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Necessary optimality conditions for DC infinite programs with inequality constraints

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Abstract

In this paper, we first recall the regularity conditions introduced by Sun in [X. K. Sun, J. Math. Anal. Appl., **414** (2014), 590–611]. Then, by using these regularity conditions, we obtain some necessary optimality conditions for ε-optimal solution and exact optimal solution of a DC infinite programming problem with inequality constraints. Moreover, we also apply the obtained results to conic programming problems.

Keywords: DC infinite programming, regularity conditions, optimality, conic programming.
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1. Introduction

Let T be a nonempty (possibly infinite) index set, C be a nonempty convex subset of a locally convex space X, and let f, g, $h_t : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, $t \in T$, be proper convex functions. In this paper, we consider the following DC infinite programming with inequality constraints:

$$(P) \qquad \begin{cases} \inf \ \{f(x) - g(x)\} \\ \text{s.t. } h_t(x) \le 0, \ t \in T, \text{ and } x \in C. \end{cases}$$

Let

$$\mathcal{A} := \{x \in C : h_t(x) \le 0, \text{for all } t \in T\} \neq \emptyset.$$

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Then, the problem (P) can be rewritten as:

$$(P) \quad \inf_{x \in X} \{f(x) - g(x) + \delta_{\mathcal{A}}(x)\}.$$

This kind of DC programming problem plays an interesting and important part in the field of nonconvex optimization since it has numerous applications in Chebyshev approximation, transportation problems, cooperative games, engineering design, optimal control, robust optimization, etc.

Recently, many important results have been established for different kinds of DC programming in the last decades, see [4, 5, 6, 7, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 22, 23] and the references therein. Here, we specially mention the works on duality and optimality due to [5, 11, 12, 19, 20]. More precisely, by using the properties of the epigraph of the conjugate functions, Dinh *et al.* [5] introduced a closedness qualification condition and employed it to deriving qualified necessary optimality conditions for local minimizers of DC infinite programs in a Banach space. Fang and Zhao [11] first introduced the local and global KKT type conditions for a DC optimization problem. Then, by using properties of the subdifferentials of the involved functions, they obtained some sufficient and/or necessary conditions for these two types of optimality conditions. By using the notions of strong subdifferential and epsilon-subdifferential, Guo and Li [12] obtained necessary and sufficient optimality conditions for an epsilon-weak Pareto minimal point and an epsilon-proper Pareto minimal point of a DC vector optimization problem. Sun and Li [19] obtained weak and strong duality assertions for extended Ky Fan inequalities with DC functions. They also apply their dual problems to a convex optimization problem and a generalized variational inequality problem. By using the properties of the epigraph of the conjugate functions, Sun, *et al.* [20] introduced a closedness qualification condition and employed it to investigate duality and Farkas-type results for a DC infinite programming problem.

On the other hand, in order to characterize optimality conditions for an optimization problem, constraint qualifications involving epigraphs (first introduced in [3]) have been introduced and extensively used by many authors, see e.g. [2, 4, 5, 6, 9, 11, 21] and the references therein. Here, we just mention the work by Sun in [16]. By using the properties of the epigraph of the conjugate functions, Sun [16] introduced some new regularity conditions for (P). Then, the author investigated some characterizations of these regularity conditions. By using these regularity conditions, the author completely characterized Fenchel-Lagrange dualities and Farkas-type results for (P). However, to the best of our knowledge, there is no paper to deal with ε-optimal solution and exact optimal solution of (P) in terms of the new regularity conditions introduced by Sun [16]. So, the purpose of this paper is to use the regularity conditions introduced by Sun [16] to establish some necessary optimality conditions of ε-optimal solution and exact optimal solution for the problem (P). Moreover, we also apply the obtained results to conic programming problems.

