Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 3157–3168
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

On convergence of random iterative schemes with errors for strongly pseudo-contractive Lipschitzian maps in real Banach spaces

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Communicated by Lj. B. Ćirić

Abstract

In this work, strong convergence and stability results of a three step random iterative scheme with errors for strongly pseudo-contractive Lipschitzian maps are established in real Banach spaces. Analytic proofs are supported by providing numerical examples. Applications of random iterative schemes with errors to find solution of nonlinear random equation are also given. Our results improve and establish random generalization of results obtained by Xu and Xie [Y. Xu, F. Xie, Rostock. Math. Kolloq., **58** (2004), 93–100], Gu and Lu [F. Gu, J. Lu, Math. Commun., **9** (2004), 149–159], Liu et al. [Z. Liu, L. Zhang, S. M. Kang, Int. J. Math. Math. Sci., **31** (2002), 611–617] and many others.

Keywords: Random Iterative schemes, stability, strongly pseudo-contractive maps.
2010 MSC: 47H10, 47H06

1. Introduction and Preliminaries

The machinery of fixed point theory provides a convenient way of modelling many problems arising in non-linear analysis, probability theory and for a solution of random equations in applied sciences, see [4, 9, 11, 12, 15, 17, 18, 20, 21, 25, 27, 29, 30, 31, 33, 34, 35, 36, 38, 39, 40] and references there. With the developments in random fixed point theory, there has been a renewed interest in random iterative schemes[2, 3, 7, 8, 10]. In linear spaces, Mann and Ishikawa iterative schemes are two general iterative

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Received 2015-12-22

schemes which have been successfully applied to fixed point problems [1, 5, 6, 13, 14, 16, 19, 26, 28, 37]. Recently, many stability and convergence results of iterative schemes have been established, using Lipschitz accretive pseudo-contractive) and Lipschitz strongly accretive (or strongly pseudo-contractive) mappings in Banach spaces [9, 10, 12, 13, 22, 23, 24, 32, 37]. Since in deterministic case the consideration of error terms is an important part of an iterative scheme, therefore, we introduce a three step random iterative scheme with errors and prove that the iterative scheme is stable with respect to T with Lipschitz condition where T is a strongly accretive mapping in arbitrary real Banach space.

Let X be a real separable Banach space and let J denote the normalized duality pairing from X to 2^{X^*} given by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\| \, \|f\|, \|f\| = \|x\|\}, \quad x \in X,$$

where X^* denote the dual space of X and $\langle \cdot, \cdot\rangle$ denote the generalized duality pairing between X and X^*.

Suppose (Ω, Σ) denotes a measurable space consisting of a set Ω and sigma algebra Σ of subsets of Ω and C, a nonempty subset of X. Let $T : \Omega \times C \to C$ be a random operator, then random Mann iterative scheme with errors is defined as follows:

$$x_{n+1}(w) = (1-\alpha_n)x_n(w) + \alpha_n T(w, x_n(w)) + u_n(w), \quad \text{for each } w \in \Omega,\ n \geq 0, \tag{1.1}$$

where $0 \leq \alpha_n \leq 1$, $x_0 : \Omega \to C$, an arbitrary measurable mapping and $\{u_n(w)\}$ is a sequence of measurable mappings from Ω to C.

Also, random Ishikawa iterative scheme with errors is defined as follows:

$$\begin{aligned} x_{n+1}(w) &= (1-\alpha_n)x_n(w) + \alpha_n T(w, y_n(w)) + u_n(w), \\ y_n(w) &= (1-\beta_n)x_n(w) + \beta_n T(w, x_n(w)) + v_n(w), \quad \text{for each } w \in \Omega,\ n \geq 0, \end{aligned} \tag{1.2}$$

where $0 \leq \alpha_n, \beta_n \leq 1$, $x_0 : \Omega \to C$, an arbitrary measurable mapping and $\{u_n(w)\}$, $\{v_n(w)\}$ are sequences of measurable mappings from Ω to C.

Obviously $\{x_n(w)\}$ and $\{y_n(w)\}$ are sequences of mappings from Ω in to C.

Also, we consider the following three step random iterative scheme with errors $\langle x_n(w)\rangle$ defined by

$$\begin{aligned} x_{n+1}(w) &= (1-\alpha_n)y_n(w) + \alpha_n T(w, y_n(w)) + u_n(w), \\ y_n(w) &= (1-\beta_n)z_n(w) + \beta_n T(w, z_n(w)) + v_n(w), \\ z_n(w) &= (1-\gamma_n)x_n(w) + \gamma_n T(w, x_n(w)) + w_n(w), \text{for each } w \in \Omega,\ n \geq 0, \end{aligned} \tag{1.3}$$

where $\{u_n(w)\}$, $\{v_n(w)\}$, $\{w_n(w)\}$ are sequences of measurable mappings from Ω to C, $0 \leq \alpha_n,\ \beta_n, \gamma_n \leq 1$ and $x_0 : \Omega \to C$, an arbitrary measurable mapping.

