Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 2175–2189
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Fixed point theorems for cyclic mappings in quasi-partial b-metric spaces

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Communicated by R. Saadati

Abstract

In this paper, we introduce the concepts of qp_b-cyclic-Banach contraction mapping, qpb-cyclic-Kannan mapping and qp_b-cyclic β-quasi-contraction mapping and establish the existence and uniqueness of fixed point theorems for these mappings in quasi-partial b-metric spaces. Some examples are presented to validate our results.

Keywords: Quasi-partial b-metric space, fixed point theorems, qp_b-cyclic-Banach contraction mapping, qpb-cyclic-Kannan mapping, qp_b-cyclic β-quasi-contraction mapping.
2010 MSC: 47H09, 47H10.

1. Introduction and preliminaries

The concept of quasi-metric spaces was introduced by Wilson in [19] as a generalization of standard metric spaces. Roldán-López-de-Hierro et al. [16] gave some coincidence point theorems and obtained some very recent results in the setting of quasi-metric spaces. Matthews also generalized the standard metric spaces to partial-metric spaces by replacing the condition $d(x, x) = 0$ with the condition $d(x, x) \leqslant d(x, y)$ for all x, y ([14, 15]). Partial-metric spaces have applications in theoretical computer science [3]. Hitzler and Seda introduced dislocated metric spaces [7]. Czerwik presented the notion of b-metric space [5]. Many other generalized metric spaces, such as partial b-metric spaces, metric-like spaces and quasi-b-metric-like, were introduced (see, e.g., ([1, 8, 17, 18]) and the references therein). Especially, as a further generalization for the quasi-metric spaces and partial-metric spaces, Karapinar et al.[10] introduced the notion of quasi-partial metric space and discussed the existence of fixed points of self-mappings T on quasi-partial metric spaces. Very recently, following ([5, 10, 14]), Gupta and Gautam [6] have generalized quasi-partial metric spaces to

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Received 2015-10-18

the class of quasi-partial b-metric spaces and have focused on the fixed points of some self-mappings which have a deep relationship with T-orbitally lower semi-continuous functions introduced by Karapinar et al. in [10]. Some better results of fixed point are claimed in [6].

Corresponding to the development of spaces, many mappings have been presented since Banach contraction principle was introduced in [2]. For example, in 1974, Ćirić [4] defined quasi-contraction mappings and stated some fixed point results in which it has shown that the condition of quasi-contractivity implies all conclusions of Banach's contraction principle. We recall the concept as follows:

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a quasi-contraction mapping if there exists $\beta \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant \beta M(x, y)$$

for all $x, y \in X$, where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

We also review the concept of cyclic mapping as follows:

Let A and B be nonempty subsets of a metric space (X, d), $T : A \cup B \to A \cup B$ is called cyclic if $T(A) \subset B$ and $T(B) \subset A$.

In 1969, Kannan introduced the concept of Kannan mapping in [9]:

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a Kannan mapping if there exists $\lambda \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leqslant \lambda d(x, Tx) + \lambda d(y, Ty)$$

for all $x, y \in X$.

In 2003, Kirk et al. [12] introduced cyclic contraction mapping as follows:

Let (X, d) be a metric space. A cyclic mapping $T : A \cup B \to A \cup B$ is said to be a cyclic contraction mapping if there exists $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant \lambda d(x, y)$$

for any $x \in A$ and $y \in B$.

In 2010, Karapinar et al. [11] introduced Kannan type cyclic contraction as follows:

Let (X, d) be a metric space. A cyclic mapping $T : A \cup B \to A \cup B$ is said to be a Kannan type cyclic contraction if there exists $\lambda \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leqslant \lambda d(x, Tx) + \lambda d(y, Ty)$$

for any $x \in A$ and $y \in B$.

Recently, Klin-eam and Suanoom introduced dislocated quasi-b-metric spaces and investigated the fixed points of Geraghty type dqb-cyclic-Banach contraction mapping and dqb-cyclic-Kannan mapping [13]. Inspired and motivated by Karapinar et al. [11], Gupta et al. [6] and Klin-eam et al. [13], we introduce the notions: *qp_b-cyclic-Banach contraction mappings, qpb-cyclic-Kannan mappings* and *qp_b-cyclic β-quasi-contraction mappings*. The corresponding fixed point results for these three kinds of mappings in the setting of quasi-partial b-metric spaces (QPBMS) are provided. Our results complement and enrich the main results of Gupta et al. in the literature [6]. We also provide some examples to show the generality and effectiveness of our results.

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}_+$ denote the set of all positive integers and the set of all nonnegative real numbers, respectively. We begin with the following definition as a recall from ([7, 19]).

Definition 1.1. Let X be a nonempty set. Suppose that the mapping $d : X \times X \to [0, \infty)$ satisfies the following conditions:

(d_1) $d(x, x) = 0$ for all $x \in X$;

(d_2) $d(x,y) = d(y,x) = 0$ implies $x = y$ for all $x, y \in X$;

(d_3) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(d_4) $d(x,y) \leqslant d(x,z) + d(z,y)$ for all $x, y, z \in X$.

If d satisfies conditions (d_1), (d_2) and (d_4), then d is called a *quasi-metric* on X. If d satisfies conditions (d_2), (d_3) and (d_4), then d is called a *dislocated metric* on X. If it satisfies conditions (d_2) and (d_4), it is called a *dislocated quasi-metric*. If d satisfies conditions (d_1)-(d_4), then d is called a (standard) *metric* on X.

The concept of a quasi-partial metric space was introduced by Karapinar et al.

Definition 1.2 ([10])**.** A *quasi-partial metric* on a nonempty set X is a function $q : X \times X \to \mathbb{R}_+$, satisfying the following conditions:
(QPM1) If $q(x,x) = q(x,y) = q(x,y)$, then $x = y$.
(QPM2) $q(x,x) \leqslant q(x,y)$.
(QPM3) $q(x,x) \leqslant q(y,x)$.
(QPM4) $q(x,y) + q(z,z) \leqslant q(x,z) + q(z,y)$ for all $x, y, z \in X$.

A *quasi-partial metric space* is a pair (X, q) such that X is a nonempty set and q is a quasi-partial metric on X.

