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Common fixed point results for multi-valued mappings with some examples

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Abstract

In this paper, we define the concepts of the (CLR)-property and the (owc)-property for two single-valued mappings and two multi-valued mappings in metric spaces and give some new common fixed point results for these mappings. Also, we give some examples to illustrate the main results in this paper. Our main results extend and improve some results given by some authors.

Keywords: Weakly compatible mappings, fixed point, coincidence point, the (CLR)-property, the (owc)-property, the (CLR_f)-property.
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1. Introduction

In 1922, Banach [4] proved a theorem, which is well-known as Banach's fixed point theorem or Banach's contractive principle, that is, Every contractive mapping T from a complete metric space (X, d) into itself has a unique fixed point z of the mapping T $(Tz = z)$. Further, the sequence $\{x_n\}$ in X defined by $x_{n+1} = Tx_n$ for each $n \geq 1$ converges to the fixed point z.

Since Banach's fixed point theorem, many authors have improved, extended and generalized this theorem in many different ways. Especially, in 1969, Nadler [27] introduced the notion of a multi-valued (set-valued) contractive mapping in a complete metric space and also proved Banach's fixed point theorem for a multi-valued mapping in a complete metric space. Also, since Nadler's fixed point theorem, many authors have studied Banach's fixed point theorem for multi-valued mappings and hybrid contractive mappings (mixed contractions with single-valued and multi-valued mappings) in several ways ([5], [13], [14], [17], [26], [28], [35], [36] and references therein).

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Further, in 1976, Jungck [18] introduced the concept of commuting mappings in metric spaces and generalized Banach's fixed point theorem by using commuting mappings in metric spaces, that is, two mappings f and g from a metric space (X, d) into itself are said to be *commuting* if $fgx = gfx$ for all $x \in X$. The concept of commuting mappings has proven very useful for generalizing in the context of metric fixed point theory (see [6], [7], [11], [12], [15], [16], [18], [19], [25], [32], [37]).

In 1982, Sessa [33] first proved common fixed point theorems for weakly commuting mappings in metric spaces and, in 1986, Jungck [20] introduced the concept of compatible mappings in order to generalize the concept of weakly commuting mappings by Sessa [33] and showed that weakly commuting mappings are compatible, but the converse is not true (for more details on compatible mappings, see [9], [10], [21], [22], [29], [30]). Again, in 1996, Jungck [23] introduced the concept of weakly compatible mappings in metric spaces and proved some common fixed point theorems for weakly compatible mappings (for more results, see [2]).

Afterward, Aamri and Moutawakil [1] introduced the notion of the $(E.A)$-property, which is a special case of the tangential property due to Sastry and Murthy [32]. In 2011, Sintunaravat and Kumam [34] proved that the notion of the $(E.A)$-property always requires the completeness (or closedness) of underlying subspaces to show the existence of common fixed points of single-valued mappings and hence they coined the idea of the *common limit in the range* (shortly, the (CLR)-property), which relaxes the requirement of completeness (or closedness) of the underlying subspace. Also, they proved common fixed point results for single-valued mappings by using this concept in fuzzy metric spaces. For more details on the (CLR)-property, refer to [8], [31], [37] and therein. In 2009, Aliouche and Popa [3] proved some common fixed point theorems for occasionally weakly compatible hybrid mappings in symmetric spaces and gave some applications.

Motivated by the results mentioned above, in this paper, we introduced the notion of occasionally weakly compatible mappings (shortly, the (owc)-property) and the common limit in the range (shortly, the (CLR)-property) for four single-valued and multi-valued mappings in metric spaces and prove some coincidence point and common fixed point theorems for the hybrid contractive mappings with the (owc)-property) and the (CLR)-property. Also, we give some examples to illustrate the main results in this paper. In fact, our main results improve, extend and generalize the corresponding results given by some authors.

