to 0$ as $n \to \infty$.

A modular ρ is said to satisfy the $\Delta_2 - condition$ $(\rho \in \Delta_2)$ if for any $\varepsilon > 0$ there exist constants $K \geq 2$ and $a > 0$ such that

$$\rho(2u) \leq K\rho(u) + \varepsilon$$

for all $u \in X_\rho$ with $\rho(u) \leq a$.

If ρ satisfies the $\Delta_2 - condition$ for any $a > 0$ with $K \geq 2$ dependent on a, we say that ρ satisfies the *strong* $\Delta_2 - condition$ $(\rho \in \Delta_2^s)$.

Lemma 2.1 ([3] Lemma 2.1)**.** *If $\rho \in \Delta_2^s$, then for any $L > 0$ and $\varepsilon > 0$, there exists $\delta = \delta(L, \varepsilon) > 0$ such that*

$$|\rho(u + v) - \rho(u)| < \varepsilon,$$

whenever $u, v \in X_\rho$ with $\rho(u) \leq L$, and $\rho(v) \leq \delta$.

Lemma 2.2 ([3] Lemma 2.3)**.** *The convergences in norm and in modular are equivalent in X_ρ if $\rho \in \Delta_2$.*

Lemma 2.3 ([3] Lemma 2.4)**.** *If $\rho \in \Delta_2^s$, then for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\| x \| \geq 1 + \delta$ whenever $\rho(x) \geq 1 + \varepsilon$.*

3. Main result

In this section, we prove the property H and the uniform Opial property in generalized modular sequence spaces $C(\mathcal{Z}; p, q)$. First we shall give some results which are very important for our consideration.

Proposition 3.1. *The functional ϱ is a convex modular on $C(\mathcal{Z}; p, q)$.*

Proof. Let $x, y \in C(\mathcal{Z}; p, q)$. It is obvious that $\varrho(x) = 0$ if and only if $x = 0$ and $\varrho(\alpha x) = \varrho(x)$ for scalar α with $|\alpha| = 1$. Let $\alpha \geq 0$, $\beta \geq 0$ with $\alpha + \beta = 1$. By the convexity of the function $t \mapsto |t|^{p_r}$, for all $r \in \mathbb{N}$, we have

$$\begin{aligned}
\varrho(\alpha x + \beta y) &= \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r |\alpha q_i(\gamma x(i) + (1-\gamma)x(i-1)) + \beta q_i(\gamma y(i) + (1-\gamma)y(i-1))| \right)^{p_r} \\
&\leq \sum_{r=0}^{\infty} \left(\alpha \frac{1}{Q_{2^r}} \sum_r q_i|\gamma x(i) + (1-\gamma)x(i-1)| + \beta \frac{1}{Q_{2^r}} \sum_r q_i|\gamma y(i) + (1-\gamma)y(i-1)| \right)^{p_r} \\
&\leq \alpha \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_i|\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r} \\
&\quad + \beta \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_i|\gamma y(i) + (1-\gamma)y(i-1)| \right)^{p_r} \\
&= \alpha\varrho(x) + \beta\varrho(y).
\end{aligned}$$

□

Proposition 3.2. *For $x \in C(\mathcal{Z}; p, q)$, the modular ϱ on $C(\mathcal{Z}; p, q)$ satisfies the following properties:*

(i) *if $0 < a < 1$, then $a^M \varrho(\frac{x}{a}) \leq \varrho(x)$ and $\varrho(ax) \leq a\varrho(x)$;*
(ii) *if $a > 1$, then $\varrho(x) \leq a^M \varrho(\frac{x}{a})$;*
(iii) *if $a \geq 1$, then $\varrho(x) \leq a\varrho(x) \leq \varrho(ax)$.*

