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Smoothness property of traveling wave solutions in a modified Kadomtsev–Petviashvili equation

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Abstract

In this paper, dynamical systems theory is applied to investigate the smooth property of traveling wave solutions for a modified Kadomtsev–Petviashvili equation. The results of our study demonstrate that an abundant of smooth traveling waves arise when their corresponding orbits have intersection points with the singular straight line. In some conditions, exact parametric representations of these smooth waves in explicit or implicit forms are obtained.

Keywords: Bifurcation method, smooth wave solution, singular traveling wave system, mKP equation.
2010 MSC: 34C23, 74J30.

1. Introduction

The one-dimensional motion of solitary waves of inviscid and incompressible fluids has been the subject of research for more than one century [14]. The derivation of the well-known Korteweg-de Vries (KdV) equation

$$u_t + 6\mu u u_x + u_{xxx} = 0, \quad \mu = \pm 1 \tag{1.1}$$

was probably one of the most important results in the study of solitary waves. The KdV equation was originally formulated to model unidirectional propagation of shallow water waves in one spatial dimension [9]. In 1970, Kadomtsev and Petviashvili [8] generalized the KdV equation to two space variables and put forward the famous (2+1)-dimensional Kadomtsev–Petviashvili (KP) equation

$$(u_t + 6\mu u u_x + u_{xxx})_x + 3u_{yy} = 0, \quad \mu = \pm 1. \tag{1.2}$$

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Recently, Wu [15] presented a new (2+1)-dimensional modified Kadomtsev-Petviashvili (mKP) equation

$$u_t = (4eu^2u_{xx} - 2euu_x^2 + 6u^5)_x - 6eu^2u_y + \partial_x^{-1}u_{yy}. \tag{1.3}$$

With the help of the non-linearization of Lax pair and the Riemann-Jacobi inversion technique [2, 3, 6], the algebro-geometric solutions of the mKP equation (1.3) are obtained. Xu [16] studied the bifurcation behavior of traveling wave solutions of (1.3) and obtained some exact traveling wave solutions. In fact, the investigation of the traveling wave solutions of nonlinear partial differential equations plays an important role in the mathematical physics. For example, the wave phenomena observed in fluid dynamics, plasma, and elastic media are often modeled by the bell shaped or kink shaped traveling wave solutions. In this paper, we shall investigate the traveling wave solutions of the mKP equation (1.3) from the view point of dynamical systems (see [1, 5, 13]).

By letting $u(x, y, t) = \phi(x + y - ct) = \phi(\xi)$, integrating Eq. (1.3) with respect to ξ, we have

$$-c\phi = 4e\phi^2\phi'' - 2e\phi\phi'^2 + 6\phi^5 - 2e\phi^3 + \phi, \tag{1.4}$$

where the integral constant is taken as zero. Clearly, Eq. (1.4) is equivalent to the two-dimensional system

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \frac{ey^2 - 3\phi^4 + e\phi^2 - d}{2e\phi}, \tag{1.5}$$

where $d = (c + 1)/2$. System (1.5) has the first integral

$$H(\phi, y) = \frac{\phi^3}{e} - \phi + \frac{y^2}{\phi} - \frac{d}{e\phi} = h. \tag{1.6}$$

We notice that the right-hand side of the second equation of (1.5) is not continuous when $\phi = 0$. In other words, on such straight lines in the phase plane (ϕ, y), the function ϕ'' is not well defined. It implies that the smooth equation (1.3) sometimes has non-smooth traveling wave solutions. This phenomenon has been considered before (see [4, 7, 10, 11, 12]. In this paper, we shall show that the traveling wave solutions will not lose their smoothness in spite of their corresponding orbits have intersection points with the singular straight line $\phi = 0$. To the best of our knowledge, up until now, this phenomenon has not been studied in any other literature.

2. Bifurcations of phase portraits of system (1.5)

Imposing the transformation $d\xi = 2e\phi d\zeta$ for $\phi \neq 0$ on system (1.5) leads to the following regular system

$$\frac{d\phi}{d\zeta} = 2e\phi y, \qquad \frac{dy}{d\zeta} = ey^2 - 3\phi^4 + e\phi^2 - d. \tag{2.1}$$

Apparently, the singular line $\phi = 0$ is an invariant constant solution of (2.1). Near the straight line $\phi = 0$, the variable "ζ" is a fast variable while the variable "ξ" is a slow variable in the sense of the geometric singular perturbation theory.

