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On the solutions and periodicity of some nonlinear systems of difference equations

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Abstract

We investigate the expressions of solutions and the periodicity nature of the following system of rational difference equations of order four

$$x_{n+1} = \frac{z_{n-3}}{a_1 + b_1 z_n y_{n-1} x_{n-2} z_{n-3}},\ y_{n+1} = \frac{x_{n-3}}{a_2 + b_2 x_n z_{n-1} y_{n-2} x_{n-3}},$$
$$z_{n+1} = \frac{y_{n-3}}{a_3 + b_3 y_n x_{n-1} z_{n-2} y_{n-3}},$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1},\ y_0,\ z_{-3},\ z_{-2},\ z_{-1}$ and z_0 are arbitrary real numbers and $a_1, b_1, a_2, b_2, a_3, b_3$ are integers.

Keywords: System of difference equations, recursive sequences, stability, periodic solution, solution of difference equation.
2010 MSC: 39A10, 39A22, 39A23.

1. Introduction

Our goal is to obtain a form of the solutions and the periodicity character of the systems of rational difference equations

$$x_{n+1} = \frac{z_{n-3}}{a_1 + b_1 z_n y_{n-1} x_{n-2} z_{n-3}},\ y_{n+1} = \frac{x_{n-3}}{a_2 + b_2 x_n z_{n-1} y_{n-2} x_{n-3}},$$

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$$z_{n+1} = \frac{y_{n-3}}{a_3 + b_3 y_n x_{n-1} z_{n-2} y_{n-3}},$$

where the initial conditions are arbitrary real numbers. Also, the constants a_i, b_i, $i = 1, 2, 3$ are integer numbers. Throughout this paper, we assume that $A = x_{-3}y_{-2}z_{-1}x_0$, $B = z_{-3}x_{-2}y_{-1}z_0$ and $C = y_{-3}z_{-2}x_{-1}y_0$.

In recent years, with the wide application of computers, difference system has become one of the important theoretical bases for computer, information system, ecological balance, engineering control, biological system, economical systems, and so forth. As typical nonlinear difference equations, rational difference equations have become a research hot spot in mathematical modelling. The behavior of solutions of systems of rational difference equations has received extensive attention. Recently a great effort has been made in studying the qualitative analysis of rational difference equations and their systems; see [1]-[39].

Din et al. [9] studied the equilibrium points, local asymptotic stability of an equilibrium point, instability of equilibrium points, periodicity behavior of positive solutions, and global character of an equilibrium point of the system of rational difference equations

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}},\ y_{n+1} = \frac{\alpha_1 y_{n-3}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}.$$

Qianhong Zhang et al. [37] obtained the boundedness, persistence, and global asymptotic stability of positive solution of the system of third-order rational difference equations

$$x_{n+1} = A + \frac{x_n}{y_{n-1} y_{n-2}},\ y_{n+1} = B + \frac{y_n}{x_{n-1} x_{n-2}}.$$

El-Dessoky et al. [14] got the form of the solution of the following system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + y_n x_{n-1}},\ y_{n+1} = \frac{y_{n-1}}{1 + x_n y_{n-1}},\ z_{n+1} = \frac{z_{n-m}}{x_n y_n}.$$

In [29], Papaschinopoulos et al. studied the existence of a unique positive equilibrium, the boundedness, persistence and global attractivity of the positive solutions of the system of the following two difference equations

$$x_{n+1} = a x_n + b y_{n-1} e^{-x_n},\quad y_{n+1} = c y_n + d x_{n-1} e^{-y_n}.$$

Hui-li Ma et al [26] investigated the behavior of positive solutions for the following systems of rational difference equations

$$x_{n+1} = \frac{A}{x_n y_n^2},\ y_{n+1} = \frac{B y_n}{x_{n-1} y_{n-1}}.$$

Özban [28] investigated the periodicity of solutions of the system of difference equations

$$x_{n+1} = \frac{1}{y_{n-k}},\ y_{n+1} = \frac{y_n}{x_{n-m} y_{n-m-k}}.$$

Touafek et al. [33] investigated the periodic nature and got the form of the solutions of the following systems of rational difference equations

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3} y_{n-1}},\ y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3} x_{n-1}}.$$

Banyat Sroysang [30] studied the solutions, stability character, and asymptotic behavior of the system of a rational m-th order difference equation

$$x_{n+1} = \frac{x_{n-m+1}}{A + y_n y_{n-1} \ldots, y_{n-m+1}},\ y_{n+1} = \frac{y_{n-m+1}}{B + x_n x_{n-1} \ldots x_{n-m+1}}.$$

The behavior of positive solutions of the following system

$$x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1}y_n}, \ y_{n+1} = \frac{y_{n-1}}{1 + y_{n-1}x_n}$$

was studied by Kurbanli et al. [24].

In the same year, Kurbanli [22] studied the behavior of the solutions of the system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1}y_n - 1}, \ y_{n+1} = \frac{y_{n-1}}{y_{n-1}x_n - 1}, \ z_{n+1} = \frac{z_{n-1}}{z_{n-1}y_n - 1}.$$

Liu et al. [25] studied the behavior of the solutions of the system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1}y_n - 1}, \ y_{n+1} = \frac{y_{n-1}}{y_{n-1}x_n - 1}, \ z_{n+1} = \frac{1}{z_{n-1}x_n}.$$

In [5] Cinar obtained the periodicity of the positive solutions of the system of rational difference equations

$$x_{n+1} = \frac{1}{y_n}, \ y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}.$$

Moreover, the system of rational difference equations

$$x_{n+1} = \frac{x_n}{cy_n + a}, \ y_{n+1} = \frac{y_n}{dx_n + b},$$

was studied by Clark et al. [6, 7].

2. The First Case $a_i = b_i = 1, \ i = 1, \ 2, \ 3$

We investigate, in this section, the solutions of the following system of three difference equations

$$x_{n+1} = \frac{z_{n-3}}{1 + z_n y_{n-1} x_{n-2} z_{n-3}}, \ y_{n+1} = \frac{x_{n-3}}{1 + x_n z_{n-1} y_{n-2} x_{n-3}}, \ z_{n+1} = \frac{y_{n-3}}{1 + y_n x_{n-1} z_{n-2} y_{n-3}}, \tag{2.1}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers.

Theorem 2.1. *Assume that* $\{x_n, y_n, z_n\}$ *are solutions of system* (2.1). *Then for* $n = 0, 1, 2, \ldots,$ *all the solutions of system* (2.1) *are given by the following formulae*

$$x_{12n-3} = x_{-3} \prod_{i=0}^{n-1} \frac{(1+(12i)A)(1+(12i+4)A)(1+(12i+8)A)}{(1+(12i+1)A)(1+(12i+5)A)(1+(12i+9)A)},$$

$$x_{12n-2} = x_{-2} \prod_{i=0}^{n-1} \frac{(1+(12i+1)B)(1+(12i+5)B)(1+(12i+9)B)}{(1+(12i+2)B)(1+(12i+6)B)(1+(12i+10)B)},$$

$$x_{12n-1} = x_{-1} \prod_{i=0}^{n-1} \frac{(1+(12i+2)C)(1+(12i+6)C)(1+(12i+10)C)}{(1+(12i+3)C)(1+(12i+7)C)(1+(12i+11)C)},$$

$$x_{12n} = x_0 \prod_{i=0}^{n-1} \frac{(1+(12i+3)A)(1+(12i+7)A)(1+(12i+11)A)}{(1+(12i+4)A)(1+(12i+8)A)(1+(12i+12)A)},$$

$$x_{12n+1} = \frac{z_{-3}}{1+B} \prod_{i=0}^{n-1} \frac{(1+(12i+4)B)(1+(12i+8)B)(1+(12i+12)B)}{(1+(12i+5)B)(1+(12i+9)B)(1+(12i+13)B)},$$

$$x_{12n+2} = \frac{z_{-2}(1+C)}{(1+2C)} \prod_{i=0}^{n-1} \frac{(1+(12i+5)C)(1+(12i+9)C)(1+(12i+13)C)}{(1+(12i+6)C)(1+(12i+10)C)(1+(12i+14)C)},$$

$$x_{12n+3} = \frac{z_{-1}(1+2A)}{(1+3A)} \prod_{i=0}^{n-1} \frac{(1+(12i+6)A)(1+(12i+10)A)(1+(12i+14)A)}{(1+(12i+7)A)(1+(12i+11)A)(1+(12i+15)A)},$$

$$x_{12n+4} = \frac{z_{0}(1+3B)}{(1+4B)} \prod_{i=0}^{n-1} \frac{(1+(12i+7)B)(1+(12i+11)B)(1+(12i+15)B)}{(1+(12i+8)B)(1+(12i+12)B)(1+(12i+16)B)},$$

$$x_{12n+5} = \frac{y_{-3}(1+4C)}{(1+C)(1+5C)} \prod_{i=0}^{n-1} \frac{(1+(12i+8)C)(1+(12i+12)C)(1+(12i+16)C)}{(1+(12i+9)C)(1+(12i+13)C)(1+(12i+17)C)},$$

$$x_{12n+6} = \frac{y_{-2}(1+A)(1+5A)}{(1+2A)(1+6A)} \prod_{i=0}^{n-1} \frac{(1+(12i+9)A)(1+(12i+13)A)(1+(12i+17)A)}{(1+(12i+10)A)(1+(12i+14)A)(1+(12i+18)A)},$$