The paper is organized as follows. In Section 2, we recall some notions and give some preliminary results. In Section 3, we first recall some new regularity conditions introduced in [16]. Then, by using these new regularity conditions, we establish some necessary and/or sufficient optimality conditions of ε-optimal solution and exact optimal solution for DC and convex programs. In Section 4, we apply the proposed approach to investigate optimality conditions of ε-optimal solution and exact optimal solution for conic programming problems.

2. Mathematical Preliminaries

Throughout this paper, let X be a real locally convex vector space with its continuous dual space X^*, endowed with the weak* topology $w(X^*, X)$. We always use the notation $\langle \cdot, \cdot \rangle$ for the canonical paring between X and X^*. Let D be a set in X, the interior (resp. closure, convex hull, convex cone hull) of D is denoted by $\mathrm{int}D$ (resp. $\mathrm{cl}D$, $\mathrm{co}D$, $\mathrm{cone}D$). Thus if $W \subseteq X^*$, then $\mathrm{cl}W$ denotes the weak* closure of W. We shall adopt the convention that $\mathrm{cone}D = \{0\}$ when D is an empty set. Let $D^* = \{x^* \in X^* : \langle x^*, x\rangle \geq 0, \forall x \in D\}$ be the dual cone of D. The indicator function $\delta_D : X \to \overline{\mathbb{R}}$ of X is defined by

$$\delta_D(x) = \begin{cases} 0, & \text{if } x \in D, \\ +\infty, & \text{if } x \notin D. \end{cases}$$

The support function $\sigma_D : X^* \to \overline{\mathbb{R}}$ of D is defined by

$$\sigma_D(x^*) = \sup_{x\in D} \langle x^*, x\rangle.$$

Further, let $\mathbb{R}^T$ be the product space of $\lambda = (\lambda_t)_{t\in T}$ with $\lambda_t \in \mathbb{R}$ for all $t \in T$, let $\mathbb{R}^{(T)}$ be collection of $\lambda \in \mathbb{R}^T$ with $\lambda_t \neq 0$ for finitely many $t \in T$, and let $\mathbb{R}^{(T)}_+$ be the positive cone in $\mathbb{R}^{(T)}$ defined by

$$\mathbb{R}^{(T)}_+ := \left\{\lambda \in \mathbb{R}^{(T)} : \lambda_t \geq 0 \text{ for all } t \in T\right\}.$$

Given $u \in \mathbb{R}^T$ and $\lambda \in \mathbb{R}^{(T)}$, and denoting supp $\lambda := \{t \in T : \lambda_t \neq 0\}$, we have

$$\langle \lambda, u\rangle := \sum_{t\in T} \lambda_t u_t = \sum_{t\in \text{supp } \lambda} \lambda_t u_t.$$

Let $f : X \to \overline{\mathbb{R}}$ be an extended real-valued function. The effective domain and the epigraph are defined by

$$\text{dom } f = \{x \in X : f(x) < +\infty\}$$

and

$$\text{epi } f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\},$$

respectively. f is said to be proper, if its effective domain is nonempty. The conjugate function $f^* : X^* \to \overline{\mathbb{R}}$ of f is defined by

$$f^*(x^*) = \sup_{x\in X} \{\langle x^*, x\rangle - f(x)\}.$$

The lower semicontinuous hull cl $f : X \to \overline{\mathbb{R}}$ of f is defined by

$$\text{epi } (\text{cl } f) = \text{cl } (\text{epi } f),$$

where the topological closure is taken with respect to the product topology. Let $\bar{x} \in \text{dom } f$. For any $\varepsilon \geq 0$, the ε-subdifferential of f at $\bar{x}$ is the convex set defined by

$$\partial_\varepsilon f(\bar{x}) := \{x^* \in X^* : f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x}\rangle - \varepsilon, \forall x \in X\}.$$

When $\bar{x} \notin \text{dom } f$, we define that $\partial_\varepsilon f(\bar{x}) = \emptyset$. If $\varepsilon = 0$, the set $\partial f(\bar{x}) := \partial_0 f(\bar{x})$ is the classical subdifferential of convex analysis, that is,

$$\partial f(\bar{x}) = \{x^* \in X^* : f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x}\rangle, \forall x \in X\}.$$