Proof. (i) Let $0 < a < 1$. Then we have

$$\begin{aligned}
\varrho(x) &= \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_{r} q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r} \\
&= \sum_{r=0}^{\infty} \left(\frac{a}{Q_{2^r}} \sum_{r} q_i \left| \frac{\gamma x(i) + (1-\gamma)x(i-1)}{a} \right| \right)^{p_r} \\
&= \sum_{r=0}^{\infty} a^{p_r} \left(\frac{1}{Q_{2^r}} \sum_{r} q_i \left| \frac{\gamma x(i) + (1-\gamma)x(i-1}{a} \right| \right)^{p_r} \\
&\geq \sum_{r=0}^{\infty} a^{M} \left(\frac{1}{Q_{2^r}} \sum_{r} q_i \left| \frac{\gamma x(i) + (1-\gamma)x(i-1}{a} \right| \right)^{p_r} \\
&= a^M \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_{r} q_i \left| \frac{\gamma x(i) + (1-\gamma)x(i-1)}{a} \right| \right)^{p_r} \\
&= a^M \varrho\left(\frac{x}{a}\right).
\end{aligned}$$

By convexity of modular ϱ, we have $\varrho(ax) \leq a\varrho(x)$, so property (i) is proved.
(ii) Let $a > 1$. Then

$$\begin{aligned}
\varrho(x) &= \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_{r} q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r} \\
&= \sum_{r=0}^{\infty} a^{p_r} \left(\frac{1}{Q_{2^r}} \sum_{r} q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r} \\
&\leq a^M \sum_{r=0}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_{r} q_i \left| \frac{\gamma x(i) + (1-\gamma)x(i-1)}{a} \right| \right)^{p_r} \\
&= a^M \varrho\left(\frac{x}{a}\right).
\end{aligned}$$

Hence property (ii) is satisfied. (iii) follows from the convexity of ϱ. □

By a similar proof of these presented in ([7, 24, 25]), we get the following Proposition.

Proposition 3.3. *For any $x \in C(\mathcal{Z}; p, q)$, we have*

(i) *if $\|x\| < 1$, then $\varrho(x) \leq \|x\|$;*
(ii) *if $\|x\| > 1$, then $\varrho(x) \geq \|x\|$;*
(iii) *$\|x\| = 1$ if and only if $\varrho(x) = 1$;*
(iv) *$\|x\| < 1$ if and only if $\varrho(x) < 1$;*
(v) *$\|x\| > 1$ if and only if $\varrho(x) > 1$.*

Proposition 3.4. *For any $x \in C(\mathcal{Z}; p, q)$, we have*

(i) *if $0 < a < 1$ and $\|x\| > a$, then $\varrho(x) > a^M$;*
(ii) *if $a \geq 1$ and $\|x\| < a$, then $\varrho(x) < a^M$.*

Proposition 3.5. *Let $\{x_n\}$ be a sequence in $C(\mathcal{Z};p,q)$.*

(i) *If $\|x_n\| \to 1$ as $n \to \infty$, then $\varrho(x_n) \to 1$ as $n \to \infty$.*
(ii) *If $\varrho(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$ as $n \to \infty$.*

Lemma 3.6. *Let $x \in C(\mathcal{Z};p,q)$ and $(x_n) \subseteq C(\mathcal{Z};p,q)$. If $\varrho(x_n) \to \varrho(x)$ as $n \to \infty$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$, then $x_n \to x$ as $n \to \infty$.*

Proof. Let $\varepsilon > 0$ be given. We put that,

$$\varrho_0(x) = \sum_{r=0}^{r_0} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r}$$

and

$$\varrho_1(x) = \sum_{r=r_0+1}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r}.$$

Since $\varrho(x) < \infty$, there exists $r_0 \in \mathbb{N}$ such that

$$\varrho_1(x) < \frac{\varepsilon}{3 \cdot 2^{M+1}}. \tag{3.1}$$

Since, $\varrho(x_n) - \varrho_0(x_n) \to \varrho(x) - \varrho_0(x)$ and $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that

$$\varrho_1(x_n - x) = \varrho(x_n) - \varrho_0(x_n) \quad < \quad \varrho(x) - \varrho_0(x) + \frac{\varepsilon}{3 \cdot 2^M} \tag{3.2}$$

and

$$\varrho_0(x_n - x) \leq \frac{\varepsilon}{3} \tag{3.3}$$

for all $n \geq n_0$. It follows from (3.1), (3.2), and (3.3) that for all $n \geq n_0$ we have