For each metric $q : X \times X \to \mathbb{R}_+$, the function $d_q : X \times X \to \mathbb{R}_+$ defined by

$$d_q(x,y) = q(x,y) + q(y,x) - q(x,x) - q(y,y)$$

is a (standard) metric on X.

The next Lemma shows the relationship between the quasi-partial metric and the standard metric.

Lemma 1.3 ([10])**.** *Let (X, q) be a quasi-partial metric space and (X, d_q) be the corresponding metric space. Then (X, q) is complete if and only if (X, d_q) is complete.*

For each metric $q : X \times X \to \mathbb{R}_+$, the function $d_{qm} : X \times X \to \mathbb{R}_+$ defined by

$$d_{qm}(x,y) = q(x,y) - q(x,x)$$

is a dislocated quasi-metric.

Gupta et al. [6] introduced the concept of quasi-partial b-metric space and gave some properties on such spaces in this section.

Definition 1.4 ([6])**.** A *quasi-partial b-metric* on a nonempty set X is a function $qp_b : X \times X \to \mathbb{R}_+$ such that for some real number $s \geqslant 1$ and all $x, y, z \in X$:
(QPb_1) If $qp_b(x,x) = qp_b(x,y) = qp_b(y,y)$, then $x = y$,
(QPb_2) $qp_b(x,x) \leqslant qp_b(x,y)$,
(QPb_3) $qp_b(x,x) \leqslant qp_b(y,x)$,
(QPb_4) $qp_b(x,y) \leqslant s[qp_b(x,z) + qp_b(z,y)] - qp_b(z,z)$.

A *quasi-partial b-metric space* (QPBMS) is a pair (X, qp_b) such that X is a nonempty set and qp_b is a generalization of quasi-partial metric on X.

Example 1.5. Let $X = [0, \frac{\pi}{8}]$. Define the metric

$$qp_b(x,y) = \sin 2|x-y| + x$$

for any $(x,y) \in X \times X$.
It can be demonstrated that (X, qp_b) is a quasi-partial b-metric space. Actually, if $qp_b(x,x) = qp_b(x,y) =$

$qp_b(y,y)$, that is, $x = \sin 2|x-y| + x = y$, then it is obvious that (QPb$_1$) holds for any $(x,y) \in X \times X$. In addition, $\sin 2|x-y| \geqslant 0$ and $\sin 2|x-y| \geqslant |x-y|$ when $|x-y| \in [0, \frac{\pi}{8}]$, then

$$qp_b(x,x) = x \leqslant \sin 2|x-y| + x = qp_b(x,y)$$

and

$$\begin{aligned} qp_b(x,x) &= x \\ &= |x-y+y| \\ &\leqslant |x-y| + |y| \\ &\leqslant \sin 2|y-x| + y \\ &= qp_b(y,x) \end{aligned}$$

are true, hence (QPb$_2$) and (QPb$_3$) hold for any $(x,y) \in X \times X$. Moreover, when $2(|x-z|+|z-y|) \in [0, \frac{\pi}{2}]$, $\sin 2(|x-z|+|z-y|) \leqslant 2(|x-z|+|z-y|)$, we get

$$\begin{aligned} qp_b(x,y) + qp_b(z,z) &= \sin 2|x-y| + x + z \\ &\leqslant \sin 2(|x-z|+|z-y|) + x + z \\ &\leqslant 2(|x-z|+|z-y|) + x + z \\ &\leqslant 2\sin 2|x-z| + 2\sin 2|z-y| + x + z \\ &= 2(\sin 2|x-z| + \sin 2|z-y| + x + z) \\ &\leqslant s\big(qp_b(x,z) + qp_b(z,y)\big) \end{aligned}$$

for all $x,y,z \in X$ and $s \geqslant 2$, (QPb$_4$) holds, hence (X, qp_b) is a quasi-partial b-metric space with $s \geqslant 2$.

Lemma 1.6 ([6])**.** *Every quasi-partial metric space is a quasi-partial b-metric, but the converse is not true.*

Each quasi-partial b-metric qp_b on X induces a topology $\mathscr{T}_{qp_b}$ on X whose base is the family of open qp_b-balls $\{B_{qp_b}(x,\delta) : x \in X, \delta > 0\}$, where $B_{qp_b}(x,\delta) = \{y \in X : |qp_b(x,y) - qp_b(x,x)| < \delta\}$.

Next we define *convergent sequence*, *Cauchy sequence*, *completeness of space* and *continuity* in quasi-partial b-metric spaces.

Definition 1.7 ([6])**.** Let (X, qp_b) be a quasi-partial b-metric. Then:

(i) A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ converges to $x \in X$ if and only if
$$qp_b(x,x) = \lim_{n\to\infty} qp_b(x,x_n) = \lim_{n\to\infty} qp_b(x_n,x).$$

(ii) A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is called a *Cauchy sequence* if and only if $\lim_{n,m\to\infty} qp_b(x_m,x_n)$ and $\lim_{n,m\to\infty} qp_b(x_n,x_m)$ exist (and are finite).

(iii) The quasi-partial b-metric space (X, qp_b) is said to be *complete* if every Cauchy sequence $\{x_n\}_{n=0}^{\infty} \subset X$ converges with respect to $\mathscr{T}_{qp_b}$ to a point $x \in X$ such that $qp_b(x,x) = \lim_{m,n\to\infty} qp_b(x_m,x_n) = \lim_{m,n\to\infty} qp_b(x_n,x_m)$.

(iv) A mapping $f : X \to X$ is said to be *continuous* at $x \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $f(B(x_0,\delta)) \subset B(f(x_0),\epsilon)$.

We denote simply qp_b-converges to x by $x_n \xrightarrow{qp_b} x$. Under a special case, we state the uniqueness of the limit of a sequence in a quasi-partial b-metric space, which is very useful in the proof of the main theorems.