Putting $\beta_n = 0$, $v_n = 0$ in (1.2) and $\beta_n = 0$, $v_n = 0$, $\gamma_n = 0$, $w_n = 0$ in (1.3), we get random Mann iterative scheme with errors (1.1).

Now we give some definitions and lemmas, which will be used in the proofs of our main results.

Definition 1.1. A mapping $g : \Omega \to C$ is said to be measurable if $g^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of X.

Definition 1.2. A function $F : \Omega \times C \to C$ is said to be a random operator if $F(\cdot, x) : \Omega \to C$ is measurable for every $x \in C$.

Definition 1.3. A measurable mapping $p : \Omega \to C$ is said to be random fixed point of the random operator $F : \Omega \times C \to C$, if $F(w, p(w)) = p(w)$ for all $w \in \Omega$.

Definition 1.4. A random operator $F : \Omega \times C \to C$ is said to be continuous if for fixed $w \in \Omega$, $F(w, \cdot) : C \to C$ is continuous.

In the sequel, I denotes the identity operator on X, $D(T)$ and $R(T)$ denote the domain and the range of T, respectively.

Definition 1.5. Let $T : \Omega \times X \to X$ be a mapping. Then

(i) T is said to be Lipschitzian, if for any $x, y \in X$ and $w \in \Omega$, there exists $L > 0$ such that

$$\|T(w,x) - T(w,y)\| \leq L\|x-y\|; \tag{1.4}$$

(ii) T is said to be nonexpansive, if for any $x, y \in X$ and $w \in \Omega$,

$$\|T(w,x) - T(w,y)\| \leq \|x-y\|; \tag{1.5}$$

(iii) $T : \Omega \times X \to X$ is strongly pseudo-contractive [9, 12] if and only if for all $x, y \in X, w \in \Omega$ and for all $r > 0$, $k \in (0,1)$, the following inequality holds:

$$\|x-y\| \leq \|(x-y) + r[(I-T-kI)(w,x) - (I-T-kI)(w,y)]\|, \tag{1.6}$$

or equivalently iff for all $x, y \in X$, there exists $j(x-y) \in J(x-y)$, such that

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \leq k\|x-y\|^2;$$

(iv) T is said to be strongly accretive [9, 12], if and only if for all $x, y \in X$ and for all $r > 0$, $k \in (0,1)$, the following inequality holds:

$$\|x-y\| \leq \|(x-y) + r[(T-kI)(w,x) - (T-kI)(w,y)]\|, \tag{1.7}$$

or equivalently iff for all $x, y \in X$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq k\|x-y\|^2;$$

(v) If T is accretive and $R(I + \lambda T) = X$ for any $\lambda > 0$, then T is called m-accretive [25, 31].

A mapping $T : \Omega \times X \to X$ is said to be strongly pseudo-contractive if $I - T$ is strongly accretive, hence the fixed point theory for strongly accretive mappings is connected with fixed point theory for strongly pseudo-contractive mappings. It is well known that if T is Lipschitz strongly pseudo-contractive mapping [11], then T has a unique fixed point.

Lemma 1.6 ([25])**.** *Suppose X is an arbitrary real Banach space, $T : D(T) \subset X \to X$ is accretive and continuous, and $D(T) = X$. Then T is m-accretive.*

Lemma 1.7 ([31])**.** *Suppose X is an arbitrary real Banach space, $T : D(T) \subset X \to X$ is an m-accretive mapping. Then the equation $x + Tx = f$ has a unique solution in $D(T)$ for any $f \in X$.*

Lemma 1.8 ([13])**.** *Let $\{x_n\}$ be a sequence of real numbers satisfying the following inequality:*

$$x_{n+1} \leq \delta x_n + \sigma_n, \quad n \geq 1,$$

where $x_n \geq 0$, $\sigma_n \geq 0$ and $\lim_{n\to\infty} \sigma_n = 0$, $0 \leq \delta < 1$. Then $x_n \to 0$ as $n \to \infty$.

Definition 1.9 ([2])**.** Let $T : \Omega \times C \to C$ be a random operator, where C is a nonempty closed convex subset of a real separable Banach space X. Let $x_0 : \Omega \to C$ be any measurable mapping. The sequence $\{x_{n+1}(w)\}$ of measurable mappings from Ω to C, for $n = 0, 1, 2, \ldots$ generated by the certain random iterative scheme involving a random operator T is denoted by $\{T, x_n(w)\}$ for each $w \in \Omega$. Suppose that $x_n(w) \to p(w)$ as $n \to \infty$ for each $w \in \Omega$, where $p \in RF(T)$. Let $\{p_n(w)\}$ be any arbitrary sequence of measurable mappings from Ω to C. Define the sequence of measurable mappings $k_n : \Omega \to R$ by $k_n(w) = d(p_n(w), \{T, p_n(w)\})$. If for each $w \in \Omega$, $k_n(w) \to 0$ as $n \to \infty$ implies $p_n(w) \to p(w)$ as $n \to \infty$ for each $w \in \Omega$, then the random iterative scheme is said to be stable with respect to the random operator T.