Now we consider the equilibrium points and their properties of system (2.1). Let

$$f(\phi) = -3\phi^4 + e\phi^2 - d = 0. \tag{2.2}$$

To investigate the equilibrium points of (2.1), we need to find all zeros of the function $f(\phi)$. Let $M(\phi_\epsilon, y_\epsilon)$ be the coefficient matrix of the linearized system of (2.1) at the equilibrium point $(\phi_\epsilon, y_\epsilon)$. Then

$$M(\phi_\epsilon, y_\epsilon) = \begin{pmatrix} 2ey_\epsilon & 2e\phi_\epsilon \\ f'(\phi_\epsilon) & 2ey_\epsilon \end{pmatrix}. \tag{2.3}$$

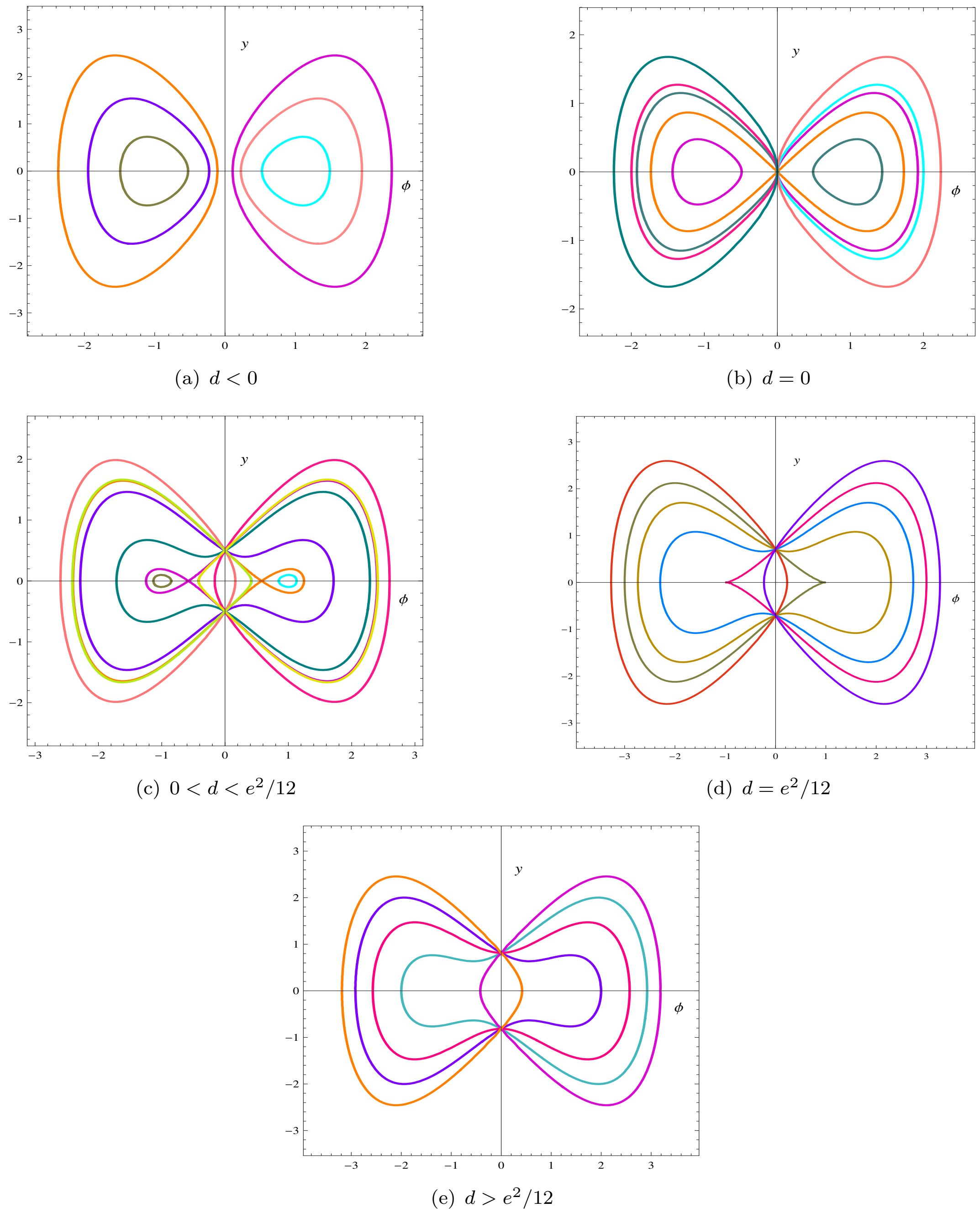
At this equilibrium point, we have

$$J(\phi_\epsilon, y_\epsilon) = \det M(\phi_\epsilon, y_\epsilon) = 4e^2 y_\epsilon^2 - 2e\phi_\epsilon f'(\phi_\epsilon), \tag{2.4}$$