Lemma 1.8. *Let (X, qp_b) be a quasi-partial b-metric space and $\{x_n\}_{n=0}^{\infty}$ be a sequence in X. If $x_n \xrightarrow{qp_b} x$, $x_n \xrightarrow{qp_b} y$ and $qp_b(x,x) = qp_b(y,y) = 0$, then $x = y$.*

Proof. Assume that $x_n \xrightarrow{qp_b} x$ and $x_n \xrightarrow{qp_b} y$ in (X, qp_b), then

$$qp_b(x,x) = \lim_{n\to\infty} qp_b(x_n, x) = \lim_{n\to\infty} qp_b(x, x_n) = 0$$

and

$$qp_b(y,y) = \lim_{n\to\infty} qp_b(x_n, y) = \lim_{n\to\infty} qp_b(y, x_n) = 0.$$

Using (QPb_4), we have

$$\begin{aligned} qp_b(x,y) &\leqslant s[qp_b(x,x_n) + qp_b(x_n,y)] - qp_b(x_n,x_n) \\ &\leqslant s[qp_b(x,x_n) + qp_b(x_n,y)] \end{aligned}$$

for every $n \in \mathbb{N}$. Taking limit as $n \to \infty$ in the above inequality, we have

$$\begin{aligned} qp_b(x,y) &\leqslant s[\lim_{n\to\infty} qp_b(x,x_n) + \lim_{n\to\infty} qp_b(x_n,y)] \\ &= 0. \end{aligned}$$

Therefore we get $qp_b(x,x) = qp_b(x,y) = qp_b(y,y) = 0$ which implies from the property (QPb_1) that $x = y$. □

Remark 1.9. Generally, the limit of a sequence in a quasi-partial b-metric space is not unique.

2. qp_b-cyclic-Banach contraction mapping in quasi-partial b-metric spaces

In this section, we extend fixed point theorem for Banach contraction mappings in standard metric spaces to *qp_b-cyclic-Banach contraction mappings* in the setting of quasi-partial b-metric spaces.

Definition 2.1. Let A and B be nonempty subsets of a quasi-partial b-metric space (X, qp_b). A cyclic mapping $T : A \cup B \to A \cup B$ is said to be a *qp_b-cyclic-Banach contraction mapping* if there exists $k \in [0,1)$ such that if $s \geqslant 1, sk < 1$, then

$$qp_b(Tx, Ty) \leqslant kqp_b(x,y) \tag{2.1}$$

holds both for $x \in A, y \in B$ and for $x \in B, y \in A$.

Theorem 2.2. *Let A and B be two nonempty closed subsets of a complete quasi-partial b-metric space (X, qp_b) and T be a cyclic mapping which is a qp_b-cyclic-Banach contraction. Then $A \cap B$ is nonempty and T has a unique fixed point in $A \cap B$.*

Proof. Let $x \in A$, noting the contractive condition of the theorem, we have

$$\begin{aligned} qp_b(T^2x, Tx) &= qp_b(T(Tx), Tx) \\ &\leqslant kqp_b(Tx, x) \end{aligned}$$

and

$$\begin{aligned} qp_b(Tx, T^2x) &= qp_b(Tx, T(Tx)) \\ &\leqslant kqp_b(x, Tx). \end{aligned}$$

Let $\alpha = \max\{qp_b(x, Tx), qp_b(Tx, x)\}$, thus

$$qp_b(Tx, T^2x) \leqslant k\alpha, \quad qp_b(T^2x, Tx,) \leqslant k\alpha. \tag{2.2}$$

Moreover, applying inequality (2.2), we have

$$qp_b(T^2x, T^3x) \leqslant k^2\alpha, \quad qp_b(T^3x, T^2x,) \leqslant k^2\alpha. \tag{2.3}$$

Hence

$$qp_b(T^nx, T^{n+1}x) \leqslant k^n\alpha, \quad qp_b(T^{n+1}x, T^nx,) \leqslant k^n\alpha \tag{2.4}$$

for every $n \in \mathbb{N}$.

Let $m, n \in \mathbb{N}$ and $m < n$, using (QPb$_4$)

$$\begin{aligned} qp_b(T^mx, T^nx) &\leqslant s[qp_b(T^mx, T^{m+1}x) + qp_b(T^{m+1}x, T^nx)] - qp_b(T^{m+1}x, T^{m+1}x) \\ &\leqslant s[qp_b(T^mx, T^{m+1}x) + qp_b(T^{m+1}x, T^nx)] \\ &\leqslant sqp_b(T^mx, T^{m+1}x) + s^2qp_b(T^{m+1}x, T^{m+2}x) + s^2qp_b(T^{m+2}x, T^nx) \\ &\leqslant sqp_b(T^mx, T^{m+1}x) + s^2qp_b(T^{m+1}x, T^{m+2}x) + \ldots + s^{n-m}qp_b(T^{n-1}x, T^nx). \end{aligned}$$

Noting $sk < 1$ and applying (2.4),

$$\begin{aligned} qp_b(T^mx, T^nx) &\leqslant (sk^m + s^2k^{m+1} + \ldots + s^{n-m}k^{n-1})\alpha \\ &= sk^m\frac{1-(sk)^{n-m}}{1-sk}\alpha \\ &\leqslant \frac{sk^m}{1-sk}\alpha. \end{aligned}$$

Taking limit as $m, n \to \infty$ in the above inequality, we have

$$\lim_{m,n\to\infty} qp_b(T^mx, T^nx) \leqslant 0,$$

thus

$$\lim_{m,n\to\infty} qp_b(T^mx, T^nx) = 0. \tag{2.5}$$

Similarly, we obtain

$$\begin{aligned} qp_b(T^nx, T^mx) &\leqslant s[qp_b(T^nx, T^{m+1}x) + qp_b(T^{m+1}x, T^mx)] - qp_b(T^{m+1}x, T^{m+1}x) \\ &\leqslant s[qp_b(T^nx, T^{m+1}x) + qp_b(T^{m+1}x, T^mx)] \\ &\leqslant s^2qp_b(T^nx, T^{m+2}x) + s^2qp_b(T^{m+2}x, T^{m+1}x) \\ &\quad + sqp_b(T^{m+1}x, T^mx) - sqp_b(T^{m+2}x, T^{m+2}x) \\ &\leqslant s^2qp_b(T^nx, T^{m+2}x) + s^2qp_b(T^{m+2}x, T^{m+1}x) + sqp_b(T^{m+1}x, T^mx) \\ &\leqslant s^{n-m}qp_b(T^nx, T^{n-1}x) + s^{n-m-1}qp_b(T^{n-1}x, T^{n-2}x) + \ldots + sqp_b(T^{m+1}x, T^mx) \\ &\leqslant (sk^m + s^2k^{m+1} + \ldots + s^{n-m}k^{n-1})\alpha \\ &= sk^m\frac{1-(sk)^{n-m}}{1-sk}\alpha \\ &\leqslant \frac{sk^m}{1-sk}\alpha. \end{aligned}$$