$$x_{12n+7} = \frac{y_{-1}(1+2B)(1+6B)}{(1+3B)(1+7B)} \prod_{i=0}^{n-1} \frac{(1+(12i+10)B)(1+(12i+14)B)(1+(12i+18)B)}{(1+(12i+11)B)(1+(12i+15)B)(1+(12i+19)B)},$$

$$x_{12n+8} = \frac{y_{0}(1+3C)(1+7C)}{(1+4C)(1+8C)} \prod_{i=0}^{n-1} \frac{(1+(12i+11)C)(1+(12i+15)C)(1+(12i+19)C)}{(1+(12i+12)C)(1+(12i+16)C)(1+(12i+20)C)},$$

$$y_{12n-3} = y_{-3} \prod_{i=0}^{n-1} \frac{(1+(12i)C)(1+(12i+4)C)(1+(12i+8)C)}{(1+(12i+1)C)(1+(12i+5)C)(1+(12i+9)C)},$$

$$y_{12n-2} = y_{-2} \prod_{i=0}^{n-1} \frac{(1+(12i+1)A)(1+(12i+5)A)(1+(12i+9)A)}{(1+(12i+2)A)(1+(12i+6)A)(1+(12i+10)A)},$$

$$y_{12n-1} = y_{-1} \prod_{i=0}^{n-1} \frac{(1+(12i+2)B)(1+(12i+6)B)(1+(12i+10)B)}{(1+(12i+3)B)(1+(12i+7)B)(1+(12i+11)B)},$$

$$y_{12n} = y_{0} \prod_{i=0}^{n-1} \frac{(1+(12i+3)C)(1+(12i+7)C)(1+(12i+11)C)}{(1+(12i+4)C)(1+(12i+8)C)(1+(12i+12)C)},$$

$$y_{12n+1} = \frac{x_{-3}}{1+A} \prod_{i=0}^{n-1} \frac{(1+(12i+4)A)(1+(12i+8)A)(1+(12i+12)A)}{(1+(12i+5)A)(1+(12i+9)A)(1+(12i+13)A)},$$

$$y_{12n+2} = \frac{x_{-2}(1+B)}{(1+2B)} \prod_{i=0}^{n-1} \frac{(1+(12i+5)B)(1+(12i+9)B)(1+(12i+13)B)}{(1+(12i+6)B)(1+(12i+10)B)(1+(12i+14)B)},$$

$$y_{12n+3} = \frac{x_{-1}(1+2C)}{(1+3C)} \prod_{i=0}^{n-1} \frac{(1+(12i+6)C)(1+(12i+10)C)(1+(12i+14)C)}{(1+(12i+7)C)(1+(12i+11)C)(1+(12i+15)C)},$$

$$y_{12n+4} = \frac{x_{0}(1+3A)}{(1+4A)} \prod_{i=0}^{n-1} \frac{(1+(12i+7)A)(1+(12i+11)A)(1+(12i+15)A)}{(1+(12i+8)A)(1+(12i+12)A)(1+(12i+16)A)},$$

$$y_{12n+5} = \frac{z_{-3}(1+4B)}{(1+B)(1+5B)} \prod_{i=0}^{n-1} \frac{(1+(12i+8)B)(1+(12i+12)B)(1+(12i+16)B)}{(1+(12i+9)B)(1+(12i+13)B)(1+(12i+17)B)},$$

$$y_{12n+6} = \frac{z_{-2}(1+C)(1+5C)}{(1+2C)(1+6C)} \prod_{i=0}^{n-1} \frac{(1+(12i+9)C)(1+(12i+13)C)(1+(12i+17)C)}{(1+(12i+10)C)(1+(12i+14)C)(1+(12i+18)C)},$$

$$y_{12n+7} = \frac{z_{-1}(1+2A)(1+6A)}{(1+3A)(1+7A)} \prod_{i=0}^{n-1} \frac{(1+(12i+10)A)(1+(12i+14)A)(1+(12i+18)A)}{(1+(12i+11)A)(1+(12i+15)A)(1+(12i+19)A)},$$

$$y_{12n+8} = \frac{z_{0}(1+3B)(1+7B)}{(1+4B)(1+8B)} \prod_{i=0}^{n-1} \frac{(1+(12i+11)B)(1+(12i+15)B)(1+(12i+19)B)}{(1+(12i+12)B)(1+(12i+16)B)(1+(12i+20)B)},$$

$$z_{12n-3} = z_{-3}\prod_{i=0}^{n-1}\frac{(1+(12i)B)(1+(12i+4)B)(1+(12i+8)B)}{(1+(12i+1)B)(1+(12i+5)B)(1+(12i+9)B)},$$

$$z_{12n-2} = z_{-2}\prod_{i=0}^{n-1}\frac{(1+(12i+1)C)(1+(12i+5)C)(1+(12i+9)C)}{(1+(12i+2)C)(1+(12i+6)C)(1+(12i+10)C)},$$

$$z_{12n-1} = z_{-1}\prod_{i=0}^{n-1}\frac{(1+(12i+2)A)(1+(12i+6)A)(1+(12i+10)A)}{(1+(12i+3)A)(1+(12i+7)A)(1+(12i+11)A)},$$

$$z_{12n} = z_{0}\prod_{i=0}^{n-1}\frac{(1+(12i+3)B)(1+(12i+7)B)(1+(12i+11)B)}{(1+(12i+4)B)(1+(12i+8)B)(1+(12i+12)B)},$$

$$z_{12n+1} = \frac{y_{-3}}{1+C}\prod_{i=0}^{n-1}\frac{(1+(12i+4)C)(1+(12i+8)C)(1+(12i+12)C)}{(1+(12i+5)C)(1+(12i+9)C)(1+(12i+13)C)},$$

$$z_{12n+2} = \frac{y_{-2}(1+A)}{(1+2A)}\prod_{i=0}^{n-1}\frac{(1+(12i+5)A)(1+(12i+9)A)(1+(12i+13)A)}{(1+(12i+6)A)(1+(12i+10)A)(1+(12i+14)A)},$$

$$z_{12n+3} = \frac{y_{-1}(1+2B)}{(1+3B)}\prod_{i=0}^{n-1}\frac{(1+(12i+6)B)(1+(12i+10)B)(1+(12i+14)B)}{(1+(12i+7)B)(1+(12i+11)B)(1+(12i+15)B)},$$

$$z_{12n+4} = \frac{y_{0}(1+3C)}{(1+4C)}\prod_{i=0}^{n-1}\frac{(1+(12i+7)C)(1+(12i+11)C)(1+(12i+15)C)}{(1+(12i+8)C)(1+(12i+12)C)(1+(12i+16)C)},$$

$$z_{12n+5} = \frac{x_{-3}(1+4A)}{(1+A)(1+5A)}\prod_{i=0}^{n-1}\frac{(1+(12i+8)A)(1+(12i+12)A)(1+(12i+16)A)}{(1+(12i+9)A)(1+(12i+13)A)(1+(12i+17)A)},$$

$$z_{12n+6} = \frac{x_{-2}(1+B)(1+5B)}{(1+2B)(1+6B)}\prod_{i=0}^{n-1}\frac{(1+(12i+9)B)(1+(12i+13)B)(1+(12i+17)B)}{(1+(12i+10)B)(1+(12i+14)B)(1+(12i+18)B)},$$

$$z_{12n+7} = \frac{x_{-1}(1+2C)(1+6C)}{(1+3C)(1+7C)}\prod_{i=0}^{n-1}\frac{(1+(12i+10)C)(1+(12i+14)C)(1+(12i+18)C)}{(1+(12i+11)C)(1+(12i+15)C)(1+(12i+19)C)},$$

$$z_{12n+8} = \frac{x_{0}(1+3A)(1+7A)}{(1+4A)(1+8A)}\prod_{i=0}^{n-1}\frac{(1+(12i+11)A)(1+(12i+15)A)(1+(12i+19)A)}{(1+(12i+12)A)(1+(12i+16)A)(1+(12i+20)A)},$$

where $\prod_{i=0}^{-1} t_i = 1$ *and* $A,\ B,\ C \notin \{-\frac{1}{n},\ n = 1,\ 2,\ \dots\}$.