2. Convergence and Stability Results

In this section, we establish the convergence and stability results of three step random iterative scheme with errors (1.3) using strongly pseudo-contractive mapping under some parametrical restrictions.

Theorem 2.1. *Let X be a real Banach space, $T : \Omega \times X \to X$ be a strongly pseudo-contractive Lipschitzian random mapping with a Lipschitz constant $L \geq 1$. Let $\{x_n(w)\}$ be the random iterative scheme with errors defined by* (1.3)*, with the following restrictions:*

(i) $\beta_n(L-1) + \gamma_n(L-1)^2 + \beta_n\gamma_n(L-1)^2 < \alpha_n\{k - (2-k)\alpha_n L(1+L)\}(1-t)$, $(n \geq 0)$;

(ii) $\lim_{n\to\infty} u_n(w) = 0$, $\lim_{n\to\infty} v_n(w) = 0$, $\lim_{n\to\infty} w_n(w) = 0$.

Then the sequence $\{x_n(w)\}$ converges strongly to a unique random fixed point $p(w)$ of T.

Proof. From (1.3), we have

$$\begin{aligned}(x_{n+1}(w) - p(w)) &+ \alpha_n[(I - T - kI)x_{n+1}(w) - (I - T - kI)p(w)]\\ &= (1-\alpha_n)(y_n(w) - p(w)) + \alpha_n[(I - T - kI)x_{n+1}(w)\\ &\quad + T(w, y_n(w))] - \alpha_n(I - kI)p(w) + u_n(w).\end{aligned} \tag{2.1}$$

Since T is strongly pseudo-contractive and Lipschitzian mapping, so using (2.1) and (1.6), we get

$$\begin{aligned}\|x_{n+1}(w) - p(w)\| &\leq \|x_{n+1}(w) - p(w) + \alpha_n[(I - T - kI)x_{n+1}(w) - (I - T - kI)p(w)]\|\\ &\leq (1-\alpha_n)\|y_n(w) - p(w)\| + \alpha_n\|T(w, y_n(w)) - T(w, x_{n+1}(w))\|\\ &\quad + \alpha_n I(1-k)\|x_{n+1}(w) - p(w)\| + \|u_n(w)\|\\ &= (1-\alpha_n)\|y_n(w) - p(w)\| + \alpha_n\|T(w, y_n(w))\\ &\quad - T(w, x_{n+1}(w))\| + \alpha_n(1-k)\|x_{n+1}(w) - p(w)\| + \|u_n(w)\|,\end{aligned}$$

which implies

$$\begin{aligned}[1 - \alpha_n(1-k)]\|x_{n+1}(w) - p(w)\| &\leq (1-\alpha_n)\|y_n(w) - p(w)\|\\ &\quad + \alpha_n\|T(w, y_n(w)) - T(w, x_{n+1}(w))\| + \|u_n(w)\|,\end{aligned}$$

or

$$\begin{aligned}\|x_{n+1}(w) - p(w)\| &\leq \frac{(1-\alpha_n)}{[1-\alpha_n(1-k)]}\|y_n(w) - p(w)\| + \frac{\alpha_n}{[1-\alpha_n(1-k)]}\|T(w, y_n(w))\\ &\quad - T(w, x_{n+1}(w))\| + \frac{1}{[1-\alpha_n(1-k)]}\|u_n(w)\|.\end{aligned} \tag{2.2}$$

Now,

$$1 - \frac{1-\alpha_n}{1-\alpha_n(1-k)} = \frac{1-(1-\alpha_n k)}{1-\alpha_n(1-k)} \geq 1 - (1-\alpha_n k),$$

implies

$$\frac{1-\alpha_n}{1-\alpha_n(1-k)} \leq 1 - \alpha_n k, \tag{2.3}$$

and

$$1 - \frac{\alpha_n}{1-\alpha_n(1-k)} = \frac{1-\alpha_n(2-k)}{1-\alpha_n(1-k)} \geq 1 - \alpha_n(2-k),$$

implies

$$\frac{\alpha_n}{1-\alpha_n(1-k)} \le \alpha_n(2-k), \tag{2.4}$$

and

$$\frac{1}{1-\alpha_n(1-k)} \le \frac{1}{k}. \tag{2.5}$$

Using (2.3), (2.4) and (2.5), (2.2) yields

$$\begin{aligned} \|x_{n+1}(w)-p(w)\| \le (1-\alpha_n k)\|y_n(w)-p(w)\| + \alpha_n(2-k)\|T(w,y_n(w)) \\ - T(w,x_{n+1}(w))\| + \frac{\|u_n(w)\|}{k}\ . \end{aligned} \tag{2.6}$$