It is easy to prove that for any $\bar{x} \in \text{dom } f$ and $x^* \in X^*$,

$$f(\bar{x}) + f^*(x^*) \leq \langle x^*, \bar{x}\rangle + \varepsilon \Leftrightarrow x^* \in \partial_\varepsilon f(\bar{x}).$$

Moreover, following [13], we have

$$\text{epi } f^* = \bigcup_{\varepsilon\geq 0} \{(x^*, \langle x^*, \bar{x}\rangle + \varepsilon - f(\bar{x})) : x^* \in \partial_\varepsilon f(\bar{x})\}.$$

Let E be a convex set of X. The ε-normal set to E at a point $\bar{x} \in E$ is defined by

$$N^\varepsilon_E(\bar{x}) := \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \leq \varepsilon, \text{ for any } x \in E\}.$$

If $\varepsilon = 0$, $N^0_E(\bar{x})$ is the normal cone $N_E(\bar{x})$ of convex analysis. Moreover, it is easy to see that $N^\varepsilon_E(\bar{x}) = \partial_\varepsilon \delta_E(\bar{x})$.

Now, let us recall the following result which will be used in the following section.

Lemma 2.1 ([1])**.** *Let $f : X \to \overline{\mathbb{R}}$ be a proper convex function. Then $(u, c) \in$ epi f^* if and only if $u(x) - c \leq f(x)$, for any $x \in X$.*

3. ε-optimal solution for (P)

n this section, we first recall some new regularity conditions introduced in [16]. Then, by using these regularity conditions, we obtain some necessary optimality conditions for ε-optimal solution and exact optimal solution of (P). To this aim, we will make use of the following characteristic set K defined by

$$K := \bigcap_{x^*\in X^*} \left(\text{epi } f^* + \text{cone } \left(\bigcup_{t\in T} \text{epi } h_t^* \right) + \text{epi } \delta_C^* - (x^*, g^*(x^*)) \right).$$

Definition 3.1 ([16], Definition 4.1)**.** The family $(f, g, \delta_C, h_t : t \in T)$ is said to satisfy the closedness condition (CC), iff

$$\text{epi } (f - g + \delta_{\mathcal{A}})^* = K.$$

In order to characterize ε-optimal solution of (P) in terms of this constraint qualification, we need the following lemma.

Lemma 3.2. *$\bar{x} \in \mathcal{A} \cap \text{dom } (f - g)$ is an ε-optimal solution of (P) if and only if*

$$(0, g(\bar{x}) - f(\bar{x}) + \varepsilon) \in \text{epi } (f - g + \delta_{\mathcal{A}})^*.$$

Proof. Note that $\bar{x} \in \mathcal{A}$ is an ε-optimal solution of (P) if and only if for any $x \in X$,

$$(f - g + \delta_{\mathcal{A}})(x) \geq (f - g + \delta_{\mathcal{A}})(\bar{x}) - \varepsilon = f(\bar{x}) - g(\bar{x}) - \varepsilon.$$

Thus, by Lemma 2.1, we know that $\bar{x} \in \mathcal{A}$ is an ε-optimal solution of (P) if and only if

$$(0, g(\bar{x}) - f(\bar{x}) + \varepsilon) \in \text{epi } (f - g + \delta_{\mathcal{A}})^*.$$

The proof is complete. □

The following result provides a new necessary optimality condition for ε-optimal solution of (P) under the constraint qualification introduced in Definition 3.1.