Proof. For $n = 0$, the assertion holds. Now suppose that $n > 0$ and that our assertion holds for $n - 1$ that is,

$$x_{12n-7} = \frac{y_{-3}(1+4C)}{(1+C)(1+5C)}\prod_{i=0}^{n-2}\frac{(1+(12i+8)C)(1+(12i+12)C)(1+(12i+16)C)}{(1+(12i+9)C)(1+(12i+13)C)(1+(12i+17)C)},$$

$$x_{12n-6} = \frac{y_{-2}(1+A)(1+5A)}{(1+2A)(1+6A)}\prod_{i=0}^{n-2}\frac{(1+(12i+9)A)(1+(12i+13)A)(1+(12i+17)A)}{(1+(12i+10)A)(1+(12i+14)A)(1+(12i+18)A)},$$

$$x_{12n-5} = \frac{y_{-1}(1+2B)(1+6B)}{(1+3B)(1+7B)}\prod_{i=0}^{n-2}\frac{(1+(12i+10)B)(1+(12i+14)B)(1+(12i+18)B)}{(1+(12i+11)B)(1+(12i+15)B)(1+(12i+19)B)},$$

$$x_{12n-4} = \frac{y_{0}(1+3C)(1+7C)}{(1+4C)(1+8C)}\prod_{i=0}^{n-2}\frac{(1+(12i+11)C)(1+(12i+15)C)(1+(12i+19)C)}{(1+(12i+12)C)(1+(12i+16)C)(1+(12i+20)C)},$$

$$y_{12n-7} = \frac{z_{-3}(1+4B)}{(1+B)(1+5B)} \prod_{i=0}^{n-2} \frac{(1+(12i+8)B)(1+(12i+12)B)(1+(12i+16)B)}{(1+(12i+9)B)(1+(12i+13)B)(1+(12i+17)B)},$$

$$y_{12n-6} = \frac{z_{-2}(1+C)(1+5C)}{(1+2C)(1+6C)} \prod_{i=0}^{n-2} \frac{(1+(12i+9)C)(1+(12i+13)C)(1+(12i+17)C)}{(1+(12i+10)C)(1+(12i+14)C)(1+(12i+18)C)},$$

$$y_{12n-5} = \frac{z_{-1}(1+2A)(1+6A)}{(1+3A)(1+7A)} \prod_{i=0}^{n-2} \frac{(1+(12i+10)A)(1+(12i+14)A)(1+(12i+18)A)}{(1+(12i+11)A)(1+(12i+15)A)(1+(12i+19)A)},$$

$$y_{12n-4} = \frac{z_{0}(1+3B)(1+7B)}{(1+4B)(1+8B)} \prod_{i=0}^{n-2} \frac{(1+(12i+11)B)(1+(12i+15)B)(1+(12i+19)B)}{(1+(12i+12)B)(1+(12i+16)B)(1+(12i+20)B)},$$

$$z_{12n-7} = \frac{x_{-3}(1+4A)}{(1+A)(1+5A)} \prod_{i=0}^{n-2} \frac{(1+(12i+8)A)(1+(12i+12)A)(1+(12i+16)A)}{(1+(12i+9)A)(1+(12i+13)A)(1+(12i+17)A)},$$

$$z_{12n-6} = \frac{x_{-2}(1+B)(1+5B)}{(1+2B)(1+6B)} \prod_{i=0}^{n-2} \frac{(1+(12i+9)B)(1+(12i+13)B)(1+(12i+17)B)}{(1+(12i+10)B)(1+(12i+14)B)(1+(12i+18)B)},$$

$$z_{12n-5} = \frac{x_{-1}(1+2C)(1+6C)}{(1+3C)(1+7C)} \prod_{i=0}^{n-2} \frac{(1+(12i+10)C)(1+(12i+14)C)(1+(12i+18)C)}{(1+(12i+11)C)(1+(12i+15)C)(1+(12i+19)C)},$$

$$z_{12n-4} = \frac{x_{0}(1+3A)(1+7A)}{(1+4A)(1+8A)} \prod_{i=0}^{n-2} \frac{(1+(12i+11)A)(1+(12i+15)A)(1+(12i+19)A)}{(1+(12i+12)A)(1+(12i+16)A)(1+(12i+20)A)}.$$

From (2.1), we see that

$$\begin{aligned}
x_{12n-3} =& \frac{z_{12n-7}}{1+z_{12n-4}y_{12n-5}x_{12n-6}z_{12n-7}} \\
=& \frac{\frac{x_{-3}(1+4A)}{(1+A)(1+5A)} \prod_{i=0}^{n-2} \frac{(1+(12i+8)A)(1+(12i+12)A)(1+(12i+16)A)}{(1+(12i+9)A)(1+(12i+13)A)(1+(12i+17)A)}}{\left(\begin{array}{c} 1+\frac{x_0(1+3A)(1+7A)}{(1+4A)(1+8A)} \prod_{i=0}^{n-2} \frac{(1+(12i+11)A)(1+(12i+15)A)(1+(12i+19)A)}{(1+(12i+12)A)(1+(12i+16)A)(1+(12i+20)A)} \\ \frac{z_{-1}(1+2A)(1+6A)}{(1+3A)(1+7A)} \prod_{i=0}^{n-2} \frac{(1+(12i+10)A)(1+(12i+14)A)(1+(12i+18)A)}{(1+(12i+11)A)(1+(12i+15)A)(1+(12i+19)A)} \\ \frac{y_{-2}(1+A)(1+5A)}{(1+2A)(1+6A)} \prod_{i=0}^{n-2} \frac{(1+(12i+9)A)(1+(12i+13)A)(1+(12i+17)A)}{(1+(12i+10)A)(1+(12i+14)A)(1+(12i+18)A)} \\ \frac{x_{-3}(1+4A)}{(1+A)(1+5A)} \prod_{i=0}^{n-2} \frac{(1+(12i+8)A)(1+(12i+12)A)(1+(12i+16)A)}{(1+(12i+9)A)(1+(12i+13)A)(1+(12i+17)A)} \end{array}\right)} \\
=& \frac{\frac{x_{-3}(1+4A)}{(1+A)(1+5A)} \prod_{i=0}^{n-2} \frac{(1+(12i+8)A)(1+(12i+12)A)(1+(12i+16)A)}{(1+(12i+9)A)(1+(12i+13)A)(1+(12i+17)A)}}{\left(1+\frac{A}{(1+8A)} \prod_{i=0}^{n-2} \frac{(1+(12i+8)A)}{(1+(12i+20)A)}\right)} \\
=& \frac{x_{-3}(1+4A)}{(1+A)(1+5A)\left(1+\frac{A}{(1+(12n-4)A)}\right)} \prod_{i=0}^{n-2} \frac{(1+(12i+8)A)(1+(12i+12)A)(1+(12i+16)A)}{(1+(12i+9)A)(1+(12i+13)A)(1+(12i+17)A)} \\
=& \frac{x_{-3}(1+4A)(1+(12n-4)A)}{(1+A)(1+5A)(1+(12n-3)A)} \prod_{i=0}^{n-2} \frac{(1+(12i+8)A)(1+(12i+12)A)(1+(12i+16)A)}{(1+(12i+9)A)(1+(12i+13)A)(1+(12i+17)A)} \\
=& x_{-3} \prod_{i=0}^{n-1} \frac{(1+(12i)A)(1+(12i+4)A)(1+(12i+8)A)}{(1+(12i+1)A)(1+(12i+5)A)(1+(12i+9)A)}.
\end{aligned}$$

$$y_{12n-1} = \frac{x_{12n-5}}{1+x_{12n-2}z_{12n-3}y_{12n-4}x_{12n-5}}$$

$$=\frac{\frac{y_{-1}(1+2B)(1+6B)}{(1+3B)(1+7B)}\prod_{i=0}^{n-2}\frac{(1+(12i+10)B)(1+(12i+14)B)(1+(12i+18)B)}{(1+(12i+11)B)(1+(12i+15)B)(1+(12i+19)B)}}{\left(\begin{array}{c}1+x_{-2}\prod_{i=0}^{n-1}\frac{(1+(12i+1)B)(1+(12i+5)B)(1+(12i+9)B)}{(1+(12i+2)B)(1+(12i+6)B)(1+(12i+10)B)}\\ z_{-3}\prod_{i=0}^{n-1}\frac{(1+(12i)B)(1+(12i+4)B)(1+(12i+8)B)}{(1+(12i+1)B)(1+(12i+5)B)(1+(12i+9)B)}\\ \frac{z_0(1+3B)(1+7B)}{(1+4B)(1+8B)}\prod_{i=0}^{n-2}\frac{(1+(12i+11)B)(1+(12i+15)B)(1+(12i+19)B)}{(1+(12i+12)B)(1+(12i+16)B)(1+(12i+20)B)}\\ \frac{y_{-1}(1+2B)(1+6B)}{(1+3B)(1+7B)}\prod_{i=0}^{n-2}\frac{(1+(12i+10)B)(1+(12i+14)B)(1+(12i+18)B)}{(1+(12i+11)B)(1+(12i+15)B)(1+(12i+19)B)}\end{array}\right)}$$

$$=\frac{\frac{y_{-1}(1+2B)(1+6B)}{(1+3B)(1+7B)}\prod_{i=0}^{n-2}\frac{(1+(12i+10)B)(1+(12i+14)B)(1+(12i+18)B)}{(1+(12i+11)B)(1+(12i+15)B)(1+(12i+19)B)}}{\left[\begin{array}{c}1+z_{-3}x_{-2}z_0y_{-1}(1+2B)(1+6B)\prod_{i=0}^{n-1}\frac{1}{(1+(12i+2)B)(1+(12i+6)B)(1+(12i+10)B)}\\ \prod_{i=0}^{n-2}(1+(12i+10)B)(1+(12i+14)B)(1+(12i+18)B)\end{array}\right]}$$

$$=\frac{\frac{y_{-1}(1+2B)(1+6B)}{(1+3B)(1+7B)}\prod_{i=0}^{n-2}\frac{(1+(12i+10)B)(1+(12i+14)B)(1+(12i+18)B)}{(1+(12i+11)B)(1+(12i+15)B)(1+(12i+19)B)}}{\left[1+\frac{z_{-3}x_{-2}z_0y_{-1}}{(1+(12n-2)B)}\right]}$$

$$=\frac{y_{-1}(1+2B)(1+6B)}{(1+3B)(1+7B)}\frac{(1+(12n-2)B)}{(1+(12n-1)B)}\prod_{i=0}^{n-2}\frac{(1+(12i+10)B)(1+(12i+14)B)(1+(12i+18)B)}{(1+(12i+11)B)(1+(12i+15)B)(1+(12i+19)B)}$$

$$=y_{-1}\prod_{i=0}^{n-1}\frac{(1+(12i+2)B)(1+(12i+6)B)(1+(12i+10)B)}{(1+(12i+3)B)(1+(12i+7)B)(1+(12i+11)B)}.$$