Now, using Lipschitz condition on T and using (1.3), we get

$$\begin{aligned} \|(T(w,x_{n+1}(w))-T(w,y_n(w)))\| &\le L\|x_{n+1}(w)-y_n(w)\| \\ &\le L\alpha_n\|y_n(w)-T(w,y_n(w))\| + L\|u_n(w)\| \\ &\le L\alpha_n\|y_n(w)-p(w)\| + L\alpha_n\|T(w,y_n(w))-p(w)\| + L\|u_n(w)\| \\ &\le L\alpha_n(1+L)\|y_n(w)-p(w)\| + L\|u_n(w)\|. \end{aligned} \tag{2.7}$$

Also, from (1.3), we have the following estimate:

$$\begin{aligned} \|y_n(w)-p(w)\| &\le (1-\beta_n)\|z_n(w)-p(w)\| + \beta_n\|T(w,z_n(w)-p(w))\| + \|v_n(w)\| \\ &\le (1-\beta_n)\|z_n(w)-p(w)\| + \beta_n L\|(z_n(w)-p(w))\| + \|v_n(w)\| \\ &= [1+\beta_n(L-1)]\|z_n(w)-p(w)\| + \|v_n(w)\| \\ &= [1+\beta_n(L-1)]\|(1-\gamma_n)x_n(w)+\gamma_n T(w,x_n(w)) + w_n(w) - p(w)\| + \|v_n(w)\| \\ &\le [1+\beta_n(L-1)][(1-\gamma_n)\|x_n(w)-p(w)\| + \gamma_n\|T(w,x_n(w))-p(w)\|] \\ &\quad + [1+\beta_n(L-1)]\|w_n(w)\| + \|v_n(w)\| \\ &\le [1+\beta_n(L-1)][(1-\gamma_n)\|x_n(w)-p(w)\| + L\gamma_n\|x_n(w)-p(w)\|] + \|v_n(w)\| \\ &\quad + [1+\beta_n(L-1)]\|w_n(w)\| \\ &= [1+\beta_n(L-1)](1-\gamma_n+L\gamma_n)\|x_n(w)-p(w)\| \\ &\quad + \|v_n(w)\| + [1+\beta_n(L-1)]\|w_n(w)\|. \end{aligned} \tag{2.8}$$

Using estimate (2.8), (2.7) becomes

$$\begin{aligned} \|T(w,y_n(w))-T(w,x_{n+1}(w))\| &\le L\alpha_n(1+L)[1+\beta_n(L-1)](1-\gamma_n+L\gamma_n)\|x_n(w)-p(w)\| \\ &\quad + L\alpha_n(1+L)\|v_n(w)\| + L\|u_n(w)\| \\ &\quad + L\alpha_n(1+L)[1+\beta_n(L-1)]\|w_n(w)\|. \end{aligned} \tag{2.9}$$

Putting values of estimates (2.8) and (2.9) in (2.6), we get

$$\begin{aligned} &\|x_{n+1}(w)-p(w)\| \\ &\quad \le (1-\alpha_n k)[1+\beta_n(L-1)](1-\gamma_n+L\gamma_n)\|x_n(w)-p(w)\| \\ &\qquad + \alpha_n^2(2-k)L(1+L)[1+\beta_n(L-1)](1-\gamma_n+L\gamma_n)\|x_n(w)-p(w)\| \\ &\qquad + [1-\alpha_n k + L\alpha_n^2(2-k)(1+L)]\|v_n(w)\| + [L\alpha_n(2-k)+\frac{1}{k}]\|u_n(w)\| \\ &\qquad + [1-\alpha_n k + L\alpha_n^2(1+L)(2-k)][1+\beta_n(L-1)]\|w_n(w)\| \\ &\quad = \{(1-\alpha_n k)[1+\beta_n(L-1)](1-\gamma_n+L\gamma_n) \\ &\qquad + (2-k)L\alpha_n^2(1+L)[1+\beta_n(L-1)](1-\gamma_n+L\gamma_n)\}\|x_n(w)-p(w)\| \\ &\qquad + [1-\alpha_n k + L\alpha_n^2(2-k)(1+L)]\|v_n(w)\| + [L\alpha_n(2-k)+\frac{1}{k}]\|u_n(w)\| \\ &\qquad + [1-\alpha_n k + L\alpha_n^2(1+L)(2-k)][1+\beta_n(L-1)]\|w_n(w)\| \end{aligned}$$