Taking limit as $m, n \to \infty$ in the above inequality, we have

$$\lim_{m,n\to\infty} qp_b(T^nx, T^mx) \leqslant 0,$$

thus

$$\lim_{m,n\to\infty} qp_b(T^nx, T^mx) = 0. \tag{2.6}$$

Eqs. (2.5) and (2.6) indicate that sequence $\{T^n x\}_{n=1}^{\infty}$ is a Cauchy sequence.

Since (X, qp_b) is complete, therefore $\{T^n x\}_{n=1}^{\infty}$ converges to some $\omega \in X$, that is,

$$\begin{aligned} qp_b(\omega,\omega) &= \lim_{n\to\infty} qp_b(T^n x, \omega) = \lim_{n\to\infty} qp_b(\omega, T^n x) \\ &= \lim_{m,n\to\infty} qp_b(T^n x, T^m x) = \lim_{m,n\to\infty} qp_b(T^m x, T^n x) = 0. \end{aligned} \tag{2.7}$$

Observe that $\{T^{2n}x\}_{n=0}^{\infty}$ is a sequence in A and $\{T^{2n-1}x\}_{n=1}^{\infty}$ is a sequence in B in a way that both sequences converge to ω. Also, note that A and B are closed, we have $\omega \in A \cap B$. On the other hand,

$$qp_b(T^n x, T\omega) \leqslant k qp_b(T^{n-1}x, \omega).$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} qp_b(T^n x, T\omega) \leqslant k \lim_{n\to\infty} qp_b(T^{n-1}x, \omega) = 0,$$

hence

$$\lim_{n\to\infty} qp_b(T^n x, T\omega) = 0. \tag{2.8}$$

Similarly, it can be derived

$$\lim_{n\to\infty} qp_b(T\omega, T^n x) = 0. \tag{2.9}$$

In addition, by the contractive condition of theorem and in combination with (2.7), we get

$$qp_b(T\omega, T\omega) \leqslant k qp_b(\omega,\omega) = 0$$

implies

$$qp_b(T\omega, T\omega) = 0. \tag{2.10}$$

Equations (2.8), (2.9) and (2.10) show that the sequence $\{T^n x\}_{n=1}^{\infty}$ is also convergent to $T\omega$. Applying Lemma 1.8, we obtain $T\omega = \omega$.

Assume that there exists another fixed point ω^* of T in $A \cup B$, that is, $T\omega^* = \omega^*$, then from the contractive condition (2.1),

$$qp_b(\omega*, \omega) = qp_b(T\omega^*, T\omega) \leqslant k qp_b(\omega*, \omega).$$

Since $k \in [0, 1)$, we get $qp_b(\omega*, \omega) = 0$. In addition, note that

$$qp_b(\omega^*, \omega^*) = qp_b(T\omega^*, T\omega^*) \leqslant k qp_b(\omega^*, \omega^*)$$

implies

$$qp_b(\omega^*, \omega^*) = 0. \tag{2.11}$$

It follows from $qp_b(\omega,\omega) = qp_b(\omega^*,\omega) = qp_b(\omega^*,\omega^*) = 0$ that $\omega = \omega^*$.

Analogously, when $x \in B$, the same results can be stated. □

Example 2.3. Let $X = [-\frac{\pi}{4}, \frac{\pi}{4}]$ and $T : A \cup B \to A \cup B$ defined by $Tx = -\frac{\sin x}{4}$, where $A = [-\frac{\pi}{4}, 0]$ and $B = [0, \frac{\pi}{4}]$. Define the metric

$$qp_b(x, y) = |x - y| + |x|$$

for any $(x, y) \in X \times X$.

First, we will show that (X, qp_b) is a quasi-partial b-metric space. If $qp_b(x, x) = qp_b(x, y) = qp_b(y, y)$, that is, $|x| = |x - y| + |x| = |y|$, then it is obvious that (QPb_1) holds for any $(x, y) \in X \times X$. And (QPb_2) is true due to

$$qp_b(x, x) = |x| \leqslant |x - y| + |x| = qp_b(x, y).$$

In addition,

$$\begin{aligned} qp_b(x,x) &= |x| \\ &= |x-y+y| \\ &\leqslant |x-y|+|y| \\ &= qp_b(y,x), \end{aligned} \tag{2.12}$$

which implies that (QPb_3) holds for any $(x,y)\in X\times X$. Moreover, we observe that for any $x,y,z\in X$,

$$\begin{aligned} qp_b(x,y)+qp_b(z,z) &= |x-y|+|x|+|z| \\ &\leqslant |x-z|+|z-y|+|x|+|z| \\ &\leqslant s\big(qp_b(x,z)+qp_b(z,y)\big), \end{aligned}$$

where $s\geqslant 1$, (QPb_4) holds, hence (X,qp_b) is a quasi-partial b-metric space with $s\geqslant 1$.