Also, we see from (2.1) that

$$z_{12n}=\frac{y_{12n-4}}{1+y_{12n-1}x_{12n-2}z_{12n-3}y_{12n-4}}$$

$$=\frac{\left(\frac{z_0(1+3B)(1+7B)}{(1+4B)(1+8B)}\prod_{i=0}^{n-2}\frac{(1+(12i+11)B)(1+(12i+15)B)(1+(12i+19)B)}{(1+(12i+12)B)(1+(12i+16)B)(1+(12i+20)B)}\right)}{\left[\begin{array}{c}1+\left(y_{-1}\prod_{i=0}^{n-1}\frac{(1+(12i+2)B)(1+(12i+6)B)(1+(12i+10)B)}{(1+(12i+3)B)(1+(12i+7)B)(1+(12i+11)B)}\right)\\ \left(x_{-2}\prod_{i=0}^{n-1}\frac{(1+(12i+1)B)(1+(12i+5)B)(1+(12i+9)B)}{(1+(12i+2)B)(1+(12i+6)B)(1+(12i+10)B)}\right)\\ \left(z_{-3}\prod_{i=0}^{n-1}\frac{(1+(12i)B)(1+(12i+4)B)(1+(12i+8)B)}{(1+(12i+1)B)(1+(12i+5)B)(1+(12i+9)B)}\right)\\ \left(\frac{z_0(1+3B)(1+7B)}{(1+4B)(1+8B)}\prod_{i=0}^{n-2}\frac{(1+(12i+11)B)(1+(12i+15)B)(1+(12i+19)B)}{(1+(12i+12)B)(1+(12i+16)B)(1+(12i+20)B)}\right)\end{array}\right]}$$

$$=\frac{\left(\frac{z_0(1+3B)(1+7B)}{(1+4B)(1+8B)}\prod_{i=0}^{n-2}\frac{(1+(12i+11)B)(1+(12i+15)B)(1+(12i+19)B)}{(1+(12i+12)B)(1+(12i+16)B)(1+(12i+20)B)}\right)}{\left[1+z_0y_{-1}x_{-2}z_{-3}\prod_{i=0}^{n-1}\frac{1}{(1+(12i+11)B)}\prod_{i=0}^{n-2}(1+(12i+11)B)\right]}$$

$$=\frac{\left(\frac{z_0(1+3B)(1+7B)}{(1+4B)(1+8B)}\prod_{i=0}^{n-2}\frac{(1+(12i+11)B)(1+(12i+15)B)(1+(12i+19)B)}{(1+(12i+12)B)(1+(12i+16)B)(1+(12i+20)B)}\right)}{\left[1+\frac{B}{(1+(12n-1)B)}\right]}$$

$$z_{12n}=\frac{z_0(1+3B)(1+7B)(1+(12n-1)B)}{(1+4B)(1+8B)(1+(12n)B)}\prod_{i=0}^{n-2}\frac{(1+(12i+11)B)(1+(12i+15)B)(1+(12i+19)B)}{(1+(12i+12)B)(1+(12i+16)B)(1+(12i+20)B)}$$

$$=z_0 \prod_{i=0}^{n-1} \tfrac{(1+(12i+3)B)(1+(12i+7)B)(1+(12i+11)B)}{(1+(12i+4)B)(1+(12i+8)B)(1+(12i+12)B)}.$$

Also, we can prove the other relations. This completes the proof. □

Lemma 2.2. *Let $\{x_n, y_n, z_n\}$ be a positive solution of system (2.1); then $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are bounded and converge to zero.*

Proof. It follows from (2.1) that

$$\begin{aligned} x_{n+1} =& \frac{z_{n-3}}{1+z_n y_{n-1} x_{n-2} z_{n-3}} < z_{n-3}, \ y_{n+1} = \frac{x_{n-3}}{1+x_n z_{n-1} y_{n-2} x_{n-3}} < x_{n-3}, \\ z_{n+1} =& \frac{y_{n-3}}{1+y_n x_{n-1} z_{n-2} y_{n-3}} < y_{n-3}. \end{aligned}$$

Thus

$$\begin{aligned} x_{n+5} <& z_{n+1}, \ y_{n+5} < x_{n+1}, \quad z_{n+5} < y_{n+1} \Rightarrow x_{n+5} < y_{n-3}, \ y_{n+5} < z_{n-3}, \ z_{n+5} < x_{n-3} \\ \Rightarrow& x_{n+9} < y_{n+1} < x_{n-3}, \ y_{n+9} < z_{n+1} < y_{n-3}, \quad z_{n+9} < x_{n+1} < z_{n-3}. \end{aligned}$$

Then the subsequences $\{x_{12n+i}\}_{n=0}^{\infty}$, $i = -3, -2, -1, 0, 1, 2, \ldots, 8$, are decreasing and bounded from above by $M = \max\{x_{-3}, x_{-2}, x_{-1}, x_0, \ldots, x_8\}$. Also, the subsequences

$$\{y_{12n+i}\}_{n=0}^{\infty} \text{ and } \{z_{12n+i}\}_{n=0}^{\infty}, i = -3, -2, -1, 0, 1, 2, \ldots, 8$$

are decreasing and bounded from above by $L = \max\{y_{-3}, y_{-2}, \ldots, y_8\}$, $N = \max\{z_{-3}, z_{-2}, \ldots, z_8\}$ respectively. This completes the proof. □

Lemma 2.3. *If x_i, y_i, z_i, $i = -3, -2, -1, 0$, are arbitrary real numbers and $\{x_n, y_n, z_n\}$ a solution of system (2.1), then the following statements are true:*

(i) If $x_{-3} = 0$, then we have $x_{12n-3} = y_{12n+1} = z_{12n+5} = 0$ and $x_{12n} = y_{12n+4} = z_{12n+8} = x_0$, $x_{12n+6} = y_{12n-2} = z_{12n+2} = y_{-2}$, $x_{12n+3} = y_{12n+7} = z_{12n-1} = z_{-1}$.

(ii) If $x_{-2} = 0$, then we have $x_{12n-2} = y_{12n+2} = z_{12n+6} = 0$, and $x_{12n+1} = y_{12n+5} = z_{12n-3} = z_{-3}$, $x_{12n+4} = y_{12n+8} = z_{12n} = z_0$, $x_{12n+7} = y_{12n-1} = z_{12n+3} = y_{-1}$.

(iii) If $x_{-1} = 0$, then we have $x_{12n-1} = y_{12n+3} = z_{12n+7} = 0$, and $x_{12n+2} = y_{12n+6} = z_{12n-2} = z_{-2}$, $x_{12n+5} = y_{12n-3} = z_{12n+1} = y_{-3}$, $x_{12n+8} = y_{12n} = z_{12n+4} = y_0$.

(iv) If $x_0 = 0$, then we have $x_{12n} = y_{12n+4} = z_{12n+8} = 0$ and $x_{12n-3} = y_{12n+1} = z_{12n+5} = x_{-3}$, $x_{12n+3} = y_{12n+7} = z_{12n-1} = z_{-1}$, $x_{12n+6} = y_{12n-2} = z_{12n+2} = y_{-2}$.

(v) If $y_{-3} = 0$, then we have $x_{12n+5} = y_{12n-3} = z_{12n+1} = 0$ and $x_{12n-1} = y_{12n+3} = z_{12n+7} = x_{-1}$, $x_{12n+2} = y_{12n+6} = z_{12n-2} = z_{-2}$, $x_{12n+8} = y_{12n} = z_{12n+4} = y_0$.

(vi) If $y_{-2} = 0$, then we have $x_{12n+6} = y_{12n-2} = z_{12n+2} = 0$ and $x_{12n-3} = y_{12n+1} = z_{12n+5} = x_{-3}$, $x_{12n} = y_{12n+4} = z_{12n+8} = x_0$, $x_{12n+3} = y_{12n+7} = z_{12n-1} = z_{-1}$.

(vii) If $y_{-1} = 0$, then we have $x_{12n+7} = y_{12n-1} = z_{12n+3} = 0$, and $x_{12n-2} = y_{12n+2} = z_{12n+6} = x_{-2}$, $x_{12n+1} = y_{12n+5} = z_{12n-3} = z_{-3}$, $x_{12n+4} = y_{12n+8} = z_{12n} = z_0$.

(viii) If $y_0 = 0$, then we have $x_{12n+8} = y_{12n} = z_{12n+4} = 0$ and $x_{12n-1} = y_{12n+3} = z_{12n+7} = x_{-1}$, $x_{12n+2} = y_{12n+6} = z_{12n-2} = z_{-2}$, $x_{12n+5} = y_{12n-3} = z_{12n+1} = y_{-3}$.

(ix) If $z_{-3} = 0$, then we have $x_{12n+1} = y_{12n+5} = z_{12n-3} = 0$ and $x_{12n-2} = y_{12n+2} = z_{12n+6} = x_{-2}$, $x_{12n+4} = y_{12n+8} = z_{12n} = z_0$, $x_{12n+7} = y_{12n-1} = z_{12n+3} = y_{-1}$.

(x) *If* $z_{-2} = 0$, *then we have* $x_{12n+2} = y_{12n+6} = z_{12n-2} = 0$ *and* $x_{12n-1} = y_{12n+3} = z_{12n+7} = x_{-1}$, $x_{12n+5} = y_{12n-3} = z_{12n+1} = y_{-3}$, $x_{12n+8} = y_{12n} = z_{12n+4} = y_0$.