Theorem 3.3. *Let $\bar{x} \in \mathcal{A} \cap \text{dom } (f - g)$. Assume that the family $(f, g, \delta_C, h_t : t \in T)$ satisfies the closedness condition (CC) and $\bar{x}$ is an ε-optimal solution of (P). Then, for any $x^* \in \partial g(\bar{x})$, there exist $\lambda \in \mathbb{R}_+^{(T)}$, η, ζ, $\epsilon_t \geq 0$ as $t \in \text{supp}\lambda$, such that*

$$x^* \in \partial_\eta f(\bar{x}) + \sum_{t\in\text{supp}\lambda} \lambda_t \partial_{\epsilon_t} h_t(\bar{x}) + N_C^\zeta(\bar{x}) \tag{3.1}$$

and

$$\eta + \sum_{t\in\text{supp}\lambda} \lambda_t(\epsilon_t - h_t(\bar{x})) + \zeta = \varepsilon. \tag{3.2}$$

Proof. By Lemma 3.2, we know that $\bar{x}$ is an ε-optimal solution of (P) if and only if

$$(0, g(\bar{x}) - f(\bar{x}) + \varepsilon) \in \text{epi } (f - g + \delta_{\mathcal{A}})^*.$$

Since the family $(f, g, \delta_C, h_t : t \in T)$ satisfies the closedness condition (CC), then

$$\text{epi } (f - g + \delta_{\mathcal{A}})^* = K.$$

Thus, if $\bar{x}$ is an ε-optimal solution of (P), then

$$(0, g(\bar{x}) - f(\bar{x}) + \varepsilon) \in K = \bigcap_{x^*\in X^*} \left(\text{epi } f^* + \text{cone } \left(\bigcup_{t\in T} \text{epi } h_t^* \right) + \text{epi } \delta_C^* - (x^*, g^*(x^*)) \right).$$

Moreover, for any $x^* \in \partial g(\bar{x})$, we have

$$(0, g(\bar{x}) - f(\bar{x}) + \varepsilon) \in \text{epi } f^* + \text{cone} \left(\bigcup_{t \in T} \text{epi } h_t^* \right) + \text{epi } \delta_C^* - (x^*, g^*(x^*)),$$

i.e.,

$$(x^*, g(x^*) + g(\bar{x}) - f(\bar{x}) + \varepsilon) \in \text{epi } f^* + \text{cone} \left(\bigcup_{t \in T} \text{epi } h_t^* \right) + \text{epi } \delta_C^*.$$

Then, there exist $\lambda \in \mathbb{R}_+^{(T)}$, $(u^*, \alpha_1) \in \text{epi } f^*$, $(w^*, \alpha_2) \in \text{epi } \delta_C^*$ and $(v_t^*, \beta_t) \in \text{epi } h_t^*$ with $t \in \text{supp } \lambda$ such that

$$(x^*, g^*(x^*) + g(\bar{x}) - f(\bar{x}) + \varepsilon) = (u^*, \alpha_1) + \sum_{t \in \text{supp } \lambda} \lambda_t (v_t^*, \beta_t) + (w^*, \alpha_2),$$

which means that

$$x^* = u^* + \sum_{t \in \text{supp } \lambda} \lambda_t v_t^* + w^* \tag{3.3}$$

and

$$g^*(x^*) + g(\bar{x}) - f(\bar{x}) + \varepsilon = \alpha_1 + \alpha_2 + \sum_{t \in \text{supp } \lambda} \lambda_t \beta_t. \tag{3.4}$$

Since

$$\begin{aligned} \text{epi } f^* &= \bigcup_{\eta \geq 0} \{(u^*, u^*(\bar{x}) + \eta - f(\bar{x})) : u^* \in \partial_\eta f(\bar{x})\}, \\ \text{epi } h_t^* &= \bigcup_{\epsilon_t \geq 0} \{(v_t^*, v_t^*(\bar{x}) + \epsilon_t - h_t(\bar{x})) : v_t^* \in \partial_{\epsilon_t} h_t(\bar{x})\} \end{aligned}$$

and

$$\text{epi } \delta_C^* = \bigcup_{\zeta \geq 0} \{(w^*, w^*(\bar{x}) + \zeta) : w^* \in \partial_\zeta \delta_C(\bar{x})\},$$