2. Preliminaries

Throughout this paper, let (X, d) be a metric space and $B(X)$ be the family of all nonempty bounded subsets of X. We define the functions $\delta(A, B)$ and $D(A, B)$ by

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$$

and

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$$

for all $A, B \in B(X)$. If A consists of a single point a, then we write $\delta(A, B) = \delta(a, B)$. If $A = \{a\}$ and $B = \{b\}$, then we write $\delta(A, B) = d(a, b)$. It follows immediately from the definition of δ that

$$\begin{aligned}
&\delta(A, B) = \delta(B, A) \geq 0,\\
&\delta(A, B) \leq \delta(A, C) + \delta(C, B),\\
&\delta(A, B) = 0 \iff A = B = \{a\},\\
&\delta(A, A) = diam\, A
\end{aligned}$$

for all $A, B, C \in B(X)$. Let $CB(X)$ denote the class of all nonempty bounded closed subsets of X and H be the Hausdorff metric with respect to d, that is,

$$H(A, B) = \left\{ \sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A) \right\}$$

for all $A, B \in CB(X)$, where

$$d(x, A) := \inf\{d(x, y) : y \in A\}.$$

Forward, we denote by $Fix(T)$ the set of all fixed points of a multi-valued mapping T, that is,

$$Fix(T) = \{x \in X : x \in Tx\}.$$

Definition 2.1 ([19])**.** Let (X, d) be a metric space. Two mappings $f, g : X \to X$ are said to be *compatible* (or *asymptotically commuting*) if

$$\lim_{n\to\infty} d(gfx_n, fgx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in X$.

In 1989, Kaneko and Sessa [24] introduced the notion of compatible mappings with single-valued and multi-valued mappings as follows:

Definition 2.2 ([24])**.** Let (X, d) be a metric space. Two mappings $f : X \to X$ and $S : X \to CB(X)$ are said to be *compatible* if $fSx \in CB(X)$ for all $x \in X$ and

$$\lim_{n\to\infty} H(Sfx_n, fSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = A$$

for some $A \in CB(X)$ and

$$\lim_{n\to\infty} fx_n = t \in A$$

for some $t \in X$.

Definition 2.3 ([20])**.** Let (X, d) be a metric space. A single-valued mapping $f : X \to X$ and a multi-valued mapping $S : X \to CB(X)$ are said to be *weakly compatible* if they commute at their coincidence points, i.e., if $fSx = Sfx$ whenever $fx \in Sx$.

It is easy to see that compatible mappings are weakly compatible, but the converse is not true.

Definition 2.4 ([34])**.** Let (X, d) be a metric space. Two mappings $f, g : X \to X$ are said to satisfy the *common limit in the range of* f with respect to g (shortly, the (CLR_f)-property with respect to g) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = fu$$

for some $u \in X$.

Example 2.5. Let $X = [1, \infty)$ with usual metric. Define two single-valued mappings $f, g : X \to X$ by

$$fx = \frac{x}{2}, \quad gx = 2x$$

for all $x \in X$, respectively. Consider the sequence $\{x_n\}$ in X defined by $x_n = \frac{1}{n}$ for each $n \geq 1$. Then we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = f(0).$$

Therefore, the mappings f and g satisfy the (CLR_f)-property with respect to g.

The following is the definition of (CLR_f)–property for a hybrid pairs of single-valued and multi-valued mappings in metric spaces.

Definition 2.6. Let (X, d) be a metric space. A single-valued mapping $f : X \to X$ and a multi-valued mapping $S : X \to CB(X)$ are said to satisfy the *common limit in the range of* f with respect to S (shortly, the (CLR_f)-property with respect to S) if there exists a sequence $\{x_n\}$ in X and $A \in CB(X)$ such that

$$\lim_{n\to\infty} fx_n = f(u) \in A = \lim_{n\to\infty} Sx_n$$

for some $u \in X$.

Now, we give an example for two mappings with the (CLR_f)-property with respect to S.

Example 2.7. Let $X = [1, \infty)$ with the usual metric. Define two mappings $f : X \to X$ and $S : X \to CB(X)$ by

$$fx = x + 2, \quad Sx = [1, x + 2]$$

for all $x \in X$, respectively. Consider the sequence $\{x_n\}$ in X defined by $x_n = \frac{1}{n}$ for each $n \geq 1$. Clearly, we have

$$\lim_{n\to\infty} fx_n = 2 = f0 \in [1, 2] = \lim_{n\to\infty} Sx_n.$$

Therefore, two mappings f and S satisfy the (CLR_f)-property with respect to S.