(xi) *If* $z_{-1} = 0$, *then we have* $x_{12n+3} = y_{12n+7} = z_{12n-1} = 0$ *and* $x_{12n-3} = y_{12n+1} = z_{12n+5} = x_{-3}$, $x_{12n} = y_{12n+4} = z_{12n+8} = x_0$, $x_{12n+6} = y_{12n-2} = z_{12n+2} = y_{-2}$.

(xii) *If* $z_0 = 0$, *then we have* $x_{12n+4} = y_{12n+8} = z_{12n} = 0$ *and* $x_{12n-2} = y_{12n+2} = z_{12n+6} = x_{-2}$, $x_{12n+1} = y_{12n+5} = z_{12n-3} = z_{-3}$, $x_{12n+7} = y_{12n-1} = z_{12n+3} = y_{-1}$.

Proof. The proof follows from the form of the solutions of system (2.1). □

Example 2.4. For confirming the results of this section, we consider a numerical example for the difference system (2.1) with the initial conditions $x_{-3} = 0.6$, $x_{-2} = -0.3$, $x_{-1} = 0.8$, $x_0 = -1.3$, $y_{-3} = 0.2$, $y_{-2} = -4$, $y_{-1} = 0.2$, $y_0 = 0.7$, $z_{-3} = 0.8$, $z_{-2} = 1.1$, $z_{-1} = 0.02$ and $z_0 = 0.4$. (See Fig. 1).

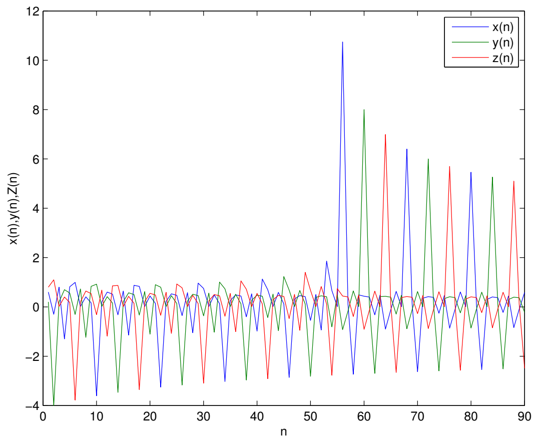

Figure 1: Plot of system (2.1).

Example 2.5. We consider interesting numerical example for the difference system (2.1) with the initial conditions $x_{-3} = 0.6$, $x_{-2} = -0.3$, $x_{-1} = 0$, $x_0 = -1.3$, $y_{-3} = 2$, $y_{-2} = -4$, $y_{-1} = 0.2$, $y_0 = 0.7$, $z_{-3} = 0.8$, $z_{-2} = 1.1$, $z_{-1} = 0$, and $z_0 = 0.4$ (See Fig. 2).

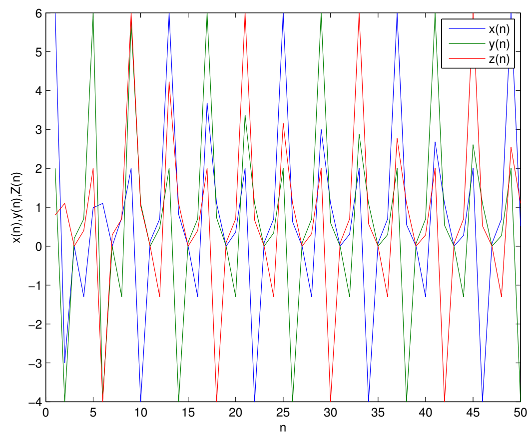

Figure 2: Plot of periodicity of system (2.1).

3. The Second Case $a_i = -1,\ b_i = 1,\ i = 1,\ 2,\ 3$

In this section, we obtain the form of the solutions of the system of three difference equations

$$x_{n+1} = \frac{z_{n-3}}{-1 + z_n y_{n-1} x_{n-2} z_{n-3}},\ y_{n+1} = \frac{x_{n-3}}{-1 + x_n z_{n-1} y_{n-2} x_{n-3}},\ z_{n+1} = \frac{y_{n-3}}{-1 + y_n x_{n-1} z_{n-2} y_{n-3}}, \tag{3.1}$$

where $n \in \mathbb{N}_0$ and the initial conditions are arbitrary real numbers with $A, B, C \neq 1$.

The following theorem is devoted to the expression of the form of the solutions of system (3.1).

Theorem 3.1. *Let $\{x_n, y_n, z_n\}_{n=-3}^{+\infty}$ be a solution of system* (3.1). *Then $\{x_n\}_{n=-3}^{+\infty}$, $\{y_n\}_{n=-3}^{+\infty}$ and $\{z_n\}_{n=-3}^{+\infty}$ are given by the formulae, for $n = 0, 1, 2, \ldots$,*

$$\begin{aligned}
x_{12n-3} =& \frac{x_{-3}}{(-1+A)^{3n}},\ x_{12n-2} = x_{-2}(-1+B)^{3n},\ x_{12n-1} = \frac{x_{-1}}{(-1+C)^{3n}},\\
x_{12n} =& x_0(-1+A)^{3n},\ x_{12n+1} = \frac{z_{-3}}{(-1+B)^{3n+1}},\ x_{12n+2} = z_{-2}(-1+C)^{3n+1},\\
x_{12n+3} =& \frac{z_{-1}}{(-1+A)^{3n+1}},\ x_{12n+4} = z_0(-1+B)^{3n+1},\ x_{12n+5} = \frac{y_{-3}}{(-1+C)^{3n+2}},\\
x_{12n+6} =& y_{-2}(-1+A)^{3n+2},\ x_{12n+7} = \frac{y_{-1}}{(-1+B)^{3n+2}},\ x_{12n+8} = y_0(-1+C)^{3n+2},\\
y_{12n-3} =& \frac{y_{-3}}{(-1+C)^{3n}},\ \ y_{12n-2} = y_{-2}(-1+A)^{3n},\ y_{12n-1} = \frac{y_{-1}}{(-1+B)^{3n}},\\
y_{12n} =& y_0(-1+C)^{3n},\ y_{12n+1} = \frac{x_{-3}}{(-1+A)^{3n+1}},\ y_{12n+2} = x_{-2}(-1+B)^{3n+1},\\
y_{12n+3} =& \frac{x_{-1}}{(-1+C)^{3n+1}},\ y_{12n+4} = x_0(-1+A)^{3n+1},\ y_{12n+5} = \frac{z_{-3}}{(-1+B)^{3n+2}},\\
y_{12n+6} =& z_{-2}(-1+C)^{3n+2},\ y_{12n+7} = \frac{z_{-1}}{(-1+A)^{3n+2}},\ y_{12n+8} = z_0(-1+B)^{3n+2},\\
z_{12n-3} =& \frac{z_{-3}}{(-1+B)^{3n}},\ z_{12n-2} = z_{-2}(-1+C)^{3n},\ z_{12n-1} = \frac{z_{-1}}{(-1+A)^{3n}},\\
z_{12n} =& z_0(-1+B)^{3n},\ z_{12n+1} = \frac{y_{-3}}{(-1+C)^{3n+1}},\ z_{12n+2} = y_{-2}(-1+A)^{3n+1},\\
z_{12n+3} =& \frac{y_{-1}}{(-1+B)^{3n+1}},\ z_{12n+4} = y_0(-1+C)^{3n+1},\ z_{12n+5} = \frac{x_{-3}}{(-1+A)^{3n+2}},\\
z_{12n+6} =& x_{-2}(-1+B)^{3n+2},\ z_{12n+7} = \frac{x_{-1}}{(-1+C)^{3n+2}},\ z_{12n+8} = x_0(-1+A)^{3n+2}.
\end{aligned}$$

Proof. For $n = 0$ the assertion holds. Now suppose that $n > 0$ and that our assertion holds for $n - 1$, that is,

$$\begin{aligned}
x_{12n-7} =& \frac{y_{-3}}{(-1+C)^{3n-1}},\ x_{12n-6} = y_{-2}(-1+A)^{3n-1},\\
x_{12n-5} =& \frac{y_{-1}}{(-1+B)^{3n-1}},\ x_{12n-4} = y_0(-1+C)^{3n-1},\\
y_{12n-7} =& \frac{z_{-3}}{(-1+B)^{3n-1}},\ y_{12n-6} = z_{-2}(-1+C)^{3n-1},\\
y_{12n-5} =& \frac{z_{-1}}{(-1+A)^{3n-1}},\ y_{12n-4} = z_0(-1+B)^{3n-1},\\
z_{12n-7} =& \frac{x_{-3}}{(-1+A)^{3n-1}},\ z_{12n-6} = x_{-2}(-1+B)^{3n-1},\\
z_{12n-5} =& \frac{x_{-1}}{(-1+C)^{3n-1}},\ z_{12n-4} = x_0(-1+A)^{3n-1}.
\end{aligned}$$