$$
\begin{aligned}
&= [1+\beta_n(L-1)](1-\gamma_n+L\gamma_n)[(1-\alpha_n k)\\
&\quad + L\alpha_n^2(2-k)(1+L)]\|x_n(w)-p(w)\|\\
&\quad + [1-\alpha_n k+L\alpha_n^2(2-k)(1+L)]\|v_n(w)\| + [L\alpha_n(2-k)+\frac{1}{k}]\|u_n(w)\|\\
&\quad + [1-\alpha_n k+L\alpha_n^2(1+L)(2-k)][1+\beta_n(L-1)]\|w_n(w)\|\\
&= [1+\beta_n(L-1)](1-\gamma_n+L\gamma_n)\times[1-\alpha_n\{k-(2-k)\alpha_n L(1+L)\}]\|x_n(w)-p(w)\|\\
&\quad + [1-\alpha_n k+L\alpha_n^2(2-k)(1+L)]\|v_n(w)\| + [L\alpha_n(2-k)+\frac{1}{k}]\|u_n(w)\|\\
&\quad + [1-\alpha_n k+L\alpha_n^2(1+L)(2-k)][1+\beta_n(L-1)]\|w_n(w)\|\\
&\le 1-[\alpha_n\{k-(2-k)\alpha_n L(1+L)\}-\gamma_n(L-1)-\beta_n(L-1)-\gamma_n\beta_n(L-1)^2]\|x_n(w)-p(w)\|\\
&\quad + [1-\alpha_n k+L\alpha_n^2(2-k)(1+L)]\|v_n(w)\| + [L\alpha_n(2-k)+\frac{1}{k}]\|u_n(w)\|\\
&\quad + [1-\alpha_n k+L\alpha_n^2(1+L)(2-k)][1+\beta_n(L-1)]\|w_n(w)\|.
\end{aligned}
\tag{2.10}
$$

Using condition (i) and (2.10), we have

$$
\begin{aligned}
\|x_{n+1}(w)-p(w)\| &\le 1-\alpha_n\{k-(2-k)\alpha_n L(1+L)\}\\
&\quad + \alpha_n\{k-(2-k)\alpha_n L(1+L)\}(1-t)\|x_n(w)-p(w)\|\\
&\quad + [1-\alpha_n k+L\alpha_n^2(2-k)(1+L)]\|v_n(w)\| + [L\alpha_n(2-k)+\frac{1}{k}]\|u_n(w)\|\\
&\quad + [1-\alpha_n k+L\alpha_n^2(1+L)(2-k)][1+\beta_n(L-1)]\|w_n(w)\|\\
&= [1-\alpha_n\{k-(2-k)\alpha_n L(1+L)\}t]\|x_n(w)-p(w)\|\\
&\quad + [1-\alpha_n k+L\alpha_n^2(2-k)(1+L)]\|v_n(w)\|\\
&\quad + [L\alpha_n(2-k)+\frac{1}{k}]\|u_n(w)\|\\
&\quad + [1-\alpha_n k+L\alpha_n^2(1+L)(2-k)][1+\beta_n(L-1)]\|w_n(w)\|.
\end{aligned}
\tag{2.11}
$$

If we let $\alpha_n \ge \alpha$, $\forall\ n \in N$, then (2.11) reduces to

$$
\begin{aligned}
\|x_{n+1}(w)-p(w)\| &\le [1-\alpha\{k-(2-k)\alpha L(1+L)\}t]\|x_n(w)-p(w)\|\\
&\quad + [1+L(2-k)(1+L)]\|v_n(w)\| + [L(2-k)\\
&\quad + \frac{1}{k}]\|u_n(w)\| + L[1+2L(1+L)]\|w_n(w)\|.
\end{aligned}
\tag{2.12}
$$

Now, if we put $[1-\alpha\{k-(2-k)\alpha L(1+L)\}t]=\delta$ and

$$
[1+L(2-k)(1+L)]\|v_n(w)\| + \left[L(2-k)+\frac{1}{k}\right]\|u_n(w)\| + L[1+2L(1+L)]\|w_n(w)\| = \sigma_n,
$$

then (2.12) becomes

$$
\|x_{n+1}(w)-p(w)\| \le \delta\|x_n(w)-p(w)\| + \sigma_n\ . \tag{2.13}
$$

Therefore, using conditions (ii) and Lemma 1.8, inequality (2.13) yields $\lim_{n\to\infty}\|x_{n+1}(w)-p(w)\|=0$, that is $\{x_n(w)\}$ defined by (1.3) converges strongly to a random fixed point $p(w)$ of T. □

Theorem 2.2. *Let X be a real Banach space, $T:\Omega\times X\to X$ be a strongly pseudo-contractive Lipschitzian random mapping with a Lipschitz constant $L\ge 1$. Let $\{x_n(w)\}$ be the random iterative scheme with errors defined by* (1.3)*, with the following restrictions:*

(i) $\beta_n(L-1)+\gamma_n(L-1)^2+\beta_n\gamma_n(L-1)^2 < \alpha_n\{k-(2-k)\alpha_n L(1+L)\}(1-t)$, $(n\ge 0)$;

(ii) $\lim_{n\to\infty} u_n(w) = 0$, $\lim_{n\to\infty} v_n(w) = 0$, $\lim_{n\to\infty} w_n(w) = 0$.

Then the sequence $\{x_n(w)\}$ *is stable. Moreover,* $\lim_{n\to\infty} p_n(w) = p(w)$ *implies* $\lim_{n\to\infty} k_n(w) = 0$.