Definition 2.8. Let (X, d) be a metric space. Two mappings $f, g : X \to X$ are said to be *occasionally weakly compatible* (shortly, (owc)-property) if there exists a point $u \in X$ such that $fu = gu$ and $fgu = gfu$.

Definition 2.9 ([2])**.** A single-valued mapping $f : X \to X$ and a multi-valued mapping $S : X \to CB(X)$ is said to be *occasionally weakly compatible* (shortly, (owc)-property) if $fSx \subset Sfx$ for some $x \in X$ with $fx \in Sx$.

Definition 2.10. Let $f, g : X \to X$ be single-valued mappings and $S, T : X \to 2^X$ be multi-valued mappings.

(1) A point $x \in X$ is said to be a *coincidence point* of f and S if $fx \in Sx$. We denote by $C(f, S)$ the set of all coincidence points of f and S;
(2) A point $x \in X$ is said to be a *common fixed point* of f, g, S and T if $x = fx = gx \in Sx$ and $x = fx = gx \in Tx$.

3. Common fixed points for mappings with the (owc)-property

Now, we prove the main result in this section.

Theorem 3.1. *Let (X, d) be a metric space. Let $f, g : X \to X$ be single-valued mappings and $S, T : X \to B(X)$ be multi-valued mappings satisfying the following conditions:*
(1) *the pairs (S, f) and (T, g) are the (owc)-property;*
(2) *for all $x, y \in X$,*

$$\delta^p(Sx, Ty) \leq \varphi\Big(\max\Big\{d^p(fx, gy), \frac{d^p(fx, Sx)d^p(gy, Ty)}{1 + d^p(fx, gy)}, \frac{d^p(fx, Ty)d^p(gy, Sx)}{1 + d^p(fx, gy)}\Big\}\Big),$$

where $p \geq 1$ and $\varphi : [0, \infty) \to [0, \infty)$ is a function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$. Then f, g, S and T have a unique common fixed point in X.

Proof. Since the pairs (S, f) and (T, g) satisfy the (owc)-property, there exist $u, v \in X$ such that

$$fu \in Su, \quad fSu \subset Sfu, \quad gv \in Tv, \quad gTv \subset Tgv,$$

which implies that $ffu \in Sfu$ and $ggv \in Tgv$.

Now, we prove that $fu = gv$. In fact, if $fu \neq gv$, then, using the condition (2), we have

$$\delta^p(Su,Tv) \le \varphi\Big(\max\Big\{d^p(fu,gv), \frac{d^p(fu,Su)d^p(gv,Tv)}{1+d^p(fu,gv)}, \frac{d^p(fu,Tv)d^p(gv,Su)}{1+d^p(fu,gv)}\Big\}\Big)$$
$$= \varphi\Big(\max\Big\{d^p(fu,gv), \frac{d^p(fu,Tv)d^p(gv,Su)}{1+d^p(fu,gv)}\Big\}\Big).$$

Since $fu \in Su$ and $gv \in Tv$, we have

$$\frac{d^p(fu,Tv)d^p(gv,Su)}{1+d^p(fu,gv)} \le \frac{d^p(fu,gv)d^p(gv,fu)}{1+d^p(fu,gv)} < d^p(fu,gv)$$

and hence

$$\delta^p(Su,Tv) \le \varphi(d^p(fu,gv)).$$

Thus it follows from the property of φ that

$$d^p(fu,gv) \le \delta^p(Su,Tv) \le \varphi(d^p(fu,gv)) < d^p(fu,gv),$$

which is a contradiction and so $fu = gv$.