Now it follows from (3.1) that

$$\begin{aligned}
x_{12n-3} =& \frac{z_{12n-7}}{-1+z_{12n-4}y_{12n-5}x_{12n-6}z_{12n-7}} \\
=& \frac{\left(\frac{x_{-3}}{(-1+A)^{3n-1}}\right)}{-1+\left(\left(x_0(-1+A)^{3n-1}\right)\left(\frac{z_{-1}}{(-1+A)^{3n-1}}\right)\left(y_{-2}(-1+A)^{3n-1}\right)\left(\frac{x_{-3}}{(-1+A)^{3n-1}}\right)\right)} \\
=& \frac{\left(\frac{x_{-3}}{(-1+A)^{3n-1}}\right)}{-1+A} = \frac{x_{-3}}{(-1+A)^{3n}}. \\
y_{12n-2} =& \frac{x_{12n-6}}{-1+x_{12n-3}z_{12n-4}y_{12n-5}x_{12n-6}} \\
=& \frac{\left(y_{-2}(-1+A)^{3n-1}\right)}{-1+\left(\left(\frac{x_{-3}}{(-1+A)^{3n}}\right)\left(x_0(-1+A)^{3n-1}\right)\left(\frac{z_{-1}}{(-1+A)^{3n-1}}\right)\left(y_{-2}(-1+A)^{3n-1}\right)\right)} \\
=& \frac{\left(y_{-2}(-1+A)^{3n-1}\right)}{-1+\left(\frac{A}{(-1+A)}\right)} = y_{-2}(-1+A)^{3n}. \\
z_{12n} =& \frac{y_{12n-4}}{-1+y_{12n-1}x_{12n-2}z_{12n-3}y_{12n-4}} \\
=& \frac{\left(z_0(-1+B)^{3n-1}\right)}{-1+\left(\left(\frac{y_{-1}}{(-1+B)^{3n}}\right)\left(x_{-2}(-1+B)^{3n}\right)\left(\frac{z_{-3}}{(-1+B)^{3n}}\right)\left(z_0(-1+B)^{3n-1}\right)\right)} \\
=& \frac{z_0(-1+B)^{3n-1}}{-1+\frac{B}{(-1+B)}} = z_0(-1+B)^{3n}
\end{aligned}$$

wherefrom

$$\begin{aligned}
x_{12n+1} =& \frac{z_{12n-3}}{-1+z_{12n}y_{12n-1}x_{12n-2}z_{12n-3}} \\
=& \frac{\left(\frac{z_{-3}}{(-1+B)^{3n}}\right)}{-1+\left(\left(z_0(-1+B)^{3n}\right)\left(\frac{y_{-1}}{(-1+B)^{3n}}\right)\left(x_{-2}(-1+B)^{3n}\right)\left(\frac{z_{-3}}{(-1+B)^{3n}}\right)\right)} \\
=& \frac{\left(\frac{z_{-3}}{(-1+B)^{3n}}\right)}{-1+B} = \frac{z_{-3}}{(-1+B)^{3n+1}}. \\
y_{12n+3} =& \frac{x_{12n-1}}{-1+x_{12n+2}z_{12n+1}y_{12n}x_{12n-1}} \\
=& \frac{\left(\frac{x_{-1}}{(-1+C)^{3n}}\right)}{-1+\left(\left(z_{-2}(-1+C)^{3n+1}\right)\left(\frac{y_{-3}}{(-1+C)^{3n+1}}\right)\left(y_0(-1+C)^{3n}\right)\left(\frac{x_{-1}}{(-1+C)^{3n}}\right)\right)} \\
=& \frac{\left(\frac{x_{-1}}{(-1+C)^{3n}}\right)}{-1+C} = \frac{x_{-1}}{(-1+C)^{3n+1}}. \\
z_{12n+4} =& \frac{y_{12n}}{-1+y_{12n+3}x_{12n+2}z_{12n+1}y_{12n}} \\
=& \frac{\left(y_0(-1+C)^{3n}\right)}{-1+\left(\frac{x_{-1}}{(-1+C)^{3n+1}}z_{-2}(-1+C)^{3n+1}\frac{y_{-3}}{(-1+C)^{3n+1}}y_0(-1+C)^{3n}\right)} \\
=& \frac{y_0(-1+C)^{3n}}{-1+\left(\frac{C}{(-1+C)}\right)} = y_0(-1+C)^{3n+1}.
\end{aligned}$$

Also, we can prove the other relations. This completes the proof. □

Lemma 3.2. *The solutions of system* (3.1) *are unbounded except in the cases from the following two theorems.*

Theorem 3.3. *System* (3.1) *has a periodic solution of period twelve iff* $A = B = C = 2$ *and it can be written in the following form*

$$\begin{aligned}
\{x_n\} &= \{x_{-3}, x_{-2}, x_{-1}, x_0, z_{-3}, z_{-2}, z_{-1}, z_0, y_{-3}, y_{-2}, y_{-1}, y_0, x_{-3}, x_{-2}, x_{-1}, x_0, \dots\},\\
\{y_n\} &= \{y_{-3}, y_{-2}, y_{-1}, y_0, x_{-3}, x_{-2}, x_{-1}, x_0, z_{-3}, z_{-2}, z_{-1}, z_0, y_{-3}, y_{-2}, y_{-1}, y_0, \dots\},\\
\{z_n\} &= \{z_{-3}, z_{-2}, z_{-1}, z_0, y_{-3}, y_{-2}, y_{-1}, y_0, x_{-3}, x_{-2}, x_{-1}, x_0, z_{-3}, z_{-2}, z_{-1}, z_0, \dots\}.
\end{aligned}$$

Proof. First suppose that there exists a periodic solution of period twelve

$$\begin{aligned}
\{x_n\} &= \{x_{-3}, x_{-2}, x_{-1}, x_0, z_{-3}, z_{-2}, z_{-1}, z_0, y_{-3}, y_{-2}, y_{-1}, y_0, x_{-3}, x_{-2}, x_{-1}, x_0, \dots\},\\
\{y_n\} &= \{y_{-3}, y_{-2}, y_{-1}, y_0, x_{-3}, x_{-2}, x_{-1}, x_0, z_{-3}, z_{-2}, z_{-1}, z_0, y_{-3}, y_{-2}, y_{-1}, y_0, \dots\},\\
\{z_n\} &= \{z_{-3}, z_{-2}, z_{-1}, z_0, y_{-3}, y_{-2}, y_{-1}, y_0, x_{-3}, x_{-2}, x_{-1}, x_0, z_{-3}, z_{-2}, z_{-1}, z_0, \dots\}
\end{aligned}$$

of system (3.1). From its form, we see that

$$\begin{aligned}
x_{-3} &= \tfrac{x_{-3}}{(-1+A)^{3n}},\ x_{-2} = x_{-2}(-1+B)^{3n},\ x_{-1} = \tfrac{x_{-1}}{(-1+C)^{3n}},\ x_0 = x_0(-1+A)^{3n},\\
z_{-3} &= \tfrac{z_{-3}}{(-1+B)^{3n+1}},\ z_{-2} = z_{-2}(-1+C)^{3n+1}, z_{-1} = \tfrac{z_{-1}}{(-1+A)^{3n+1}},\ z_0 = z_0(-1+B)^{3n+1},\\
y_{-3} &= \tfrac{y_{-3}}{(-1+C)^{3n+2}},\ y_{-2} = y_{-2}(-1+A)^{3n+2},\ y_{-1} = \tfrac{y_{-1}}{(-1+B)^{3n+2}},\ y_0 = y_0(-1+C)^{3n+2},\\
y_{-3} &= \tfrac{y_{-3}}{(-1+C)^{3n}},\ y_{-2} = y_{-2}(-1+A)^{3n},\ y_{-1} = \tfrac{y_{-1}}{(-1+B)^{3n}},\ y_0 = y_0(-1+C)^{3n},\\
x_{-3} &= \tfrac{x_{-3}}{(-1+A)^{3n+1}},\ x_{-2} = x_{-2}(-1+B)^{3n+1},\ x_{-1} = \tfrac{x_{-1}}{(-1+C)^{3n+1}},\ x_0 = x_0(-1+A)^{3n+1},\\
z_{-3} &= \tfrac{z_{-3}}{(-1+B)^{3n+2}},\ z_{-2} = z_{-2}(-1+C)^{3n+2},\ z_{-1} = \tfrac{z_{-1}}{(-1+A)^{3n+2}},\ z_0 = z_0(-1+B)^{3n+2},\\
z_{-3} &= \tfrac{z_{-3}}{(-1+B)^{3n}},\ z_{-2} = z_{-2}(-1+C)^{3n},\ z_{-1} = \tfrac{z_{-1}}{(-1+A)^{3n}},\ z_0 = z_0(-1+B)^{3n},\\
y_{-3} &= \tfrac{y_{-3}}{(-1+C)^{3n+1}},\ y_{-2} = y_{-2}(-1+A)^{3n+1},\ y_{-1} = \tfrac{y_{-1}}{(-1+B)^{3n+1}},\ y_0 = y_0(-1+C)^{3n+1},\\
x_{-3} &= \tfrac{x_{-3}}{(-1+A)^{3n+2}},\ x_{-2} = x_{-2}(-1+B)^{3n+2},\ x_{-1} = \tfrac{x_{-1}}{(-1+C)^{3n+2}},\ x_0 = x_0(-1+A)^{3n+2},
\end{aligned}$$

where from we get

$$A = B = C = 2.$$

Second, assume now that $A = B = C = 2$. Then we see from the form of the solution of system (3.1) that

$$\begin{aligned}
x_{12n-3} &= x_{-3},\ x_{12n-2} = x_{-2},\ x_{12n-1} = x_{-1},\ x_{12n} = x_0,\ x_{12n+1} = z_{-3},\ x_{12n+2} = z_{-2},\\
x_{12n+3} &= z_{-1},\ x_{12n+4} = z_0,\ x_{12n+5} = y_{-3},\ x_{12n+6} = y_{-2},\ x_{12n+7} = y_{-1},\ x_{12n+8} = y_0,\\
y_{12n-3} &= y_{-3},\ y_{12n-2} = y_{-2},\ y_{12n-1} = y_{-1},\ y_{12n} = y_0,\ y_{12n+1} = x_{-3},\ y_{12n+2} = x_{-2},\\
y_{12n+3} &= x_{-1},\ y_{12n+4} = x_0,\ y_{12n+5} = z_{-3},\ y_{12n+6} = z_{-2},\ y_{12n+7} = z_{-1},\ y_{12n+8} = z_0,\\
z_{12n-3} &= z_{-3},\ z_{12n-2} = z_{-2},\ z_{12n-1} = z_{-1},\ z_{12n} = z_0,\ z_{12n+1} = y_{-3},\ z_{12n+2} = y_{-2},\\
z_{12n+3} &= y_{-1},\ z_{12n+4} = y_0,\ z_{12n+5} = x_{-3},\ z_{12n+6} = x_{-2},\ z_{12n+7} = x_{-1},\ z_{12n+8} = x_0.
\end{aligned}$$