Proof. Suppose that $\{p_n(w)\} \subset X$, be an arbitrary sequence,

$$k_n(w) = \|p_{n+1}(w) - (1-\alpha_n)q_n(w) - \alpha_n T(w, q_n(w)) - u_n(w)\|,$$

where

$$\begin{aligned} q_n(w) &= (1-\beta_n)r_n(w) + \beta_n T(w, r_n(w)) + v_n(w), \\ r_n(w) &= (1-\gamma_n)p_n(w) + \gamma_n T(w, p_n(w)) + w_n(w), \end{aligned}$$

such that $\lim_{n\to\infty} k_n(w) = 0$. Then

$$\begin{aligned} \|p_{n+1}(w) - T(w, p(w))\| &= \|p_{n+1}(w) - (1-\alpha_n)q_n(w) - \alpha_n T(w, q_n(w)) - u_n(w)\| \\ &\quad + \|(1-\alpha_n)q_n(w) + \alpha_n T(w, q_n(w)) + u_n(w) - T(w, p(w))\| \\ &= k_n(w) + \|s_n(w) - T(w, p(w))\|, \end{aligned} \tag{2.14}$$

where

$$s_n(w) = (1-\alpha_n)q_n(w) + \alpha_n T(w, q_n(w)) + u_n(w)\ . \tag{2.15}$$

From (2.15), we have

$$\begin{aligned} &s_n(w) - p(w) + \alpha_n[(I - T - kI)T(w, s_n(w)) - (I - T - kI)p(w)] \\ &\quad = (1-\alpha_n)(q_n(w) - p(w)) + \alpha_n[(I - T - kI)s_n(w) + T(w, q_n(w))] - \alpha_n(I - kI)p(w) + u_n(w), \end{aligned}$$

which further implies

$$\begin{aligned} \|s_n(w) - p(w)\| &\le \|s_n(w) - p(w) + \alpha_n[(I - T - kI)s_n(w) - (I - T - kI)p(w)]\| \\ &\le (1-\alpha_n)\|(q_n(w) - p(w))\| + \alpha_n\|(T(w, q_n(w)) - T(w, s_n(w)))\| \\ &\quad + \alpha_n(1-k)\|(s_n(w) - p(w))\| + \|u_n(w)\|. \end{aligned} \tag{2.16}$$

Rearranging terms in (2.16) and using estimates (2.3)–(2.5), we get

$$\begin{aligned} \|s_n(w) - p(w)\| &\le (1 - \alpha_n k)\|(q_n(w) - p(w))\| \\ &\quad + \alpha_n(2-k)\|T(w, q_n(w)) - T(w, s_n(w))\| + \frac{\|u_n(w)\|}{k}. \end{aligned} \tag{2.17}$$

Following the same procedure as in Theorem 2.1, similar to estimate (2.12), we have the following estimate

$$\begin{aligned} \|s_n(w) - p(w)\| &\le [1 - \alpha\{k - (2-k)\alpha L(1+L)\}t]\|p_n(w) - p(w)\| + [1 + L(2-k)(1+L)]\|v_n(w)\| \\ &\quad + \left[L(2-k) + \frac{1}{k}\right]\|u_n(w)\| + L[1 + 2L(1+L)]\|w_n(w)\|. \end{aligned} \tag{2.18}$$

Inequality (2.18) together with inequality (2.14) yields

$$\begin{aligned} \|p_{n+1}(w) - T(w, p(w))\| &\le [1 - \alpha\{k - (2-k)\alpha L(1+L)\}t]\|p_n(w) - p(w)\| \\ &\quad + [1 + L(2-k)(1+L)]\|v_n(w)\| + \left[L(2-k) + \frac{1}{k}\right]\|u_n(w)\| \\ &\quad + L[1 + 2L^2(1+L)]\|w_n(w)\| + k_n. \end{aligned} \tag{2.19}$$

Putting $[1-\alpha\{k-(2-k)\alpha L(1+L)\}t]=\delta$ and

$$[1+L(2-k)(1+L)]\|v_n(w)\|+\left[L(2-k)+\frac{1}{k}\right]\|u_n(w)\|+L[1+2L(1+L)]\|w_n(w)\|+k_n=\sigma_n,$$

and using condition (ii), and Lemma 1.8, inequality (2.19) yields $\lim_{n\to\infty}\|p_{n+1}(w)-p(w)\|=0$. i.e $\lim_{n\to\infty} p_{n+1}(w)=p(w)$. Hence given iterative scheme is T stable.