$$T(\phi_\epsilon, y_\epsilon) = \mathrm{trace}(M(\phi_\epsilon, y_\epsilon)) = 4e y_\epsilon. \tag{2.5}$$

By the theory of planar dynamical systems, we know that for an equilibrium point $(\phi_\epsilon, y_\epsilon)$ of a planar integrable system, $(\phi_\epsilon, y_\epsilon)$ is a saddle point if $J < 0$, a node point if $T^2 > 4J > 0$ (stable if $T < 0$, unstable if $T > 0$), a center point if $T = 0 < J$, a degenerate singular point if $J = 0$.

From the above qualitative analysis, we can obtain the following conclusions.



(a) $d < 0$

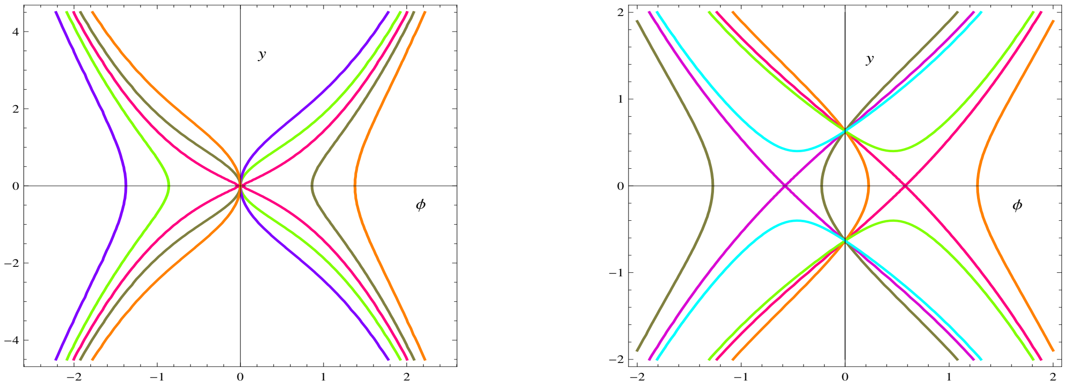
(b) $d = 0$

(c) $0 < d < e^2/12$

(d) $d = e^2/12$

(e) $d > e^2/12$

Figure 1: Bifurcation of phase portraits of system (2.1) for $e > 0$.



(a) $d = 0$ (b) $d < 0$

Figure 2: Bifurcation of phase portraits of system (2.1) for $e < 0$.

Proposition 2.1. *Denote that* $\phi_{1,2} = \pm\sqrt{e+\sqrt{e^2-12d}}\Big/\sqrt{6}$, $\phi_{3,4} = \pm\sqrt{e-\sqrt{e^2-12d}}\Big/\sqrt{6}$ *and* $Y_\pm = \pm\sqrt{d/e}$, *then we have*

(1) *For* $d < 0$ *and* $e > 0$, *system* (2.1) *has two center points* $E_{1,2}(\phi_{1,2}, 0)$ *on the* ϕ*-axis.*
(2) *For* $d = 0$ *and* $e > 0$, *system* (2.1) *has two center points* $E_{1,2}(\phi_{1,2}, 0)$ *on the* ϕ*-axis and a degenerate singular point* $O(0, 0)$.
(3) *For* $0 < d < e^2/12$ *and* $e > 0$, *system* (2.1) *has two saddle points* $E_{1,2}(\phi_{1,2}, 0)$ *and two center points* $E_{3,4}(\phi_{3,4}, 0)$ *on the* ϕ*-axis, and two node points* $N_\pm(0, Y_\pm)$ *on the* y*-axis.*
(4) *For* $d = e^2/12$ *and* $e > 0$, *system* (2.1) *has two cusps* $E_{1,3}(\phi_{1,3}, 0)$ *and* $E_{2,4}(\phi_{2,4}, 0)$ *on the* ϕ*-axis, and two node points* $N_\pm(0, Y_\pm)$ *on the* y*-axis.*
(5) *For* $d > e^2/12$ *and* $e > 0$, *system* (2.1) *has two node points* $N_\pm(0, Y_\pm)$ *on the* y*-axis.*
(6) *For* $d = 0$ *and* $e < 0$, *system* (2.1) *has a degenerate singular point* $O(0, 0)$ *at the origin.*
(7) *For* $d < 0$ *and* $e < 0$, *system* (2.1) *has two saddle points* $E_{1,2}(\phi_{1,2}, 0)$ *on the* ϕ*-axis, and two saddle points* $N_\pm(0, Y_\pm)$ *on the* y*-axis.*