Next, we prove that fu is a fixed point of f. Suppose that $ffu \neq fu$. Then, by using the condition (2), we have

$$d^p(ffu,fu) = d^p(ffu,gv) \le \delta^p(Sfu,Tv)$$
$$\le \varphi\Big(\max\Big\{d^p(ffu,gv), \frac{d^p(ffu,Sfu)d^p(gv,Tv)}{1+d^p(fSu,gv)}, \frac{d^p(ffu,Tv)d^p(gv,Sfu)}{1+d^p(fSu,gv)}\Big\}\Big).$$

Since $ffu \in Sfu$ and $gv \in Tv$, we have

$$\frac{d^p(ffu,Tv)d^p(gv,Sfu)}{1+d^p(fSu,gv)} \le d^p(ffu,Tv) < d^p(ffu,gv)$$

and hence

$$\delta^p(Sfu,Tv) \le \varphi(d^p(ffu,gv)).$$

Thus it follows from the property of φ that

$$d^p(ffu,fu) = d^p(ffu,gv) \le \delta^p(Sfu,Tv) \le \varphi(d^p(ffu,gv)) < d^p(ffu,gv) = d^p(ffu,fu),$$

which is a contradiction and so $ffu = fu$. Similarly, we can prove $fu = gfu = ffu$. Thus we have

$$fu = ffu \in Sfu$$

and

$$fu = gfu = ggv \in Tgv = Tfu.$$

Therefore, fu is a common fixed point of f, g, S and T. Moreover, by the condition (2), we have

$$\delta^p(Sfu,Tfu) \le \varphi\Big(\max\Big\{d^p(ffu,gfu), \frac{d^p(ffu,Sfu)d^p(gfu,Tfu)}{1+d^p(ffu,gfu)}, \frac{d^p(ffu,Tfu)d^p(gfu,Sfu)}{1+d^p(ffu,gfu)}\Big\}\Big)$$
$$= \varphi(\max\{0,0,0\}) = 0.$$

Therefore $Sfu = Tfu = \{fu\}$.

Next, assume that $w \neq z$ is another common fixed point of f, g, S and T. From the condition (2), we have

$$d^p(z,w) = \delta^p(Sz,Tw) \le \varphi\Big(\max\Big\{d^p(fz,gw), \frac{d^p(fz,Sz)d^p(gw,Tw)}{1+d^p(fz,gw)}, \frac{d^p(fz,Tw)d^p(gw,Sz)}{1+d^p(fz,gw)}\Big\}\Big)$$
$$= \varphi\Big(\max\Big\{d^p(z,w), 0, \frac{d^p(z,w)d^p(w,z)}{1+d^p(z,w)}\Big\}\Big)$$
$$= \varphi(d^p(z,w)) < d^p(z,w),$$

which is a contradiction. Thus the common fixed point z is unique. This completes the proof. □

If $p = 1$ in Theorem 3.1, then we have the following:

Corollary 3.2. *Let (X, d) be a metric space. Let $f, g : X \to X$ be single-valued mappings and $S, T : X \to B(X)$ be multi-valued mappings satisfying the following conditions:*
(1) *the pairs (S, f) and (T, g) satisfy the (owc)-property;*
(2) *for all $x, y \in X$,*

$$\delta(Sx, Ty) \le \varphi\Big(\max\Big\{d(fx, gy), \frac{d(fx, Sx)d(gy, Ty)}{1 + d(fx, gy)}, \frac{d(fx, Ty)d(gy, Sx)}{1 + d(fx, gy)}\Big\}\Big),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.
Then f, g, S and T have a unique common fixed point in X.

If we take $S = T$ and $f = g$ in Theorem 3.1, then we have the following:

Corollary 3.3. *Let (X, d) be a metric space. Let $f : X \to X$ be a single-valued mapping and $S : X \to B(X)$ be a multi-valued mapping satisfying the following conditions:*
(1) *the pair (S, f) satisfies the (owc)-property;*
(2) *for all $x, y \in X$,*

$$\delta^p(Sx, Sy) \le \varphi\Big(\max\Big\{d^p(fx, fy), \frac{d^p(fx, Sx)d^p(fy, Sy)}{1 + d^p(fx, fy)}, \frac{d^p(fx, Sy)d^p(fy, Sx)}{1 + d^p(fx, fy)}\Big\}\Big),$$

where $p \ge 1$ and $\varphi : [0, \infty) \to [0, \infty)$ is a function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.
Then f and S have a unique common fixed point in X.