there exist η, ζ, $\epsilon_t \geq 0$ such that

$$u^* \in \partial_\eta f(\bar{x}) \text{ and } \alpha_1 = u^*(\bar{x}) + \eta - f(\bar{x}),$$

$$v_t^* \in \partial_{\epsilon_t} h_t(\bar{x}) \text{ and } \beta_t = v_t^*(\bar{x}) + \epsilon_t - h_t(\bar{x}),$$

$$w^* \in \partial_\zeta \delta_C(\bar{x}) \text{ and } \alpha_2 = w^*(\bar{x}) + \zeta.$$

By (3.3), we get

$$x^* \in \partial_\eta f(\bar{x}) + \sum_{t \in \text{supp}\lambda} \lambda_t \partial_{\epsilon_t} h_t(\bar{x}) + N_C^\zeta(\bar{x}).$$

Moreover, by (3.3) and (3.4), we get

$$\begin{aligned} g^*(x^*) + g(\bar{x}) - f(\bar{x}) + \varepsilon &= \alpha_1 + \alpha_2 + \sum_{t \in \text{supp } \lambda} \lambda_t \beta_t \\ &= u^*(\bar{x}) + \eta - f(\bar{x}) + w^*(\bar{x}) + \zeta + \sum_{t \in \text{supp } \lambda} \lambda_t (v_t^*(\bar{x}) + \epsilon_t - h_t(\bar{x})) \\ &= x^*(\bar{x}) + \eta + \sum_{t \in \text{supp}\lambda} \lambda_t (\epsilon_t - h_t(\bar{x})) + \zeta - f(\bar{x}). \end{aligned}$$

Since $x^* \in \partial g(\bar{x})$, we have $g^*(x^*) + g(\bar{x}) = x^*(\bar{x})$. Therefore,

$$\eta + \sum_{t\in \text{supp}\lambda} \lambda_t(\epsilon_t - h_t(\bar{x})) + \zeta = \varepsilon.$$

This completes the proof. □

The following corollary establishes a necessary optimality conditions for exact optimal solution of (P). It is important to note that the result obtained is a refinement of the corresponding conditions established recently in [4, 5] under a more restrictive constraint qualification and a different method.

Corollary 3.4. *Let $\bar{x} \in \mathcal{A} \cap \text{dom } (f - g)$. Assume that the family $(f, g, \delta_C, h_t : t \in T)$ satisfies the closedness condition (CC) and $\bar{x}$ is an exact optimal solution of (P). Then, for any $x^* \in \partial g(\bar{x})$, there exists $\lambda \in \mathbb{R}_+^{(T)}$, such that*

$$x^* \in \partial f(\bar{x}) + \sum_{t\in \text{supp}\lambda} \lambda_t \partial h_t(\bar{x}) + N_C(\bar{x})$$

and

$$\lambda_t h_t(\bar{x}) = 0, t \in \text{supp}\lambda.$$

Proof. Take $\varepsilon = 0$ in Theorem 3.3. It is easy to see that if $\bar{x}$ is an exact optimal solution of (P), then, for any $x^* \in \partial g(\bar{x})$, there exist $\lambda \in \mathbb{R}_+^{(T)}$, η, ζ, $\epsilon_t \geq 0$ as $t \in \text{supp}\lambda$ such that

$$x^* \in \partial_\eta f(\bar{x}) + \sum_{t\in \text{supp}\lambda} \lambda_t \partial_{\epsilon_t} h_t(\bar{x}) + N_C^\zeta(\bar{x})$$

and

$$\eta + \sum_{t\in \text{supp}\lambda} \lambda_t(\epsilon_t - h_t(\bar{x})) + \zeta = 0. \tag{3.5}$$