Next, we verify that the mapping T is a qp_b-cyclic-Banach contraction. If $x\in A$, then $Tx\in[0,\frac{\sqrt{2}}{8}]\subset B$. If $x\in B$, then $Tx\in[-\frac{\sqrt{2}}{8},0]\subset A$. Hence the map T is cyclic on X because $T(A)\subset B$ and $T(B)\subset A$. Calculating

$$\begin{aligned} qp_b(Tx,Ty) &= \Big|\frac{\sin x}{4}-\frac{\sin y}{4}\Big|+|-\frac{\sin x}{4}| \\ &= \frac{1}{4}(|\sin x-\sin y|+|\sin x|). \end{aligned} \tag{2.13}$$

Considering function $f(u)=\sin u, u\in[-\frac{\pi}{4},\frac{\pi}{4}]$ and using the differential mean value theorem, there exists $\zeta\in[-\frac{\pi}{4},\frac{\pi}{4}]$ such that

$$f'(\zeta)=\cos\zeta=\frac{\sin x-\sin y}{x-y}$$

for any $x,y\in[-\frac{\pi}{4},\frac{\pi}{4}]$, hence

$$|\sin x-\sin y|\leqslant|x-y|.$$

Thus

$$\begin{aligned} qp_b(Tx,Ty) &= \frac{1}{4}(|\sin x-\sin y|+|\sin x|) \\ &\leqslant \frac{1}{4}|x-y|+\frac{1}{4}|x| \\ &\leqslant kqp_b(x,y) \end{aligned} \tag{2.14}$$

for all $x,y\in[-\frac{\pi}{4},\frac{\pi}{4}]$ and $\frac{1}{4}\leqslant k<1$. Choosing $s\geqslant 1$ and $\frac{1}{4}\leqslant k<1$ such that $sk<1$, T satisfies the qp_b-cyclic-Banach contraction of Theorem 2.2 and $x=0$ is the unique fixed point of T.

3. *qpb*-cyclic-Kannan mapping in quasi-partial *b*-metric spaces

In this section, we extend fixed point theorem for Kannan mappings in the setting of quasi-partial b-metric spaces.

Definition 3.1. Let A and B be nonempty subsets of a quasi-partial b-metric space (X,qp_b). A cyclic mapping $T:A\cup B\to A\cup B$ is said to be a *qp_b-cyclic-Kannan mapping* if there exists $\lambda\in[0,\frac{1}{2})$ such that if $s\geqslant 1, s\lambda<\frac{1}{2}$, then

$$qp_b(Tx,Ty)\leqslant \lambda qp_b(x,Tx)+\lambda qp_b(y,Ty) \tag{3.1}$$

holds both for $x\in A, y\in B$ and for $x\in B, y\in A$.

Theorem 3.2. *Let A and B be two nonempty closed subsets of a complete quasi-partial b-metric space (X, qp_b) and T be a cyclic mapping which is a qpb-cyclic-Kannan mapping. Then $A \cap B$ is nonempty and T has a unique fixed point in $A \cap B$.*

Proof. Let $x \in A$, considering condition (3.1), we have

$$\begin{aligned} qp_b(Tx, T^2x) &= qp_b(Tx, T(Tx)) \\ &\leqslant \lambda qp_b(x, Tx) + \lambda qp_b(Tx, T^2x), \end{aligned} \tag{3.2}$$

thus

$$qp_b(Tx, T^2x) \leqslant \frac{\lambda}{1-\lambda} qp_b(x, Tx). \tag{3.3}$$

Using (3.3), we get

$$\begin{aligned} qp_b(T^2x, Tx) &= qp_b(T(Tx), Tx) \\ &\leqslant \lambda qp_b(Tx, T^2x) + \lambda qp_b(x, Tx) \\ &\leqslant \frac{\lambda^2}{1-\lambda} qp_b(x, Tx) + \lambda qp_b(x, Tx) \\ &\leqslant \frac{\lambda}{1-\lambda} qp_b(x, Tx). \end{aligned}$$

Set $\delta = qp_b(x, Tx)$. Moreover, we have

$$qp_b(T^2x, T^3x) \leqslant \left(\frac{\lambda}{1-\lambda}\right)^2 \delta, \quad qp_b(T^3x, T^2x,) \leqslant \left(\frac{\lambda}{1-\lambda}\right)^2 \delta. \tag{3.4}$$

Hence

$$qp_b(T^nx, T^{n+1}x) \leqslant \left(\frac{\lambda}{1-\lambda}\right)^n \delta, \quad qp_b(T^{n+1}x, T^nx,) \leqslant \left(\frac{\lambda}{1-\lambda}\right)^n \delta \tag{3.5}$$

for every $n \in \mathbb{N}$.
Let $m, n \in \mathbb{N}$ and $m < n$, using (QPb$_4$)

$$\begin{aligned} qp_b(T^mx, T^nx) &\leqslant s[qp_b(T^mx, T^{m+1}x) + qp_b(T^{m+1}x, T^nx)] - qp_b(T^{m+1}x, T^{m+1}x) \\ &\leqslant s[qp_b(T^mx, T^{m+1}x) + qp_b(T^{m+1}x, T^nx)] \\ &\leqslant sqp_b(T^mx, T^{m+1}x) + s^2qp_b(T^{m+1}x, T^{m+2}x) + s^2qp_b(T^{m+2}x, T^nx) \\ &\leqslant sqp_b(T^mx, T^{m+1}x) + s^2qp_b(T^{m+1}x, T^{m+2}x) + \ldots + s^{n-m}qp_b(T^{n-1}x, T^nx). \end{aligned}$$

Setting $\gamma = \dfrac{\lambda}{1-\lambda}$ and using (3.5),

$$\begin{aligned} qp_b(T^mx, T^nx) &\leqslant (s\gamma^m + s^2\gamma^{m+1} + \ldots + s^{n-m}\gamma^{n-1})\delta \\ &= s\gamma^m \frac{1-(s\gamma)^{n-m}}{1-s\gamma}\delta. \end{aligned}$$

Because $\lambda \in [0, \frac{1}{2})$ and $s\lambda < \frac{1}{2}$, therefore $\gamma, s\gamma \in [0, 1)$. Furthermore,