If S is a single-valued mapping in Corollary 3.3, then we have the following:

Corollary 3.4. *Let (X, d) be a metric space and $f, S : X \to X$ be two single-valued mappings satisfying the following conditions:*
(1) *the pair (S, f) satisfies the (owc)-property;*
(2) *for all $x, y \in X$,*

$$d^p(Sx, Sy) \le \varphi\Big(\max\Big\{d^p(fx, fy), \frac{d^p(fx, Sx)d^p(fy, Sy)}{1 + d^p(fx, fy)}, \frac{d^p(fx, Sy)d^p(fy, Sx)}{1 + d^p(fx, fy)}\Big\}\Big),$$

where $p \ge 1$ and $\varphi : [0, \infty) \to [0, \infty)$ is a function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.
Then f and S have a unique common fixed point in X.

Example 3.5. Let $X = [0, \infty)$ be the set of real numbers with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define two single-valued mappings $S, f : X \to X$ by

$$Sx = \begin{cases} 4, & 0 \le x < 1, \\ x^4, & 1 \le x < \infty \end{cases}$$

and

$$fx = \begin{cases} 3, & 0 \le x < 1, \\ 1 - \frac{1}{x^4}x, & 1 \le x < \infty. \end{cases}$$

Then $f(1) = S(1) = 1$ and $fS(1) = 1 = Sf(1)$ and so the pair (S, f) satisfies the (owc)-property. Also, for some $k \in [0, 1)$, if we define a function $\varphi(t) = kt$ for all $t \in [0, \infty)$, then all the conditions in Corollary 3.4 are satisfied and, further, the point 1 is a unique common fixed point of S and f.

If both S and T are single-valued mappings in Theorem 3.1, then we have the following:

Corollary 3.6. *Let (X, d) be a metric space and let $f, g, S, T : X \to X$, be four mappings satisfying the following conditions:*

(1) *the pairs (S, f) and (T, g) satisfy the (owc)-property;*

(2) *for all $x, y \in X$,*

$$d^p(Sx, Ty) \le \varphi\Big(\max\Big\{d^p(fx, gy), \frac{d^p(fx, Sx)d^p(gy, Ty)}{1 + d^p(fx, gy)}, \frac{d^p(fx, Ty)d^p(gy, Sx)}{1 + d^p(fx, gy)}\Big\}\Big),$$

where $p \ge 1$ and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous monotone increasing function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.

Then f, g, S and T have a unique common fixed point in X.

If we take $\varphi = kt$ for some $[0, 1)$ in Corollary 3.6, then we have the following:

Corollary 3.7. *Let (X, d) be a metric space and let $f, g, S, T : X \to X$ be four single-valued mappings satisfying the following conditions:*

(1) *the pairs (S, f) and (T, g) satisfy the (owc)-property;*

(2) *for all $x, y \in X$,*

$$d^p(Sx, Ty) \le k \max\Big\{d^p(fx, gy), \frac{d^p(fx, Sx)d^p(gy, Ty)}{1 + d^p(fx, gy)}, \frac{d^p(fx, Ty)d^p(gy, Sx)}{1 + d^p(fx, gy)}\Big\},$$

for all $x, y \in X$.

Then f, g, S and T have a unique common fixed point in X.

Now, we give an example to illustrate Theorem 3.1.

Example 3.8. Let $X = [0, 10]$ be endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$ and

$$Sx = \begin{cases} \{0\} & \text{if } x \in [0, \frac{1}{2}], \\ [\frac{1}{32}, \frac{1}{16}] & \text{if } x \in (\frac{1}{2}, 10], \end{cases} \qquad fx = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}), \\ 10, & \text{if } x \in (\frac{1}{2}, 10], \end{cases}$$

$$Tx = \begin{cases} \{0\} & \text{if } x \in [0, \frac{1}{2}), \\ [\frac{1}{32} \frac{1}{18}] & \text{if } x \in (\frac{1}{2}, 10], \end{cases} \qquad gx = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}), \\ 5 & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Then the pairs (S, f) and (T, g) satisfy the (owc)-property because

$$f(0) \in S(0), \quad fS(0) \subseteq Sf(0), \quad g(0) \in T(0), \quad gT(0) \subseteq Tg(0).$$

Now, we verify that the mappings f, g, S, T satisfy the condition (2) of Theorem 3.1 with $\varphi(t) = \frac{1}{2}t$. We have the following cases:

(1) If $x, y \in (0, 2]$, it is obvious.