Form $\lambda \in \mathbb{R}_+^{(T)}$, η, ζ, $\epsilon_t \geq 0$, $h_t(\bar{x}) \leq 0$ and (3.5), we get

$$\eta = \epsilon_t = -(\lambda_t h_t)(\bar{x}) = \zeta = 0, \text{ for } t \in \text{supp}\lambda.$$

Then, this proof is obtained by Theorem 3.3. □

Finally, in this section, we consider a particular case of the DC problem (P) with $g(x) = 0$ when (P) reduces to the following convex infinite programming problem

$$(P_0) \qquad \begin{cases} \inf f(x) \\ \text{ s.t. } h_t(x) \leq 0, \ t \in T, \text{ and } x \in C. \end{cases}$$

Since the function $g = 0$, the characteristic set K becomes

$$\text{epi } f^* + \text{cone} \left(\bigcup_{t\in T} \text{epi } h_t^* \right) + \text{epi } \delta_C^*.$$

Definition 3.5 ([8], Definition 3.1)**.** The family $(f, \delta_C, h_t : t \in T)$ is said to satisfy the closedness condition $(CC)_0$, iff

$$\text{epi } (f + \delta_{\mathcal{A}})^* = \text{epi } f^* + \text{cone} \left(\bigcup_{t\in T} \text{epi } h_t^* \right) + \text{epi } \delta_C^*.$$

The next theorem shows that the specification of conditions (3.1) and (3.2) in this case is an necessary and sufficient optimality conditions for (P_0) under the closedness condition $(CC)_0$.

Theorem 3.6. *Let* $\bar{x} \in \mathcal{A} \cap \operatorname{dom} f$. *Assume that the family* $(f, \delta_C, h_t : t \in T)$ *satisfies the closedness condition* $(CC)_0$. *Then,* $\bar{x}$ *is an* ε*-optimal solution of* (P_0) *if and only if there exist* $\lambda \in \mathbb{R}_+^{(T)}$, η, ζ, $\epsilon_t \geq 0$ *as* $t \in \operatorname{supp}\lambda$ *such that*

$$0 \in \partial_\eta f(\bar{x}) + \sum_{t \in \operatorname{supp}\lambda} \lambda_t \partial_{\epsilon_t} h_t(\bar{x}) + N_C^\zeta(\bar{x})$$

and

$$\eta + \sum_{t \in \operatorname{supp}\lambda} \lambda_t(\epsilon_t - h_t(\bar{x})) + \zeta = \varepsilon.$$

Proof. We only need to prove the sufficient condition, since the necessary condition can be obtained easily by $g \equiv 0$ in Theorem 3.3. Suppose that there exist $\lambda \in \mathbb{R}_+^{(T)}$, η, ζ, $\epsilon_t \geq 0$ as $t \in \operatorname{supp}\lambda$ such that

$$0 \in \partial_\eta f(\bar{x}) + \sum_{t \in \operatorname{supp}\lambda} \lambda_t \partial_{\epsilon_t} h_t(\bar{x}) + N_C^\zeta(\bar{x}) \tag{3.6}$$

and

$$\eta + \sum_{t \in \operatorname{supp}\lambda} \lambda_t(\epsilon_t - h_t(\bar{x})) + \zeta = \varepsilon. \tag{3.7}$$

By (3.6), there exist $u^* \in \partial_\eta f(\bar{x})$, $v_t^* \in \partial_{\epsilon_t} h_t(\bar{x})$ and $w^* \in N_C^\zeta(\bar{x})$ such that

$$u^* + \sum_{t \in \operatorname{supp}\,\lambda} \lambda_t v_t^* + w^* = 0. \tag{3.8}$$

Let $x \in \mathcal{A}$ be arbitrary. Then, by the definitions of subdifferential and $N_C^\zeta(\bar{x})$,

$$\begin{aligned} f(x) - f(\bar{x}) &\geq u^*(x - \bar{x}) - \eta, \\ h_t(x) - h_t(\bar{x}) &\geq v_t^*(x - \bar{x}) - \epsilon_t, \\ 0 &\geq w^*(x - \bar{x}) - \zeta. \end{aligned}$$