$$qp_b(T^mx, T^nx) \leqslant \frac{s\gamma^m}{1-s\gamma}\delta.$$

Taking limit as $m, n \to \infty$ in the above inequality, we have

$$\lim_{m,n\to\infty} qp_b(T^mx, T^nx) \leqslant 0,$$

thus

$$\lim_{m,n\to\infty} qp_b(T^m x, T^n x) = 0. \tag{3.6}$$

Also,

$$\begin{aligned} qp_b(T^n x, T^m x) &\leqslant s[qp_b(T^n x, T^{m+1}x) + qp_b(T^{m+1}x, T^m x)] - qp_b(T^{m+1}x, T^{m+1}x) \\ &\leqslant s[qp_b(T^n x, T^{m+1}x) + qp_b(T^{m+1}x, T^m x)] \\ &\leqslant s^2 qp_b(T^n x, T^{m+2}x) + s^2 qp_b(T^{m+2}x, T^{m+1}x) \\ &\quad + sqp_b(T^{m+1}x, T^m x) - sqp_b(T^{m+2}x, T^{m+2}x) \\ &\leqslant s^2 qp_b(T^n x, T^{m+2}x) + s^2 qp_b(T^{m+2}x, T^{m+1}x) + sqp_b(T^{m+1}x, T^m x) \\ &\leqslant s^{n-m} qp_b(T^n x, T^{n-1}x) + s^{n-m-1} qp_b(T^{n-1}x, T^{n-2}x) + \ldots + sqp_b(T^{m+1}x, T^m x) \\ &\leqslant (s\gamma^m + s^2\gamma^{m+1} + \ldots + s^{n-m}\gamma^{n-1})\delta \\ &= s\gamma^m \frac{1-(s\gamma)^{n-m}}{1-s\gamma}\delta \\ &\leqslant \frac{s\gamma^m}{1-s\gamma}\delta. \end{aligned}$$

Taking limit as $m, n \to \infty$ in the above inequality, we have

$$\lim_{m,n\to\infty} qp_b(T^n x, T^m x) \leqslant 0,$$

thus

$$\lim_{m,n\to\infty} qp_b(T^n x, T^m x) = 0. \tag{3.7}$$

Eqs. (3.6) and (3.7) indicate that sequence $\{T^n x\}_{n=1}^{\infty}$ is a Cauchy sequence.

Since (X, qp_b) is complete, therefore $\{T^n x\}_{n=1}^{\infty}$ converges to some $\omega \in X$, that is,

$$\begin{aligned} qp_b(\omega,\omega) &= \lim_{n\to\infty} qp_b(T^n x, \omega) = \lim_{n\to\infty} qp_b(\omega, T^n x) \\ &= \lim_{m,n\to\infty} qp_b(T^n x, T^m x) = \lim_{m,n\to\infty} qp_b(T^m x, T^n x) = 0. \end{aligned} \tag{3.8}$$

Observe that $\{T^{2n}x\}_{n=0}^{\infty}$ is a sequence in A and $\{T^{2n-1}x\}_{n=1}^{\infty}$ is a sequence in B in a way that both sequences converge to ω. Note also that A and B are closed, we have $\omega \in A \cap B$. On the other hand,

$$qp_b(T^n x, T\omega) \leqslant \lambda qp_b(T^{n-1}x, T^n x) + \lambda qp_b(\omega, T\omega). \tag{3.9}$$

Taking limit as $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} qp_b(T^n x, T\omega) \leqslant \lambda qp_b(\omega, T\omega). \tag{3.10}$$

By (QPb_4),

$$\begin{aligned} \lambda qp_b(\omega, T\omega) &\leqslant s\lambda[qp_b(\omega, T^n x) + qp_b(T^n x, T\omega)] - \lambda qp_b(T^n x, T^n x) \\ &\leqslant s\lambda[qp_b(\omega, T^n x) + qp_b(T^n x, T\omega)] \end{aligned} \tag{3.11}$$

for every $n \in \mathbb{N}$. Taking limit as $n \to \infty$ in the above inequality, we get

$$\lambda qp_b(\omega, T\omega) \leqslant s\lambda \lim_{n\to\infty} qp_b(T^n x, T\omega). \tag{3.12}$$

Thus, applying (3.10) and (3.12), we obtain

$$\lim_{n\to\infty} qp_b(T^n x, T\omega) \leqslant \lambda qp_b(\omega, T\omega) \leqslant s\lambda \lim_{n\to\infty} qp_b(T^n x, T\omega). \tag{3.13}$$

Since $s\lambda \in [0, \frac{1}{2})$, we obtain

$$\lim_{n\to\infty} qp_b(T^n x, T\omega) = qp_b(\omega, T\omega) = 0. \tag{3.14}$$

Similarly, it can be derived

$$\lim_{n\to\infty} qp_b(T\omega, T^n x) = qp_b(T\omega, \omega) = 0. \tag{3.15}$$

In addition, by the contractive condition of theorem and in combination with (3.14), we get

$$\begin{aligned} qp_b(T\omega, T\omega) &\leqslant \lambda qp_b(\omega, T\omega) + \lambda qp_b(\omega, T\omega) \\ &= 2\lambda qp_b(\omega, T\omega) \\ &= 0 \end{aligned} \tag{3.16}$$

implies

$$qp_b(T\omega, T\omega) = 0. \tag{3.17}$$

Equations (3.14), (3.15) and (3.17) show that the sequence $\{T^n x\}_{n=1}^{\infty}$ is also convergent to $T\omega$. Applying Lemma 1.8, we obtain $T\omega = \omega$.

Assume that there exists another fixed point ω^* of T in $A \cup B$, that is, $T\omega^* = \omega^*$, then from the contractive condition (3.1),

$$\begin{aligned} qp_b(\omega *, \omega) &= qp_b(T\omega^*, T\omega) \\ &\leqslant \lambda qp_b(\omega^*, T\omega^*) + \lambda qp_b(\omega, T\omega) \\ &\leqslant \lambda qp_b(\omega^*, \omega^*) + \lambda qp_b(\omega, \omega). \end{aligned} \tag{3.18}$$

In addition, note that

$$\begin{aligned} qp_b(\omega, \omega) &= qp_b(T\omega, T\omega) \\ &\leqslant 2\lambda qp_b(\omega, T\omega) \\ &= 2\lambda qp_b(\omega, \omega) \end{aligned} \tag{3.19}$$

and $2\lambda \in [0, 1)$, we get $qp_b(\omega, \omega) = 0$. Similarly, we obtain that $qp_b(\omega^*, \omega^*) = 0$. Moreover, by (3.18), $qp_b(\omega^*, \omega) = 0$. It follows from $qp_b(\omega, \omega) = qp_b(\omega^*, \omega) = qp_b(\omega^*, \omega^*) = 0$ that $\omega = \omega^*$.

Analogously, when $x \in B$, the same results can be stated. □

An example of *qpb*-cyclic-Kannan mapping in quasi-partial *b*-metric space is provided to illustrate Theorem 3.2.