By using the above information, we have the phase portraits of the regular system (2.1) shown in Figure 1 and Figure 2.

3. Existence of smooth traveling wave solutions

We may ask a question: if the equilibrium points $N_\pm(0, Y_\pm)$ in the straight line solution $\phi = 0$ of (2.1) are node points, what is the dynamical behavior of orbits of the vector field defined by (1.5)? We hope to answer this question in this section.

From Proposition 2.1, it is easy to know that for $e > 0$ and $d > 0$, system (2.1) has two node points $N_\pm(0, Y_\pm)$ on the straight line $\phi = 0$. For the sake of simplicity, we shall discuss only the case $e > 0$ and $0 < d < e^2/12$, the other two cases can be discussed in a similar way. Let $h_i = H(\phi_i, 0)$, $i = 1, 2, 3, 4$, where $H(\phi, y)$ is defined by (1.6). We have the different orbits of vector fields defined by systems (2.1) and (1.5) shown in Figure 3.

We notice that the orbits of system (2.1) have the following dynamical behavior:

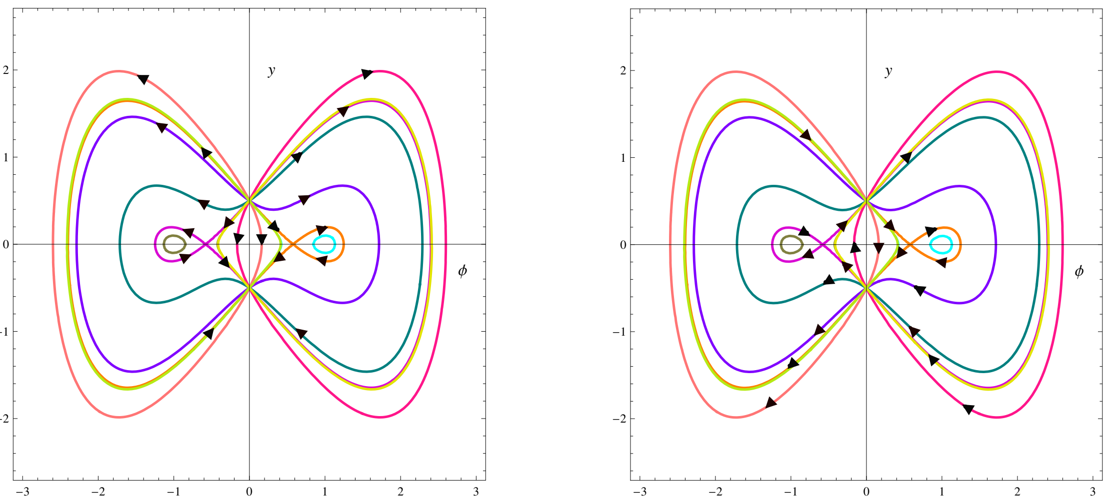
(i) Corresponding to the curves defined by $H(\phi, y) = h$, $h \in (-\infty, h_1)$ [see Figure 4(a)], there are two heteroclinic orbits connecting the equilibrium points $N_\pm(0, Y_\pm)$ on the left and right sides of the straight line $\phi = 0$. That is, the curves defined by $H(\phi, y) = h$, $h \in (-\infty, h_1)$ consist of two orbits of (2.1) and two equilibrium points N_+ and N_-.