Now, let $\lim_{n\to\infty} p_n(w)=p(w)$, then using (2.18), we have

$$\begin{aligned} k_n(w) &= \|p_{n+1}(w)-(1-\alpha_n)q_n(w)-\alpha_n T(w,q_n(w))-u_n(w)\| \\ &= \|p_{n+1}(w)-s_n(w)\| \\ &\le \|p_{n+1}(w)-p(w)\|+\|s_n(w)-p(w)\| \\ &\le \|p_{n+1}(w)-p(w)\|+[1-\alpha\{k-(2-k)\alpha L(1+L)\}t]\|p_n(w)-p(w)\| \\ &\quad +[1+L(2-k)(1+L)]\|v_n(w)\|+[L(2-k)+\frac{1}{k}]\|u_n(w)\|+L[1+2L(1+L)]\|w_n(w)\|, \end{aligned} \tag{2.20}$$

which implies $\lim_{n\to\infty} k_n(w)=0$. □

Putting $\beta_n=0$, $\gamma_n=0$, in Theorem 2.1 and Theorem 2.2, we have the following obvious corollary:

Corollary 2.3. *Let X be a real Banach space, $T:\Omega\times X\to X$ be a strongly pseudo-contractive Lipschitzian random mapping with a Lipschitz constant $L\ge 1$. Let $\{x_n(w)\}$ be the random Mann iterative scheme with errors defined by* (1.1) *with the following conditions:*

(i) $0<\alpha<\alpha_n$, $(n\ge 0)$;

(ii) $\lim_{n\to\infty} u_n(w)=0$.

Then

(i) *the sequence $\{x_n(w)\}$ converges strongly to unique fixed point $p(w)$ of T;*

(ii) *the sequence $\{x_n(w)\}$ is stable. Moreover, $\lim_{n\to\infty} p_n(w)=p(w)$ implies $\lim_{n\to\infty} k_n(w)=0$, where $\{x_n(w)\}\subseteq X$ is an arbitrary sequence.*

Now, we demonstrate the following example to prove the validity of our results.

Example 2.4. Let $\Omega=\left[\frac{1}{2},2\right]$ and Σ be the sigma algebra of Lebesgue's measurable subsets of Ω. Take $X=R$ and define random operator T from $\Omega\times X$ to X as $T(w,x)=\frac{w}{x}$. Then the measurable mapping $\xi:\Omega\to X$ defined by $\xi(w)=\sqrt{w}$, for every $w\in\Omega$, serve as a random fixed point of T. It is easy to see that the operator T is a Lipschitz random operator with Lipschitz constant $L=4$ and strongly pseudo-contractive random operator for any $k\in(0,1)$ and $\alpha_n=0.0082$, $k=0.9$, $t=0.4$, $\beta_n=\frac{1}{(1+L)^6}$, $\gamma_n=\frac{1}{(1+L)^7}$, $\|u_n\|=\frac{1}{(n+1)^2}$, $\|v_n\|=\frac{1}{(n+2)^2}$, $\|w_n\|=\frac{1}{(n+3)^2}$ satisfies all the conditions (i)–(ii) given in Theorem 2.1 and Theorem 2.2.

3. Convergence speed comparison

Let $\Omega=[0,1]$ and Σ be the sigma algebra of Lebesgue's measurable subsets of Ω. Take $X=R$ and define random operator T from $\Omega\times X$ to X as $T(w,x)=1-2\sin x$. Then the measurable mapping $\xi:\Omega\to X$ defined by $\xi(w)=0.3376$, for every $w\in\Omega$, serve as a random fixed point of T. It is easy to see that the operator T is a Lipschitz random operator with Lipschitz constant $L=2$ such that T is strongly pseudo-contractive and $\alpha_n=0.002$, $\beta_n=\frac{1}{(1+L)^7}$, $\gamma_n=\frac{1}{(1+L)^8}$, $\|u_n\|=\frac{1}{(n+1)^2}$, $\|v_n\|=\frac{1}{(n+2)^2}$, $\|w_n\|=\frac{1}{(n+3)^2}$, $k=0.9$, $r=0.2$, $t=0.5$ satisfies the conditions (i)-(ii) given in Theorem 2.1 and Theorem 2.2.

New random iterative scheme with errors is more acceptable for strongly pseudo-contractive mappings because it has better convergence rate as compared to Mann and Ishikawa iterative schemes with errors:

Taking initial approximation $x_0 = 1.8$, convergence of new three step iterative scheme with errors, Ishikawa and Mann iterative schemes with errors to the fixed point **0.3376** of operator T is shown in the following table. From table, it is obvious that in deterministic case new three step iterative scheme with errors has much better convergence rate as compared to Ishikawa and Mann iterative schemes with errors.