(2) If $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 10]$, we obtain

$$\begin{aligned} \delta(Sx, Ty) &= \frac{1}{32} < \frac{5}{2} \le \frac{1}{2}d(fx, gy) \\ &\le \frac{1}{2}\max\Big\{d(fx, gy), \frac{d(fx, Sx)d(gy, Ty)}{1 + d(fx, gy)}, \frac{d^p(fx, Ty)d^p(gy, Sx)}{1 + d^p(fx, gy)}\Big\}. \end{aligned}$$

(3) If $x, y \in [\frac{1}{2}, 10]$, we obtain

$$\begin{aligned} \delta(Sx, Ty) &= \frac{1}{16} - \frac{1}{32} = \frac{1}{32} \le \frac{8}{2} \\ &\le \frac{1}{2}\frac{d(fx, Sx)d(gy, Ty)}{1 + d(fx, gy)} \\ &\le \frac{1}{2}\max\Big\{d(fx, gy), \frac{d(fx, Sx)d(gy, Ty)}{1 + d(fx, gy)}, \frac{d^p(fx, Ty)d^p(gy, Sx)}{1 + d^p(fx, gy)}\Big\}. \end{aligned}$$

Therefore, all the conditions of Theorem 3.1 are satisfied and, further, 0 is the unique common fixed point of the mappings f, g, S and T.

4. Common Fixed Points for Mappings with the (CLR_f)-Property

In this section, we introduce the notion of (CLR_f) property for four mappings and prove some common fixed point theorems for the mappings with the (CLR_f)-property in metric spaces.

Definition 4.1. Let (X, d) be a metric space. Two single-valued mappings $f, g : X \to X$ and two multi-valued mappings $S, T : X \to CB(X)$ are said to satisfy the *common limit in the range of* f (shortly, the (CLR_f)-property) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X and $A, B \in CB(X)$ such that

$$\lim_{n\to\infty} Sx_n = A, \quad \lim_{n\to\infty} Ty_n = B$$

and

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = fu \in A \cap B$$

for some $u \in X$.

Example 4.2. Let $X = [1, \infty)$ with usual metric. Define two single-valued mappings $f, g : X \to X$ and two multi-valued mappings $S, T : X \to CB(X)$ by

$$fx = 2 + \frac{x}{3}, \quad gx = 2 + \frac{x}{2}$$

and

$$Sx = [1, 2 + x], \quad Tx = [3, 3 + \frac{x}{2}]$$

for all $x \in X$, respectively. Then the mappings f and T satisfy the (CLR_f)-property for the sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_n = 3 + \frac{1}{n}$ and $y_n = 2 + \frac{1}{n}$ for each $n \geq 1$, respectively. Indeed, we have

$$\lim_{n\to\infty} Sx_n = [1, 5] = A, \quad \lim_{n\to\infty} Ty_n = [3, 4] = B$$

and

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = 3 = f(3) \in A \cap B.$$

Therefore, the pairs (S, f) and (T, g) satisfy the (CLR_f)-property.

Theorem 4.3. *Let (X, d) be a metric space. Let $f, g : X \to X$ be two single-valued mappings and $S, T : X \to CB(X)$ be two multi-valued mappings satisfying the following conditions:*

(a) *the pairs (S, f) and (T, g) satisfy the (CLR_f)-property;*

(b) *for all $x, y \in X$,*

$$H^p(Sx, Ty) \leq \varphi\Big(\max\Big\{d^p(fx, gy), \frac{d^p(fx, Sx)d^p(gy, Ty)}{1 + d^p(fx, gy)}, \frac{d^p(fx, Ty)d^p(gy, Sx)}{1 + d^p(fx, gy)}\Big\}\Big),$$

where $p \geq 1$ and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous monotone increasing function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.