(ii) Corresponding to the curves defined by $H(\phi, y) = h,\ h \in (h_1, h_3)$ [see Figure 4(b)], there are two heteroclinic orbits connecting the equilibrium points N_+ and N_-, which lie on the left and right sides of the straight line $\phi = 0$, respectively; besides, there is a periodic orbit surrounding the center point E_1. Namely, the curves defined by $H(\phi, y) = h,\ h \in (h_1, h_3)$ consist of three orbits of (2.1) and two equilibrium points N_+ and N_-.

(iii) Corresponding to the curves defined by $H(\phi, y) = h_3$ [see Figure 4(c)], there is a heteroclinic orbit connecting the two node pints N_+ and N_- on the left side of the straight line $\phi = 0$; there are two heteroclinic orbits connecting the equilibrium points N_+, N_- and E_3 on the right side of the straight line $\phi = 0$. There is a homoclinic orbit to the saddle point E_3. The curves defined by $H(\phi, y) = h_3$ consist of four orbits of (2.1) and three equilibrium points N_+, N_- and E_3.

(iv) Corresponding to the curves defined by $H(\phi, y) = h,\ h \in (h_3, 0)$ [see Figure 4(d)], there are two heteroclinic orbits connecting the equilibrium points N_+ and N_- on the left and right sides of the straight line $\phi = 0$. The curves defined by $H(\phi, y) = h,\ h \in (h_3, 0)$ consist of two orbits of (2.1) and two equilibrium points N_+ and N_-.

By theory of planar dynamical systems, it is known that for system (2.1), when ζ is varied along the orbit defined by $H(\phi, y) = h$, a point $(\phi(\zeta), y = \phi'(\zeta))$ in this orbit tends to the equilibrium point N_+ or N_-, only if $\zeta \to +\infty$ or $-\infty$. Differing from system (2.1), for system (1.5), on the left side of the straight line $\phi = 0$, the direction of orbits of the vector field defined by (1.5) is just the inverse direction of the orbits of the vector field defined by (2.1) (see Figure 3 and Figure 4). In addition, we have,



(a) (b)

Figure 3: The different phase portraits of systems (2.1) and (1.5) for $e > 0,\ 0 < d < e^2/12$. (a) The orbits of the vector field defined by (2.1). (b) The orbits of the vector field defined by (1.5).

Lemma 3.1. *Let $(\phi(\xi), y{=}\phi'(\xi))$ be the parametric representation of an orbit γ of system* (1.5) *connecting the two points $N_\pm(0, Y_\pm)$ on the singular straight line $\phi = 0$. Then, there is a finite value $\xi = \tilde{\xi}$ such that* $\lim_{\xi \to \tilde{\xi}} \phi(\xi) = 0$.

Proof Denote $(\phi_0(\xi_0), y_0(\xi_0))$ as an initial point on the orbit γ. As ξ increases from ξ_0 to ξ, it follows from (1.6) and the first equation of (1.5) that

$$\xi - \xi_0 = \int_{\phi_0}^{\phi} \frac{d\phi}{y} = \int_{\phi_0}^{\phi} \frac{1}{\sqrt{-\phi^4/e + \phi^2 + h\phi + d/e}} d\phi. \tag{3.1}$$

Notice that

$$\lim_{\phi \to 0} \frac{1}{\sqrt{-\phi^4/e + \phi^2 + h\phi + d/e}} = \frac{1}{Y_+} \neq 0. \tag{3.2}$$

So we have

$$\lim_{\phi\to 0}\int_{\phi_0}^{\phi}\frac{1}{\sqrt{-\phi^4/e+\phi^2+h\phi+d/e}}d\phi = A_s = \text{constant}. \tag{3.3}$$

Thus, there is $\widetilde{\xi}=\xi_0+A_s$ such that $\lim_{\xi\to\widetilde{\xi}}\phi(\xi)=0$.

Lemma 3.1 and the vector field defined by (1.5) imply the following conclusion.

Theorem 3.2. *(i) The two curves shown in Figure* 4(e) *(or* 4(f) *or* 4(h)*), which connect N_+ and N_- on the left and right sides of the singular straight line $\phi=0$, is a periodic orbit of* (1.5)*. It gives rise to a smooth periodic wave solution of the mKP equation* (1.3)*.*
(ii) The four curves shown in Figure 4(g)*, which connect three points N_+, N_- and E_3 on the left and right sides of the straight line $\phi=0$, are two homoclinic orbits of system* (1.5) *to the saddle point $E_3(\phi_3,0)$. It gives rise to two smooth solitary wave solutions of the mKP equation* (1.3)*.*