Number of iterations	Three step iterative scheme with errors	Ishikawa iterative scheme with errors	Mann iterative scheme with errors
n	x_{n+1}	x_{n+1}	x_{n+1}
1	1.79283	1.79874	1.7945
2	1.78567	1.79749	1.78902
3	1.77853	1.79623	1.78353
4	1.77139	1.79497	1.77806
5	1.76426	1.79372	1.77258
6	1.75715	1.79246	1.76712
7	1.75005	1.7912	1.76166
8	1.74296	1.78995	1.75621
9	1.73588	1.78869	1.75077
10	1.72881	1.78744	1.74533
-	-	-	-
1547	0.337601	0.593217	0.337846
1548	0.337601	0.592893	0.337844
1549	0.337601	0.592569	0.337843
1550	**0.3376**	0.592246	0.337841
1551	**0.3376**	0.591923	0.33784
-	**-**	**-**	-
2019	**0.3376**	0.47716	0.337601
2020	**0.3376**	0.47698	0.337601
2021	**0.3376**	0.4768	0.337601
2022	**0.3376**	0.47662	**0.3376**
2023	**0.3376**	0.47644	**0.3376**
-	**-**	**-**	-
8888	**0.3376**	0.337601	**0.3376**
8889	**0.3376**	0.337601	**0.3376**
8890	**0.3376**	0.337601	**0.3376**
8891	**0.3376**	**0.3376**	**0.3376**
8892	**0.3376**	**0.3376**	**0.3376**

4. Applications

In this section, we apply the random iterative schemes with errors to find solution of nonlinear random equation with Lipschitz strongly accretive mappings.

Theorem 4.1. *Suppose that $A : \Omega \times X \to X$ be a Lipschitz strongly accretive mapping. Let $x^*(w)$ be a solution of random equation $A(w,x) = f$; where $f \in X$ is any given point and $S(w,x) = f + x(w) - A(w,x)$, $\forall\ x \in X$. Consider the new three step random iterative scheme with errors defined by*

$$\begin{aligned} x_{n+1}(w) &= (1-\alpha_n)y_n(w) + \alpha_n S(w, y_n(w)) + u_n(w),\\ y_n(w) &= (1-\beta_n)z_n(w) + \beta_n S(w, z_n(w)) + v_n(w),\\ z_n(w) &= (1-\gamma_n)x_n(w) + \gamma_n S(w, x_n(w)) + w_n(w), \quad \text{for each } w \in \Omega,\ n \geq 0, \end{aligned} \tag{4.1}$$

where $\{u_n(w)\}$, $\{v_n(w)\}$, $\{w_n(w)\}$ are sequences of measurable mappings from Ω to X, $0 \leq \alpha_n, \beta_n, \gamma_n \leq 1$ and $x_0 : \Omega \to X$, an arbitrary measurable mapping, satisfying

(i) $\beta_n(L-1) + \gamma_n(L-1)^2 + \beta_n\gamma_n(L-1)^2 < \alpha_n\{k - (2-k)\alpha_n L(1+L)\}(1-t)$, $(n \geq 0)$

(ii) $\lim_{n\to\infty} u_n(w) = 0$, $\lim_{n\to\infty} v_n(w) = 0$, $\lim_{n\to\infty} w_n(w) = 0$,

where $L \geq 1$ is Lipschitz constant of $S(w,x)$. Then

(1) *$\{x_n(w)\}$ converges strongly to unique solution $x^*(w)$ of $A(w,x) = f$;*

(2) *It is S-stable to approximate the solution of $A(w,x) = f$; by new three step random iterative scheme with errors* (4.1).

Proof. Since $A(w,x)$ is Lipschitz strongly accretive mapping, so $S(w,x) = f + x(w) - A(w,x)$ is Lipschitz strongly pseudo-contractive mapping. Convergence of iterative scheme (4.1) to the fixed point $x^*(w)$ of mapping $S(w,x)$ is obvious from Theorem 2.1 and it is easy to see that $x^*(w)$ is unique fixed point of S iff $x^*(w)$ is solution of random equation $A(w,x) = f$. Stability of iterative scheme (4.1) follows on the same lines as stability of iterative scheme (1.3) in Theorem 2.2. □

From Theorem 4.1, with ease we can prove the following theorem:

Theorem 4.2. *Suppose that $A : \Omega \times X \to X$ be a Lipschitz strongly accretive mapping. Let $x^*(w)$ be a solution of random equation $A(w,x) = f$; where $f \in X$ is any given point and $S(w,x) = f + x(w) - A(w,x)$, $\forall\ x \in X$. Consider the random Mann iterative scheme with errors defined by*

$$x_{n+1}(w) = (1-\alpha_n)y_n(w) + \alpha_n S(w, y_n(w)) + u_n(w), \ \text{for each } w \in \Omega,\ n \geq 0, \tag{4.2}$$

where $\{u_n(w)\}$ is a sequence of measurable mappings from Ω to X, $0 \leq \alpha_n \leq 1$ and $x_0 : \Omega \to X$, an arbitrary measurable mapping, satisfying

(i) $\alpha < \alpha_n$ $(n \geq 0)$;

(ii) $\lim_{n\to\infty} u_n(w) = 0$,

where $L \geq 1$ is Lipschitz constant of $S(w,x)$. Then

(1) *$\{x_n(w)\}$ converges strongly to unique solution $x^*(w)$ of $A(w,x) = f$;*

(2) *It is S-stable to approximate the solution of $A(w,x) = f$; by random iterative scheme with errors* (4.2).

Acknowledgements

The authors thank the editor and the referees for their valuable comments and suggestions.

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