Example 3.3. Let $X = [-\frac{1}{2}, \frac{1}{2}]$ and $T : A \cup B \to A \cup B$ defined by $Tx = -\frac{1}{8}x$, where $A = [-\frac{1}{2}, 0]$ and $B = [0, \frac{1}{2}]$. Define the metric

$$qp_b(x, y) = |x - y|^{\frac{1}{2}} + |x|$$

for any $(x, y) \in X \times X$.

If $qp_b(x, x) = qp_b(x, y) = qp_b(y, y)$, that is, $|x| = |x - y|^{\frac{1}{2}} + |x| = |y|$, then it is obvious that (QPb_1) holds for any $(x, y) \in X \times X$. In addition, $|x - y|^{\frac{1}{2}} \geqslant 0$ and $|y - x| \leqslant |y - x|^{\frac{1}{2}}$ when $|x - y| \in [0, 1]$, then

$$qp_b(x, x) = |x| \leqslant |x - y|^{\frac{1}{2}} + |x| = qp_b(x, y)$$

and

$$\begin{aligned} qp_b(x, x) &= |x| = |x - y + y| \\ &\leqslant |y - x| + |y| \\ &\leqslant |y - x|^{\frac{1}{2}} + |y| \\ &= qp_b(y, x) \end{aligned}$$

are true, then (QPb_2) and (QPb_3) hold for any $(x, y) \in X \times X$. Moreover, we observe that

$$\begin{aligned} qp_b(x,y) + qp_b(z,z) &= |x-y|^{\frac{1}{2}} + |x| + |z| \\ &\leqslant (|x-z| + |z-y|)^{\frac{1}{2}} + |x| + |z| \\ &\leqslant |x-z|^{\frac{1}{2}} + |z-y|^{\frac{1}{2}} + |x| + |z| \\ &= qp_b(x,z) + qp_b(z,y) \\ &\leqslant s[qp_b(x,z) + qp_b(z,y)] \end{aligned}$$

for any $x, y, z \in X$ and $s \geqslant 1$, (QPb_4) holds, hence (X, qp_b) is a quasi-partial b-metric space with $s \geqslant 1$.

Next, we verify that the mapping T is a qp_b-cyclic-Kannan contraction. If $x \in A$, then $Tx \in [0, \frac{1}{16}] \subset B$. If $x \in B$, then $Tx \in [-\frac{1}{16}, 0] \subset A$. Hence the map T is cyclic on X because $T(A) \subset B$ and $T(B) \subset A$. On the other hand,

$$\begin{aligned} qp_b(Tx,Ty) &= \frac{\sqrt{2}}{4}|x-y|^{\frac{1}{2}} + |-\frac{1}{8}x| \\ &\leqslant \frac{\sqrt{2}}{4}(|x| + |y|)^{\frac{1}{2}} + \frac{1}{8}|x| + \frac{1}{8}|y| \\ &\leqslant \frac{\sqrt{2}}{4}|x|^{\frac{1}{2}} + \frac{\sqrt{2}}{4}|y|^{\frac{1}{2}} + \frac{1}{8}|x| + \frac{1}{8}|y| \\ &\leqslant \frac{\sqrt{2}}{4}|\frac{9}{8}x|^{\frac{1}{2}} + \frac{\sqrt{2}}{4}|\frac{9}{8}y|^{\frac{1}{2}} + \frac{1}{8}|x| + \frac{1}{8}|y| \\ &\leqslant \frac{\sqrt{2}}{4}\big(|\frac{9}{8}x|^{\frac{1}{2}} + |x| + |\frac{9}{8}y|^{\frac{1}{2}} + |y|\big) \\ &\leqslant \frac{\sqrt{2}}{4}\big(qp_b(x,Tx) + qp_b(y,Ty)\big) \\ &\leqslant \lambda(qp_b(x,Tx) + qp_b(y,Ty)) \end{aligned} \tag{3.20}$$

for all $x, y \in X$ and $\lambda \in [\frac{\sqrt{2}}{4}, \frac{1}{2})$. Choosing s and λ such that $s\lambda < \frac{1}{2}$, T satisfies the qp_b-cyclic-Kannan mapping of Theorem 3.2 and $x = 0$ is the unique fixed point of T.

4. qp_b-cyclic β-quasi-contraction mapping in quasi-partial b-metric spaces

In this section, we extend Ćirić's fixed point theorem for quasi-contraction type mappings in the setting of quasi-partial b-metric spaces.

Let A and B be nonempty subsets of a quasi-partial b-metric space (X, qp_b). And let $T : A \cup B \to A \cup B$ is a cyclic mapping. We denote

$$M(x,y) = \max\{qp_b(x,y), qp_b(x,Tx), qp_b(y,Ty), qp_b(x,Ty), qp_b(y,Tx)\}$$

for any $x, y \in X$.

Definition 4.1. Let A and B be nonempty subsets of a quasi-partial b-metric space (X, qp_b) with $s \geqslant 1$. A cyclic mapping $T : A \cup B \to A \cup B$ is said to be a *qp_b-cyclic β-quasi-contraction mapping* if there exists $\beta \in [0, \frac{1}{2})$ such that if $\beta s \in [0, \frac{1}{2})$, then

$$qp_b(Tx,Ty) \leqslant \beta M(x,y) \tag{4.1}$$

holds both for $x \in A, y \in B$ and for $x \in B, y \in A$.

Next, we give the result for a qp_b-cyclic β-quasi-contraction mapping which is an extension of the result of Ćirić.