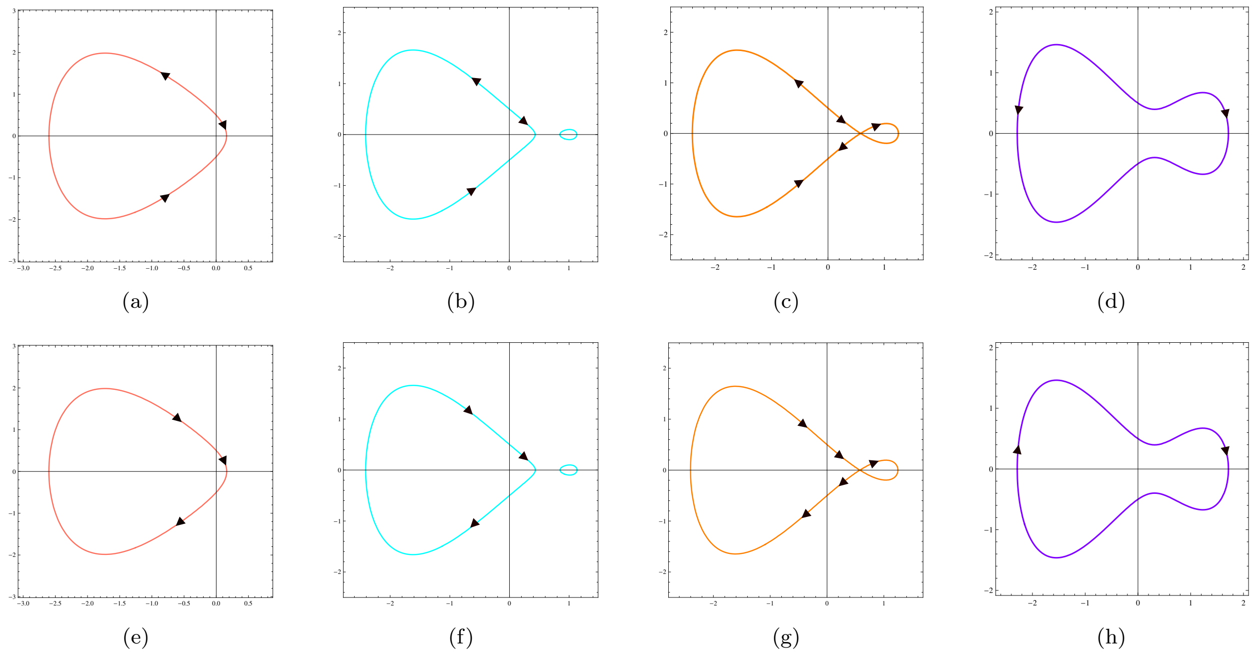


Figure 4: The vector fields defined by systems (2.1) and (1.5) for $e>0$ and $0<d<e^2/12$. (a) The level curves of (2.1) given by $H(\phi,y)=h,\ h\in(-\infty,\ h_1)$. (b) The level curves of (2.1) given by $H(\phi,y)=h,\ h\in(h_1,h_3)$. (c) The level curves of (2.1) given by $H(\phi,y)=h,\ h=h_3$. (d) The level curves of (2.1) given by $H(\phi,y)=h,\ h\in(h_3,\ 0)$. (e) The level curves of (1.5) given by $H(\phi,y)=h,\ h\in(-\infty,\ h_1)$. (f) The level curves of (1.5) given by $H(\phi,y)=h,\ h\in(h_1,h_3)$. (g) The level curves of (1.5) given by $H(\phi,y)=h,\ h=h_3$. (h) The level curves of (1.5) given by $H(\phi,y)=h,\ h\in(h_3,\ 0)$.

4. Exact smooth periodic wave and solitary wave solutions

In this section, by using the results of Figure 4 and the theory given in Section 3, we shall give the exact representations of smooth traveling wave solutions of Eq. (1.3). Here we denote $\mathrm{sn}(\cdot,k)$ and $\mathrm{cn}(\cdot,k)$ as the Jacobian elliptic functions with the modulus k.