Together with these inequalities, we get

$$\begin{aligned} f(x) - f(\bar{x}) &\geq f(x) - f(\bar{x}) + \sum_{t \in \operatorname{supp}\,\lambda} \lambda_t h_t(x) - \sum_{t \in \operatorname{supp}\,\lambda} \lambda_t h_t(\bar{x}) + \sum_{t \in \operatorname{supp}\,\lambda} \lambda_t h_t(\bar{x}) \\ &\geq \left(u^* + \sum_{t \in \operatorname{supp}\,\lambda} \lambda_t v_t^* + w^* \right)(x - \bar{x}) - \left(\eta + \sum_{t \in \operatorname{supp}\lambda} \lambda_t(\epsilon_t - h_t(\bar{x})) + \zeta \right). \end{aligned}$$

By (3.7) and (3.8), we get

$$f(x) - f(\bar{x}) \geq -\varepsilon.$$

which means that $\bar{x}$ is an ε-optimal solution of (P_0), and the proof is complete. □

Similarly, we can easily get the following result for exact optimal solution of (P_0). Let us mention that it extends the recent result in [4, 5, 9] derived by a different constraint qualification and a different method.

Corollary 3.7. *Let* $\bar{x} \in \mathcal{A} \cap \operatorname{dom} f$. *Assume that the family* $(f, \delta_C, h_t : t \in T)$ *satisfies the closedness condition* $(CC)_0$. *Then,* $\bar{x}$ *is an exact optimal solution of* (P_0) *if and only if there exists* $\lambda \in \mathbb{R}_+^{(T)}$, *such that*

$$0 \in \partial f(\bar{x}) + \sum_{t \in \operatorname{supp}\lambda} \lambda_t \partial h_t(\bar{x}) + N_C(\bar{x})$$

and

$$\lambda_t h_t(\bar{x}) = 0, t \in \operatorname{supp}\lambda.$$

4. Applications

In this section, let X and Y be real locally convex Hausdorff topological vector spaces, $C \subseteq X$ be a nonempty convex set. Let $S \subseteq Y$ be a nonempty closed convex cone which defined the partial order of Y, namely: $y_1 \leq_S y_2 \Longleftrightarrow y_2 - y_1 \in S$, for any $y_1, y_2 \in Y$. We attach an element $+\infty \notin Y$ which is a greatest element with respect to " $\leq_S$" and let $Y^\bullet = Y \cup \{+\infty\}$. The following operations are defined on $Y^\bullet$: $y + (+\infty) = (+\infty) + y = +\infty$ and $t(+\infty) = +\infty$, for any $y \in Y$ and $t \geq 0$. Let $f, g : X \to \overline{\mathbb{R}}$ be two proper convex functions, and $h : X \to Y^\bullet$ be a proper S-convex function. For any $p \in X^*$, we consider the following conic programming problem:

$$(P_1) \qquad \begin{cases} \inf \ \{f(x) - g(x)\} \\ \text{ s.t. } h(x) \in -S, \text{ and } x \in C. \end{cases}$$

Pursuing the approach given in [16, Section 5], the problem (P_1) can be viewed as an example of (P). We also use $\mathcal{A}$ to denote the solution set:

$$\mathcal{A} := \{x \in C : (\lambda h)(x) \leq 0, \forall \lambda \in S^*\} = \{x \in C : h(x) \in -S\}.$$

And the characteristic set K introduced in Section 3 becomes

$$K_1 := \bigcap_{x^* \in X^*} \left(\text{epi } f^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^* - (x^*, g(x^*)) \right).$$

Definition 4.1 ([16], Definition 5.1)**.** The family (f, g, h, δ_C) is said to satisfy the closedness condition $(CC)_1$, iff

$$\text{epi } (f - g + \delta_{\mathcal{A}})^* = K_1.$$

By using the similar methods of Sections 3, we can also use this constraint qualification to establish the corresponding results of the problem (P_1).