Thus we have a periodic solution of period twelve. This completes the proof. □

Theorem 3.4. *System* (3.1) *has a periodic solution of period four iff* $x_{-i} = y_{-i} = z_{-i},\ i = 0, 1, 2, 3,\ A = 2$ *which has the form*

$$\{x_n\} = \{y_n\} = \{z_n\} = \{x_{-3}, x_{-2}, x_{-1}, x_0, x_{-3}, x_{-2}, \dots\}.$$

Proof. The proof follows from the formulae of solutions of system (3.1). □

Example 3.5. We consider an interesting numerical example for (3.1) with the initial conditions $x_{-3} = 0.6,\ x_{-2} = -0.3,\ x_{-1} = 0.8,\ x_0 = -1.3,\ y_{-3} = 0.2,\ y_{-2} = -4,\ y_{-1} = 0.2,\ y_0 = 0.7,\ z_{-3} = 0.8,\ z_{-2} = 1.1,\ z_{-1} = 0.02$, and $z_0 = 0.4$. See Figure 3.

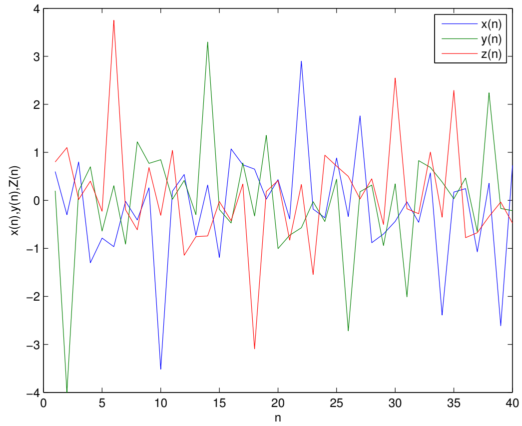

Figure 3: Plot of system (3.1).

Example 3.6. See Figure 4, when we take system (3.1) with the initial conditions $x_{-3} = 6$, $x_{-2} = 1/14$, $x_{-1} = -1/6$, $x_0 = -2$, $y_{-3} = 0.3$, $y_{-2} = -3$, $y_{-1} = 8$, $y_0 = -2$, $z_{-3} = -7$, $z_{-2} = 20$, $z_{-1} = 1/18$, and $z_0 = -0.5$.

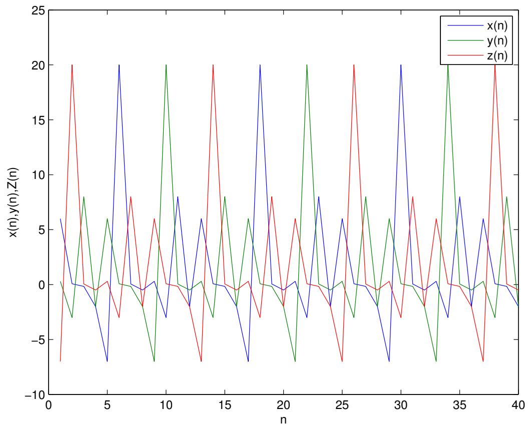

Figure 4: Plot of periodicity of system (3.1).

Example 3.7. Figure 5 shows the behavior of the solution of system (3.1) with the initial conditions $x_{-3} = y_{-3} = z_{-3} = 6$, $x_{-2} = y_{-2} = z_{-2} = -3$, $x_{-1} = y_{-1} = z_{-1} = 1/18$ and $x_0 = y_0 = z_0 = -2$.

The following cases can be treated similarly.

4. The Third Case $a_i = b_i = b_3 = 1$, $i = 1, 2$, $a_3 = -1$

Here we obtain a form of the solutions of the system of the difference equations

$$x_{n+1} = \frac{z_{n-3}}{1 + z_n y_{n-1} x_{n-2} z_{n-3}}, \; y_{n+1} = \frac{x_{n-3}}{1 + x_n z_{n-1} y_{n-2} x_{n-3}},$$
$$z_{n+1} = \frac{y_{n-3}}{-1 + y_n x_{n-1} z_{n-2} y_{n-3}}, \tag{4.1}$$

where $n = 0, 1, 2, \ldots$ and the initial conditions are arbitrary real numbers with $A \neq \pm 1, \neq \frac{1}{2}$, $B \neq \pm 1$, $\neq -\frac{1}{2}$ and $C \neq 1, \neq \frac{1}{2}, \neq \frac{1}{3}$.

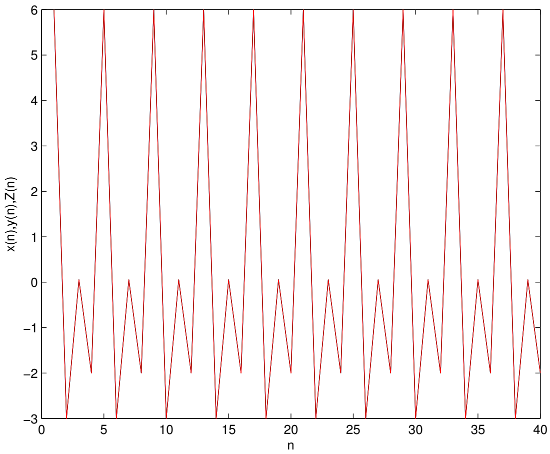

Figure 5: Plot of system (3.1) when $x_{-i} = y_{-i} = z_{-i}$, $i = 0, 1, 2, 3$, $A = 2$.

Theorem 4.1. *Let $\{x_n, y_n, z_n\}_{n=-3}^{+\infty}$ be a solution of system (4.1). Then for $n = 0, 1, 2, \ldots,$ we have*

$$x_{12n-3} = \frac{x_{-3}(-1+2A)^n}{(-1+A)^{2n}(1+A)^n}, \qquad x_{12n-2} = \frac{x_{-2}(-1+B)^n(1+B)^{2n}}{(1+2B)^n},$$

$$x_{12n-1} = \frac{x_{-1}(1-2C)^{2n}}{(-1+C)^{2n}(-1+3C)^n}, \qquad x_{12n} = \frac{x_0(-1+A)^{2n}(1+A)^n}{(-1+2A)^n},$$

$$x_{12n+1} = \frac{z_{-3}(1+2B)^n}{(-1+B)^n(1+B)^{2n+1}}, \qquad x_{12n+2} = \frac{z_{-2}(-1+C)^{2n+1}(-1+3C)^n}{(-1+2C)^{2n+1}},$$

$$x_{12n+3} = -\frac{z_{-1}(-1+2A)^n}{(-1+A)^{2n+1}(1+A)^n}, \qquad x_{12n+4} = \frac{z_0(-1+B)^n(1+B)^{2n+1}}{(1+2B)^n},$$

$$x_{12n+5} = \frac{y_{-3}(-1+2C)^{2n+1}}{(-1+C)^{2n+2}(-1+3C)^n}, \qquad x_{12n+6} = \frac{y_{-2}(-1+A)^{2n+1}(1+A)^{n+1}}{(-1+2A)^n},$$

$$x_{12n+7} = -\frac{y_{-1}(1+2B)^{n+1}}{(-1+B)^n(1+B)^{2n+2}}, \qquad x_{12n+8} = -\frac{y_0(-1+C)^{2n+1}(-1+3C)^{n+1}}{(-1+2C)^{2n+2}},$$

$$y_{12n-3} = \frac{y_{-3}(-1+2C)^{2n}}{(-1+C)^{2n}(-1+3C)^n}, \qquad y_{12n-2} = \frac{y_{-2}(-1+A)^{2n}(1+A)^n}{(-1+2A)^n},$$

$$y_{12n-1} = \frac{y_{-1}(1+2B)^n}{(-1+B)^n(1+B)^{2n}}, \qquad y_{12n} = \frac{y_0(-1+C)^{2n}(-1+3C)^n}{(-1+2C)^{2n}},$$

$$y_{12n+1} = \frac{x_{-3}(-1+2A)^n}{(-1+A)^{2n}(1+A)^{n+1}}, \qquad y_{12n+2} = \frac{x_{-2}(-1+B)^n(1+B)^{2n+1}}{(1+2B)^{n+1}},$$

$$y_{12n+3} = \frac{x_{-1}(-1+2C)^{2n+1}}{(-1+C)^{2n}(-1+3C)^{n+1}}, \qquad y_{12n+4} = \frac{x_0(-1+A)^{2n+1}(1+A)^n}{(-1+2A)^{n+1}},$$