4.1. Smooth periodic wave solutions

From Figure 4(e), we see that there is a periodic orbit passing the points $(\gamma_1,0)$ and $(\gamma_2,0)$. Its expression is

$$y=\pm\sqrt{(\gamma_1-\phi)(\phi-\gamma_2)[(\phi-\beta_1)^2+\alpha_1^2]/e}, \quad \gamma_2\le\phi\le\gamma_1, \tag{4.1}$$

where $(\gamma_1-\phi)(\phi-\gamma_2)[(\phi-\beta_1)^2+\alpha_1^2]=-\phi^4+e\phi^2+he\phi+d$. Substituting (4.1) into $\frac{d\phi}{d\xi}=y$, integrating along the periodic orbit and completing the integral, we get the smooth periodic wave solution of Eq. (1.3) as follows:

$$\phi(\xi)=\frac{\gamma_1 B+r_2 A-(\gamma_1 B-\gamma_2 A)\mathrm{cn}(\omega\xi,k)}{A+B+(A-B)\mathrm{cn}(\omega\xi,k)}, \tag{4.2}$$

where $A=\sqrt{(\gamma_1-\beta_1)^2+\alpha_1^2}$, $B=\sqrt{(\gamma_2-\beta_1)^2+\alpha_1^2}$, $\omega=\sqrt{\frac{AB}{e}}$ and $k=\sqrt{\frac{(\gamma_1-\gamma_2)^2-(A-B)^2}{4AB}}$. The wave profile of (4.2) is shown in Figure 5(a).

From Figure 4(f), we see that there are two periodic orbits passing the points $(\gamma_i,0)$, $i=1,2,3,4$. Their expressions are

$$y=\pm\sqrt{(\gamma_1-\phi)(\phi-\gamma_2)(\phi-\gamma_3)(\phi-\gamma_4)/e},\quad \gamma_2\le\phi\le\gamma_1 \text{ (or } \gamma_4\le\phi\le\gamma_3), \tag{4.3}$$

where γ_1, γ_2, γ_3, γ_4 $(\gamma_4<\gamma_3<\gamma_2<\gamma_1)$ are four real roots of the equation $\phi^4-e\phi^2-he\phi-d=0$. Substituting (4.3) into the first equation of (1.5) and integrating them, respectively, we get the smooth periodic wave solutions of Eq. (1.3) as follows:

$$\phi(\xi)=\frac{\gamma_1(\gamma_2-\gamma_4)+\gamma_4(\gamma_1-\gamma_2)\mathrm{sn}^2(\omega\xi,k)}{(\gamma_2-\gamma_4)+(\gamma_1-\gamma_2)\mathrm{sn}^2(\omega\xi,k)} \tag{4.4}$$

and

$$\phi(\xi)=\frac{\gamma_4(\gamma_1-\gamma_3)+\gamma_1(\gamma_3-\gamma_4)\mathrm{sn}^2(\omega\xi,k)}{(\gamma_1-\gamma_3)+(\gamma_3-\gamma_4)\mathrm{sn}^2(\omega\xi,k)}, \tag{4.5}$$

where $\omega=\sqrt{(\gamma_1-\gamma_3)(\gamma_2-\gamma_4)/(4e)}$ and $k=\sqrt{\frac{(\gamma_1-\gamma_2)(\gamma_3-\gamma_4)}{(\gamma_1-\gamma_3)(\gamma_2-\gamma_4)}}$. The wave profiles of (4.4) and (4.5) are shown in Figure 5(b) and Figure 5(c), respectively.

Corresponding to Figure 4(h), Eq. (1.3) has a smooth periodic wave solution which has the same representation as (4.2). The wave profile is shown in Figure 5(d).

4.2. Solitary wave solutions

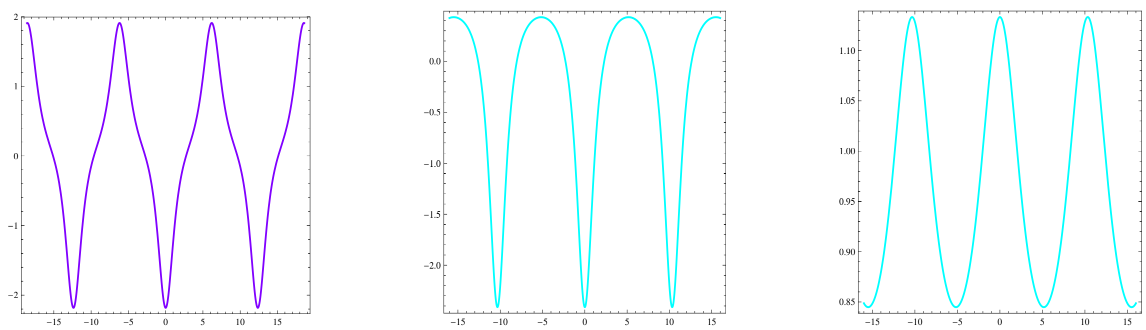
From Figure 4(g), it is seen that there are two homoclinic orbits connecting with the saddle point $(\phi_1,0)$ and passing points $(\gamma_1,0)$, $(\gamma_3,0)$. There expressions are