$$\begin{aligned}
\varrho(x_n - x) &= \varrho_0(x_n - x) + \varrho_1(x_n - x) \\
&\leq \frac{\varepsilon}{3} + \sum_{r=r_0+1}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma(x_n(i) - x(i)) + (1-\gamma)(x_n(i-1) - x(i))| \right)^{p_r} \\
&\leq \frac{\varepsilon}{3} + 2^M \sum_{r=r_0+1}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma x_n(i) + (1-\gamma)x_n(i-1)| \right)^{p_r} \\
&+ 2^M \sum_{r=r_0+1}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r} \\
&= \frac{\varepsilon}{3} + 2^M \left(\varrho(x_n) - \sum_{r=0}^{r_0} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma x_n(i) + (1-\gamma)x_n(i-1)| \right)^{p_r} \right) \\
&+ 2^M \sum_{r=r_0+1}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r} \\
&\leq \frac{\varepsilon}{3} + 2^M \left(\varrho(x) - \sum_{r=0}^{r_0} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r} + \frac{\varepsilon}{3 \cdot 2^M} \right) \\
&+ 2^M \sum_{r=r_0+1}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\varepsilon}{3} + 2^M \left(\sum_{r=r_0+1}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r} + \frac{\varepsilon}{3 \cdot 2^M} \right) \\
&+ 2^M \sum_{r=r_0+1}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r} \\
&= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + 2^{M+1} \sum_{r=r_0+1}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r} \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + 2^{M+1} \left(\frac{\varepsilon}{3 \cdot 2^{M+1}} \right) \\
&= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon.
\end{aligned}
$$

This show that $\varrho(x_n - x) \to 0$ as $n \to \infty$. Hence, by Proposition 3.5(ii), we have

$$\|x_n - x\| \to 0 \text{ as } \to \infty.$$

□

Theorem 3.7. *The space $C(\mathcal{Z}; p, q)$ has the property (H).*

Proof. Let $x \in S(C(\mathcal{Z}; p, q))$ and $(x_n) \subseteq C(\mathcal{Z}; p, q)$ be such that $\|x_n\| \to 1$ and $x_n \overset{w}{\to} x$ as $n \to \infty$. By Proposition 3.3(iii), we have $\varrho(x) = 1$, so it follows form Proposition 3.5(i) that $\varrho(x_n) \to \varrho(x)$ as $n \to \infty$. Since the mapping $\pi_i : C(\mathcal{Z}; p, q) \to \mathbb{R}$ defined by $\pi_i(y) = y(i)$, is a continuous linear functional on $C(\mathcal{Z}; p, q)$, it follows that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$. Thus by Lemma 3.6, we obtain that $x_n \to x$ as $n \to \infty$, and hence the space $C(\mathcal{Z}; p, q)$ has the property (H). □

Corollary 3.8. *For any $1 < p < \infty$, the space $ces[(p), (q)]$ has the property (H).*

Corollary 3.9 ([20]). *The space* $\mathrm{ces}(p)$ *has the property (H).*

Next, we will prove that the spaces $C(\mathcal{Z}; p, q)$ has the uniform Opial property.

Theorem 3.10. *The space $C(\mathcal{Z}; p, q)$ has the uniform Opial property.*

Proof. Take any $\varepsilon > 0$ and $x \in C(\mathcal{Z}; p, q)$ with $\|x\| \geq \varepsilon$. Let (x_n) be a weakly null sequence in $S(C(\mathcal{Z}; p, q))$. By $\sup_k p_k < \infty$ we have that $\varrho \in \Delta_2^s$, hence by Lemma 2.2 there exists $\delta \in (0, 1)$ independent of x such that $\varrho(x) > \delta$. Also, by $\varrho \in \Delta_2^s$ and Lemma 2.1 asserts that there exists $\delta_1 \in (0, \delta)$ such that

$$|\varrho(y + z) - \varrho(y)| < \frac{\delta}{4}, \tag{3.4}$$

whenever, $\varrho(y) \leq 1$ and $\varrho(z) \leq \delta_1$. Choose $r_0 \in \mathbb{N}$ such that

$$\sum_{r=r_0+1}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r} < \frac{\delta_1}{4}. \tag{3.5}$$