If $f(X)$ and $g(X)$ are closed subsets of X, then we have the following:

(1) *f and S have a coincidence point;*

(2) *g and T have a coincidence point;*

(3) *f and S have a common fixed point provided that f and S are weakly compatible at v and $ffv = fv$ for any $v \in C(f, S)$;*

(4) *g and T have a common fixed point provided that g and T are weakly compatible at v and $ggv = gv$ for any $v \in C(g, T)$;*

(5) *f, g, S and T have a common point provided that both (3) and (4) are true.*

Proof. (1) Since (S, f) and (T, g) satisfy the (CLR_f)-property, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X and $A, B \in CB(X)$ such that

$$\lim_{n\to\infty} Sx_n = A, \quad \lim_{n\to\infty} Ty_n = B, \quad \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = fu \in A \cap B$$

for some $u \in X$. Since $f(X)$ and $g(X)$ are closed, we have $fu = fv$ and $fu = gw$ for some $v, w \in X$.

Now, we show that $gw \in Tw$. In fact, suppose that $gw \notin Tw$. Then, using the condition (b) with $x = x_n$ and $y = w$, we have

$$H^p(Sx_n, Tw) \le \varphi\Big(\max\Big\{d^p(fx_n, gw), \frac{d^p(fx_n, Sx_n)d^p(gw, Tw)}{1 + d^p(fx_n, gw)}, \frac{d^p(fx_n, Tw)d^p(gw, Sx_n)}{1 + d^p(fx_n, gw)}\Big\}\Big)$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} H^p(A, Tw) &\le \varphi\Big(\max\Big\{d^p(fv, gw), \frac{d^p(fv, A)d^p(gw, Tw)}{1 + d^p(fv, gw)}, \frac{d^p(fv, Tw)d^p(gw, A)}{1 + d^p(fv, gw)}\Big\}\Big) \\ &= \varphi(\max\{0, 0, 0\}) = 0. \end{aligned}$$

Since $gw \in A$, it follows from the definition of Hausdorff metric that

$$d^p(gw, Tw) \le H^p(A, Tw) = 0,$$

which is a contradiction and so $gw \in Tw$. On the other hand, by the condition (b) again, we have

$$H^p(Sv, Ty_n) \le \varphi\Big(\max\Big\{d^p(fv, gy_n), \frac{d^p(fv, Sv)d^p(gy_n, Ty_n)}{1 + d^p(fv, gy_n)}, \frac{d^p(fv, Ty_n)d^p(gy_n, Sv)}{1 + d^p(fv, gy_n)}\Big\}\Big)$$

for all $n \in \mathbb{N}$. Similarly, by taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} H^p(Sv, B) &\le \varphi\Big(\max\Big\{d^p(fv, gw), \frac{d^p(fv, Sv)d^p(gw, B)}{1 + d^p(fv, B)}, \frac{d^p(fv, Tw)d^p(gw, Sv)}{1 + d^p(fv, fu)}\Big\}\Big) \\ &= \varphi(\max\{0, 0, 0\}) = 0. \end{aligned}$$

Since $fv \in B$, it follows from the definition of Hausdorff metric that

$$d^p(fv, Sv) \le H^p(B, Sv) = 0,$$

which is a contradiction and so $fv \in Sv$. Thus the mappings f, S have a coincidence point v and g, T have a coincidence point w. Furthermore, by virtue of the condition (b), we obtain $ffv = fv$ and $ffv \in Sfv$. Thus $u = fu \in Su$. This proves (3). A similar argument proves (4). Then (5) holds immediately. This completes the proof. □

If $f = g$ in Theorem 4.3, then we can conclude the following:

Corollary 4.4. *Let (X, d) be a metric space. Let $f : X \to X$ be a single-valued mapping and $S, T : X \to CB(X)$ be two multi-valued mappings satisfying the following conditions:*

(a) *the pairs (f, S) and (f, T) satisfy the (CLR_f)-property;*

(b) *for all $x, y \in X$,*

$$H^p(Sx, Ty) \le \varphi\Big(\max\Big\{d^p(fx, fy), \frac{d^p(fx, Sx)d^p(fy, Ty)}{1 + d^p(fx, fy)}, \frac{d^p(fx, Ty)d^p(fy, Sx)}{1 + d^p(fx, fy)}\Big\}\Big),$$

where $p \ge 1$ and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous monotone increasing function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.