Theorem 4.2. *Let A and B be two nonempty closed subsets of a complete quasi-partial b-metric space (X, qp_b) with $s \geqslant 1$ and T be a cyclic mapping which is a qp_b-cyclic β-quasi-contraction. Then $A \cap B$ is nonempty and T has a unique fixed point in $A \cap B$.*

Proof. Let $x \in A$ and denote $x_{n+1} = Tx_n = T^{n+1}x, x_0 = x$. From condition (4.1), we obtain

$$\begin{aligned} qp_b(x_n, x_{n+1}) &\leqslant \beta M(x_{n-1}, x_n) \\ &\leqslant \beta \max\Big\{qp_b(x_{n-1}, x_n), qp_b(x_{n-1}, Tx_{n-1}), qp_b(x_n, Tx_n), qp_b(x_{n-1}, Tx_n), qp_b(x_n, Tx_{n-1})\Big\} \\ &\leqslant \beta \max\Big\{qp_b(x_{n-1}, x_n), qp_b(x_{n-1}, x_n), qp_b(x_n, x_{n+1}), qp_b(x_{n-1}, x_{n+1}), qp_b(x_n, x_n)\Big\} \\ &\leqslant \beta \max\Big\{qp_b(x_{n-1}, x_n), qp_b(x_n, x_{n+1}), qp_b(x_{n-1}, x_{n+1}), qp_b(x_n, x_n)\Big\} \end{aligned}$$

for any $n \in \mathbb{N}$. Property (QPb_2) shows $qp_b(x_n, x_n) \leqslant qp_b(x_n, x_{n+1})$, so

$$qp_b(x_n, x_{n+1}) \leqslant \beta \max\Big\{qp_b(x_{n-1}, x_n), qp_b(x_n, x_{n+1}), qp_b(x_{n-1}, x_{n+1})\Big\}.$$

Furthermore, from (QPb_4), we have

$$\begin{aligned} qp_b(x_{n-1}, x_{n+1}) &\leqslant s[qp_b(x_{n-1}, x_n) + qp_b(x_n, x_{n+1})] - qp_b(x_n, x_n) \\ &\leqslant s[qp_b(x_{n-1}, x_n) + qp_b(x_n, x_{n+1})] \end{aligned}$$

with $s \geqslant 1$, hence

$$\begin{aligned} qp_b(x_n, x_{n+1}) &\leqslant \beta \max\Big\{qp_b(x_{n-1}, x_n), qp_b(x_n, x_{n+1}), qp_b(x_{n-1}, x_{n+1})\Big\} \\ &\leqslant \beta \max\Big\{qp_b(x_{n-1}, x_n), qp_b(x_n, x_{n+1}), s[qp_b(x_{n-1}, x_n) + qp_b(x_n, x_{n+1})]\Big\} \\ &= \beta s[qp_b(x_{n-1}, x_n) + qp_b(x_n, x_{n+1})]. \end{aligned}$$

Subsequently,

$$qp_b(x_n, x_{n+1}) \leqslant \frac{\beta s}{1 - \beta s} qp_b(x_{n-1}, x_n).$$

Set $k = \frac{\beta s}{1-\beta s}$. It can be derived that $0 \leqslant k < 1$ because $\beta s \in [0, \frac{1}{2})$. It follows

$$\begin{aligned} qp_b(x_n, x_{n+1}) &\leqslant k qp_b(x_{n-1}, x_n) \\ &\leqslant \ldots \leqslant k^n qp_b(x, x_1) \\ &= k^n qp_b(x, Tx). \end{aligned}$$

Similarly, we get

$$\begin{aligned} qp_b(x_{n+1}, x_n) &\leqslant k qp_b(x_n, x_{n-1}) \\ &\leqslant \ldots \leqslant k^n qp_b(x_1, x) \\ &= k^n qp_b(Tx, x). \end{aligned}$$

Letting $\alpha = \max\{qp_b(Tx, x), qp_b(x, Tx)\}$, thus

$$qp_b(x_n, x_{n+1}) \leqslant k^n \alpha, \quad qp_b(x_{n+1}, x_n) \leqslant k^n \alpha.$$

The latter process of proof for the theorem is same as Theorem 2.2, thus we omit it. This completes the proof. □

Example 4.3. Let $X = [-\frac{\pi}{16}, \frac{\pi}{16}]$ and define $qp_b : X \times X \to \mathbb{R}_+$ as

$$qp_b(x, y) = \sin 2|x - y| + |x|$$

for any $(x, y) \in X \times X$. (X, qp_b) is a quasi-partial b-metric space with $s \geqslant 2$ as claimed in Example 1.5.

Let $T : A \cup B \to A \cup B$ defined by $Tx = -\frac{x}{12}$, where $A = [-\frac{\pi}{16}, 0]$ and $B = [0, \frac{\pi}{16}]$. If $x \in A$, then $Tx \in [0, \frac{\pi}{192}] \subset B$. If $x \in B$, then $Tx \in [-\frac{\pi}{192}, 0] \subset A$. Hence the map T is cyclic on X due to $T(A) \subset B$ and $T(B) \subset A$.
Because $|x - y| \in [0, \frac{\pi}{8}]$ and when $\sin u \leqslant u \leqslant \sin 2u$, $u \in [0, \frac{\pi}{8}]$ holds, then

$$\begin{aligned} qp_b(Tx, Ty) &= \sin 2\left|\frac{x}{12} - \frac{y}{12}\right| + |-\frac{x}{12}| \\ &= \sin \frac{|x - y|}{6} + \frac{1}{12}|x| \\ &\leqslant \frac{|x - y|}{6} + \frac{1}{12}|x| \\ &\leqslant \frac{1}{6}\sin 2|x - y| + \frac{1}{12}|x| \\ &\leqslant \frac{1}{6}(\sin 2|x - y| + |x|) \\ &= \frac{1}{6}qp_b(x, y). \end{aligned} \tag{4.2}$$

In addition,
$qp_b(x, y) \leqslant M(x, y) = \max\{qp_b(x, y), qp_b(x, Tx), qp_b(y, Ty), qp_b(x, Ty), qp_b(y, Tx)\}$, thus

$$\begin{aligned} qp_b(Tx, Ty) &\leqslant \frac{1}{6}M(x, y) \\ &\leqslant \beta M(x, y) \end{aligned} \tag{4.3}$$

for $\beta \in [\frac{1}{6}, \frac{1}{2})$.

Choosing s and β such that $\beta s < \frac{1}{2}$, T satisfies the qp_b-cyclic β-quasi-contraction mapping of Theorem 4.2 and $x = 0$ is the unique fixed point of T.

Acknowledgements

Projects supported by China Postdoctoral Science Foundation (Grant No. 2014M551168) and Natural Science Foundation of Heilongjiang Province of China (Grant No. A201410).

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