$$y=\pm(\phi-\gamma_2)\sqrt{(\gamma_1-\phi)(\phi-\gamma_3)/e},\quad \gamma_2\le\phi\le\gamma_1 \text{ (or } \gamma_3\le\phi\le\gamma_2), \tag{4.6}$$

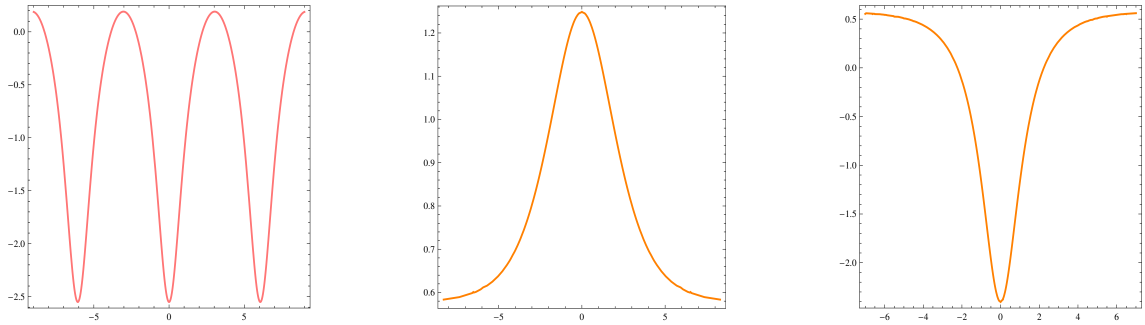
where $\gamma_3<\gamma_2=\phi_1<\gamma_1$ are three real roots of the equation $\phi^4-e\phi^2-h_3e\phi-d=0$. Substituting (4.6) into $\frac{d\phi}{d\xi}=y$, integrating it along the homoclinic orbits and completing the integral, we have

$$\pm\,\xi=\sqrt{\frac{e}{\alpha}}\ln\left|\frac{2\alpha+\beta(\phi-\gamma_2)-2\sqrt{\alpha(\gamma_1-\phi)(\phi-\gamma_3)}}{(\gamma_1-\gamma_3)(\phi-\gamma_2)}\right|, \tag{4.7}$$

where $\alpha=(\gamma_1-\gamma_2)(\gamma_2-\gamma_3)$ and $\beta=\gamma_1+\gamma_3-2\gamma_2$. The wave profiles of (4.7) are shown in Figure 5(e) and Figure 5(f).



(a) (b) (c)



(d) (e) (f)

Figure 5: The wave profiles of the mKP equation (1.3). (a) Smooth periodic wave corresponding to the curves defined by $H(\phi, y) = h,\ h \in (h_3, 0)$. (b) Smooth periodic wave corresponding to the curves defined by $H(\phi, y) = h,\ h \in (h_1, h_3)$. (c) Smooth periodic wave corresponding to the curves defined by $H(\phi, y) = h,\ h \in (h_1, h_3)$. (d) Smooth periodic wave corresponding to the curves defined by $H(\phi, y) = h,\ h \in (-\infty, h_1)$. (e) Solitary wave corresponding to the curves defined by $H(\phi, y) = h,\ h = h_3$. (f) Solitary wave corresponding to the curves defined by $H(\phi, y) = h,\ h = h_3$.

5. Conclusion

In this paper, we investigate a modified Kadomtsev–Petriashvili equation with a singular straight line. By phase space analytical technique, we find that the singular traveling wave system (1.5) and its associated regular system (2.1) define different vector fields. Different from (2.1), the orbits of system (1.5) intersecting with the singular straight line $\phi = 0$ and having the same level set given by (1.6) correspond to smooth periodic wave solutions or solitary wave solutions of Eq. (1.3). Further, we obtain some parametric representations for periodic wave solutions and solitary wave solutions of Eq. (1.3).

6. Acknowledgement

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