Theorem 4.2. *Let $\bar{x} \in \mathcal{A} \cap \text{dom } (f - g)$. Assume that the family (f, g, h, δ_C) satisfies the closedness condition $(CC)_1$ and $\bar{x}$ is an ε-optimal solution of (P_1). Then, for any $x^* \in \partial g(\bar{x})$, there exist $\lambda \in S^*$, η, ζ, $\epsilon \geq 0$ such that*

$$x^* \in \partial_\eta f(\bar{x}) + \partial_\epsilon (\lambda h)(\bar{x}) + N_C^\zeta(\bar{x})$$

and

$$\eta + \epsilon - (\lambda h)(\bar{x}) + \zeta = \varepsilon.$$

Corollary 4.3. *Let $\bar{x} \in \mathcal{A} \cap \text{dom } (f - g)$. Assume that the family (f, g, h, δ_C) satisfies the closedness condition $(CC)_1$ and $\bar{x}$ is an exact optimal solution of (P_1). Then, for any $x^* \in \partial g(\bar{x})$, there exists $\lambda \in S^*$, such that*

$$x^* \in \partial f(\bar{x}) + \partial(\lambda h)(\bar{x}) + N_C(\bar{x})$$

and

$$(\lambda h)(\bar{x}) = 0.$$

Finally, in this section, we consider a particular case of the problem (P_1) with $g(x) = 0$ when (P_1) reduces to the following convex programming problem

$$(P_2) \qquad \begin{cases} \inf f(x) \\ \text{ s.t. } h(x) \in -S, \text{ and } x \in C. \end{cases}$$

Since the function $g = 0$, the characteristic set K becomes

$$\text{epi } f^* + \bigcup_{\lambda \in S^*} \text{epi } (\lambda h)^* + \text{epi } \delta_C^*.$$

Definition 4.4 ([16], Definition 5.2)**.** The family (f, h, δ_C) is said to satisfy the closedness condition $(CC)_2$, iff

$$\text{epi } (f+\delta_{\mathcal{A}})^* = \text{epi } f^* + \bigcup_{\lambda\in S^*} \text{epi}(\lambda h)^* + \text{epi } \delta_C^*.$$

Theorem 4.5. *Let $\bar{x} \in \mathcal{A} \cap \text{dom } f$. Assume that the family (f, h, δ_C) satisfies the closedness condition $(CC)_2$. Then, $\bar{x}$ is an ε-optimal solution of (P_2) if and only if there exist $\lambda \in S^*$, η, ζ, $\epsilon \geq 0$ such that*

$$0 \in \partial_\eta f(\bar{x}) + \partial_\epsilon(\lambda h)(\bar{x}) + N_C^\zeta(\bar{x})$$

and

$$\eta + \epsilon - (\lambda h)(\bar{x}) + \zeta = \varepsilon.$$

Corollary 4.6. *Let $\bar{x} \in \mathcal{A} \cap \text{dom } f$. Assume that the family (f, h, δ_C) satisfies the closedness condition $(CC)_2$. Then, $\bar{x}$ is an exact optimal solution of (P_2) if and only if there exists $\lambda \in S^*$, such that*

$$0 \in \partial f(\bar{x}) + \partial(\lambda h)(\bar{x}) + N_C(\bar{x})$$

and

$$(\lambda h)(\bar{x}) = 0.$$

5. Conclusions

In this paper, we consider a DC infinite programming problem with inequality constraints. By using the properties of the epigraph of the conjugate functions, we first recall some notions of constraint qualifications for the DC infinite programming problem. Then, we establish some necessary optimality conditions for the ε-optimal solutions and the exact optimal solution of the DC infinite programming problem. As an special case, we also obtain some optimality conditions for convex infinite programming problems. Moreover, as applications, we obtain the corresponding results for conic programming problems.

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