If $f(X)$ is a closed subset of X, then we have the following:

(1) *f, S and T have a coincidence point;*
(2) *f, S and T have a common fixed point provided that f and S are weakly compatible, g and T are weakly commuting at v and $ffv = fv$ for any $v \in C(f, S)$.*

If $f = g$ in Theorem 4.3, then we can conclude the following:

Corollary 4.5. *Let (X, d) be a metric space. Let $f : X \to X$ be a single-valued mapping and $S : X \to CB(X)$ be a multi-valued mapping satisfying the following conditions:*
(a) *the pair (f, S) satisfies the (CLR_f)-property;*
(b) *for all $x, y \in X$,*

$$H^p(Sx, Sy) \leq \varphi\Big(\max\Big\{d^p(fx, fy), \frac{d^p(fx, Sx)d^p(fy, Sy)}{1 + d^p(fx, fy)}, \frac{d^p(fx, Sy)d^p(fy, Sx)}{1 + d^p(fx, fy)}\Big\}\Big),$$

where $p \geq 1$ and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous monotone increasing function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.

If $f(X)$ is closed subsets of X, then we have the following:

(1) *f and S have a coincidence point;*
(2) *f and S have a common fixed point provided that f and S are weakly compatible at v and $ffv = fv$ for any $v \in C(f, S)$.*

Now, we give an example to illustrate Theorem 4.3.

Example 4.6. Let $X = [1, \infty)$ with usual metric. Define two single-valued mappings $f, g : X \to X$ and two multi-valued mappings $S, T : X \to CB(X)$ by

$$fx = gx = x^2, \quad Sx = Tx = [1, x + 1]$$

for all $x \in X$, respectively. Then the pair (S, f) satisfies the (CLR_f)-property with respect to S for the sequence $\{x_n\}$ in X defined by $x_n = y_n = 1 + \frac{1}{n}$ for each $n \geq 1$. Indeed, we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 = 1 = f(1)$$

and

$$f(1) \in [1, 2] = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n.$$

Clearly, we know that the pairs f, g, S and T satisfy the condition (b) in Theorem 4.3 with $p = 1$ and $\varphi(t) = \frac{1}{2}t$. Thus all the conditions in Theorem 4.3 are satisfied. Then f and S have coincidence points in X. It is easy to see that f and S have infinitely coincidence points in X. Indeed, $C(f, S) = \Big[1, \frac{1+\sqrt{5}}{2}\Big]$. Also, we can see that f and T are weakly compatible at a point a and $ffa = fa$ for $a = 1 \in C(f, T)$. Therefore, all the conditions of Theorem 4.3 are satisfied. Therefore, f and T have a common fixed point in X. In this case, a point 1 is a unique common fixed point of f and T.

If both S and T are single-valued mappings in Theorem 4.3, then we have the following:

Corollary 4.7. *Let (X, d) be a metric space and $f, g, S, T : X \to X$ be single-valued mappings satisfying the following conditions:*
(a) *the pairs (S, f) and (T, g) satisfy the (CLR_f)-property;*
(b) *for all $x, y \in X$,*

$$d^p(Sx, Ty) \leq \varphi\Big(\max\Big\{d^p(fx, gy), \frac{d^p(fx, Sx)d^p(gy, Ty)}{1 + d^p(fx, gy)}, \frac{d^p(fx, Ty)d^p(gy, Sx)}{1 + d^p(fx, gy)}\Big\}\Big),$$

where $p \geq 1$ and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous monotone increasing function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.

If $f(X)$ and $g(X)$ are closed subsets of X, then we have the following:

(1) *f and S have a coincidence point;*
(2) *g and T have a coincidence point;*
(3) *f and S have a common fixed point provided that f and S are weakly compatible at v and $ffv = fv$ for any $v \in C(f, S)$;*
(4) *g and T have a common fixed point provided that g and T are weakly commuting at v and $ggv = gv$ for any $v \in C(g, T)$;*
(5) *f, g, S and T have a common point provided that both (3) and (4) are true.*

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