Then, we have

$$
\begin{aligned}
\delta &< \sum_{r=0}^{r_0} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r} + \sum_{r=r_0+1}^{\infty} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r} \\
&\leq \sum_{r=0}^{r_0} \left(\frac{1}{Q_{2^r}} \sum_r q_i |\gamma x(i) + (1-\gamma)x(i-1)| \right)^{p_r} + \frac{\delta_1}{4},
\end{aligned} \tag{3.6}
$$

which implies that

$$\begin{aligned}\sum_{r=0}^{r_0}\left(\frac{1}{Q_{2^r}}\sum_r q_i|\gamma x(i)+(1-\gamma)x(i-1)|\right)^{p_r} &> \delta-\frac{\delta_1}{4}\\ &> \delta-\frac{\delta}{4}\\ &= \frac{3\delta}{4}.\end{aligned} \tag{3.7}$$

Since $x_n \overset{w}{\to} 0$ and the weak convergence implies the coordinatewise convergence, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{3\delta}{4} \le \sum_{r=0}^{r_0}\left(\frac{1}{Q_{2^r}}\sum_r q_i|\gamma(x_n(i)+x(i))+(1-\gamma)(x_n(i-1)+x(i-1))|\right)^{p_r} \tag{3.8}$$

for all $n > n_0$. Again, by $x_n \overset{w}{\to} 0$, there exists $n_1 \in \mathbb{N}$ such that

$$\| x_{n|_{r_o}} \| < 1-\left(1-\frac{\delta}{4}\right)^{\frac{1}{M}} \tag{3.9}$$

for all $n \ge n_1$, where $p_r \le M$ for all $r \in \mathbb{N}$. Hence, by the triangle inequality of the norm, we get

$$\| x_{n|_{\mathbb{N}-r_o}} \| > \left(1-\frac{\delta}{4}\right)^{\frac{1}{M}}. \tag{3.10}$$

It follows from Proposition 3.3(ii) that

$$\begin{aligned} 1 &\le \varrho\left(\frac{x_{n|_{\mathbb{N}-r_o}}}{\left(1-\frac{\delta}{4}\right)^{\frac{1}{M}}}\right)\\ &= \sum_{r=r_0+1}^{\infty}\left(\frac{\frac{1}{Q_{2^r}}\sum_r q_i|\gamma x_n(i)+(1-\gamma)x_n(i-1))|}{\left(1-\frac{\delta}{4}\right)^{\frac{1}{M}}}\right)^{p_r}\\ &\le \left(\frac{1}{\left(1-\frac{\delta}{4}\right)^{\frac{1}{M}}}\right)^{M}\sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r|\gamma x_n(i)+(1-\gamma)x_n(i-1))|\right)^{p_r}\end{aligned} \tag{3.11}$$

implies that

$$\sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r||\gamma x_n(i)+(1-\gamma)x_n(i-1))|\right)^{p_k} \ge 1-\frac{\delta}{4} \tag{3.12}$$

for all $n > n_1$. By inequality (3.4), (3.5), (3.8), and (3.12), we get for any $n > n_1$ that

$$\begin{aligned}\varrho(x_n+x) =& \sum_{r=0}^{r_0}\left(\frac{1}{Q_{2^r}}\sum_r|\gamma(x_n(i)+x(i))+(1-\gamma)(x_n(i-1)+x(i-1))|\right)^{p_r}\\ &+\sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r|\gamma(x_n(i)+x(i))+(1-\gamma)(x_n(i-1)+x(i-1))|\right)^{p_k}\\ \ge& \frac{3\delta}{4}+\sum_{r=r_0+1}^{\infty}\left(\frac{1}{Q_{2^r}}\sum_r q_i|\gamma x_n(i)+(1-\gamma)x_n(i-1)|\right)^{p_k}-\frac{\delta}{4}\end{aligned}$$

$$\geq \frac{3\delta}{4} + \left(1 - \frac{\delta}{4}\right) - \frac{\delta}{4}$$
$$\geq 1 + \frac{\delta}{4}.$$

Since $\varrho \in \Delta_2^s$, by Lemma 2.3 there exists τ depending only on δ such that $\| x_n + x \| \geq 1 + \tau$, which implies that $\lim_{n\to\infty} \inf \| x_n + x \| \geq 1 + \tau$. This completes the proof. □

Corollary 3.11. *For any $1 < p < \infty$, the space $ces[(p),(q)]$ has the uniform Opial property.*

Corollary 3.12. *The space* ces(p) *has the uniform Opial property.*

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