$$y_{12n+5} = -\frac{z_{-3}(1+2B)^n}{(-1+B)^{n+1}(1+B)^{2n+1}}, \qquad y_{12n+6} = -\frac{z_{-2}(-1+C)^{2n+2}(-1+3C)^n}{(-1+2C)^{2n+1}},$$

$$y_{12n+7} = -\frac{z_{-1}(-1+2A)^n}{(-1+A)^{2n+1}(1+A)^{n+1}}, \qquad y_{12n+8} = \frac{z_0(-1+B)^n(1+B)^{2n+2}}{(1+2B)^{n+1}},$$

$$z_{12n-3} = \frac{z_{-3}(1+2B)^n}{(-1+B)^n(1+B)^{2n}}, \qquad z_{12n-2} = \frac{z_{-2}(-1+C)^{2n}(-1+3C)^n}{(-1+2C)^{2n}},$$

$$z_{12n-1} = \frac{z_{-1}(-1+2A)^n}{(-1+A)^{2n}(1+A)^n}, \qquad z_{12n} = \frac{z_0(-1+B)^n(1+B)^{2n}}{(1+2B)^n},$$

$$z_{12n+1} = \frac{y_{-3}(-1+2C)^{2n}}{(-1+C)^{2n+1}(-1+3C)^{n}}, \qquad z_{12n+2} = -\frac{y_{-2}(-1+A)^{2n}(1+A)^{n+1}}{(-1+2A)^{n}},$$
$$z_{12n+3} = -\frac{y_{-1}(1+2B)^{n+1}}{(-1+B)^{n}(1+B)^{2n+1}}, \qquad z_{12n+4} = -\frac{y_{0}(-1+C)^{2n}(-1+3C)^{n+1}}{(-1+2C)^{2n+1}},$$
$$z_{12n+5} = -\frac{x_{-3}(-1+2A)^{n+1}}{(-1+A)^{2n+1}(1+A)^{n+1}}, \qquad z_{12n+6} = \frac{x_{-2}(-1+B)^{n+1}(1+B)^{2n+1}}{(1+2B)^{n+1}},$$
$$z_{12n+7} = \frac{x_{-1}(-1+2C)^{2n+1}}{(-1+C)^{2n+1}(-1+3C)^{n+1}}, \qquad z_{12n+8} = -\frac{x_{0}(-1+A)^{2n+1}(1+A)^{n+1}}{(-1+2A)^{n+1}}.$$

Example 4.2. In the following example for system (4.1), we put the initial conditions $x_{-3} = 0.6$, $x_{-2} = -0.3$, $x_{-1} = 0.7$, $x_0 = -1.3$, $y_{-3} = 0.2$, $y_{-2} = -0.4$, $y_{-1} = 0.2$, $y_0 = 0.7$, $z_{-3} = 0.8$, $z_{-2} = 0.24$, $z_{-1} = 0.5$, and $z_0 = 0.4$. See Figure 6.

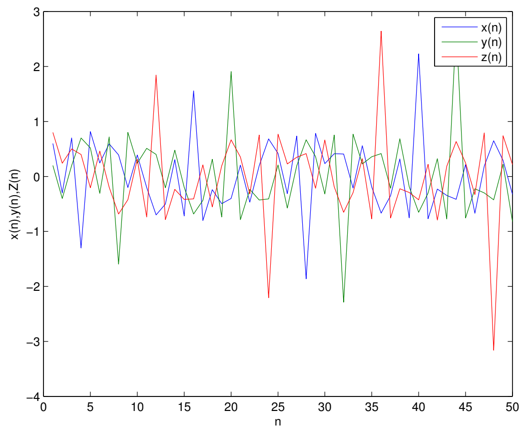

Figure 6: Plot of system (4.1).

5. Other Cases

In this section, we get a form of the solutions of the following systems of the difference equations

$$x_{n+1} = \frac{z_{n-3}}{1+z_n y_{n-1} x_{n-2} z_{n-3}}, \; y_{n+1} = \frac{x_{n-3}}{-1-x_n z_{n-1} y_{n-2} x_{n-3}},$$
$$z_{n+1} = \frac{y_{n-3}}{1-y_n x_{n-1} z_{n-2} y_{n-3}}. \tag{5.1}$$

$$x_{n+1} = \frac{z_{n-3}}{-1-z_n y_{n-1} x_{n-2} z_{n-3}}, \; y_{n+1} = \frac{x_{n-3}}{-1-x_n z_{n-1} y_{n-2} x_{n-3}},$$
$$z_{n+1} = \frac{y_{n-3}}{-1-y_n x_{n-1} z_{n-2} y_{n-3}}. \tag{5.2}$$

Theorem 5.1. *Let $\{x_n, y_n, z_n\}$ be a solution of system* (5.1)*. Then for $n = 0, 1, 2, \ldots,$ we have*

$$x_{12n-3} = \frac{x_{-3}(1+2A)^{n}}{(1+A)^{2n}(-1+A)^{n}}, \qquad x_{12n-2} = \frac{x_{-2}(-1)^{n}(1+B)^{2n}(1+3B)^{n}}{(1+2B)^{2n}},$$
$$x_{12n-1} = \frac{x_{-1}(1+2C)^{n}}{(1+C)^{2n}(-1+C)^{n}}, \qquad x_{12n} = \frac{x_{0}(1+A)^{2n}(-1+A)^{n}}{(1+2A)^{n}},$$

$$x_{12n+1} = \frac{z_{-3}(-1)^n(1+2B)^{2n}}{(1+B)^{2n+1}(1+3B)^n}, \qquad x_{12n+2} = -\frac{z_{-2}(-1+C)^{n+1}(1+C)^{2n}}{(1+2C)^n},$$

$$x_{12n+3} = \frac{z_{-1}(1+2A)^{n+1}}{(1+A)^{2n+1}(-1+A)^n}, \qquad x_{12n+4} = \frac{z_0(-1)^n(1+3B)^{n+1}(1+B)^{2n}}{(1+2B)^{2n+1}},$$

$$x_{12n+5} = -\frac{y_{-3}(1+2C)^{n+1}}{(1+C)^{2n+1}(-1+C)^{n+1}}, \qquad x_{12n+6} = -\frac{y_{-2}(1+A)^{2n+1}(-1+A)^{n+1}}{(1+2A)^{n+1}},$$

$$x_{12n+7} = \frac{y_{-1}(-1)^n(1+2B)^{2n+1}}{(1+3B)^{n+1}(1+B)^{2n+1}}, \qquad x_{12n+8} = -\frac{y_0(-1+C)^{n+1}(1+C)^{2n+1}}{(1+2C)^{n+1}},$$

$$y_{12n-3} = \frac{y_{-3}(1+2C)^n}{(1+C)^{2n}(-1+C)^n}, \qquad y_{12n-2} = \frac{y_{-2}(1+A)^{2n}(-1+A)^n}{(1+2A)^n},$$

$$y_{12n-1} = \frac{y_{-1}(-1)^n(1+2B)^{2n}}{(1+3B)^n(1+B)^{2n}}, \qquad y_{12n} = \frac{y_0(1+C)^{2n}(-1+C)^n}{(1+2C)^n},$$

$$y_{12n+1} = -\frac{x_{-3}(1+2A)^n}{(1+A)^{2n+1}(-1+A)^n}, \qquad y_{12n+2} = \frac{x_{-2}(-1)^{n+1}(1+3B)^n(1+B)^{2n+1}}{(1+2B)^{2n+1}},$$

$$y_{12n+3} = -\frac{x_{-1}(1+2C)^n}{(1+C)^{2n+1}(-1+C)^n}, \qquad y_{12n+4} = -\frac{x_0(1+A)^{2n+1}(-1+A)^n}{(1+2A)^n},$$

$$y_{12n+5} = \frac{z_{-3}(-1)^{n+1}(1+2B)^{2n+1}}{(1+3B)^n(1+B)^{2n+2}}, \qquad y_{12n+6} = \frac{z_{-2}(1+C)^{2n+1}(-1+C)^{n+1}}{(1+2C)^n},$$

$$y_{12n+7} = -\frac{z_{-1}(1+2A)^{n+1}}{(1+A)^{2n+2}(-1+A)^n}, \qquad y_{12n+8} = \frac{z_0(-1)^{n+1}(1+3B)^{n+1}(1+B)^{2n+1}}{(1+2B)^{2n+2}},$$

$$z_{12n-3} = \frac{z_{-3}(-1)^n(1+2B)^{2n}}{(1+3B)^n(1+B)^{2n}}, \qquad z_{12n-2} = \frac{z_{-2}(1+C)^{2n}(-1+C)^n}{(1+2C)^n},$$

$$z_{12n-1} = \frac{z_{-1}(1+2A)^n}{(1+A)^{2n}(-1+A)^n}, \qquad z_{12n} = \frac{z_0(-1)^n(1+3B)^n(1+B)^{2n}}{(1+2B)^{2n}},$$

$$z_{12n+1} = -\frac{y_{-3}(1+2C)^n}{(-1+C)^{n+1}(1+C)^{2n}}, \qquad z_{12n+2} = \frac{y_{-2}(-1+A)^n(1+A)^{2n+1}}{(1+2A)^{n+1}},$$

$$z_{12n+3} = \frac{y_{-1}(-1)^n(1+2B)^{2n+1}}{(1+B)^{2n}(1+3B)^{n+1}}, \qquad z_{12n+4} = \frac{y_0(1+C)^{2n+1}(-1+C)^n}{(1+2C)^{n+1}},$$

$$z_{12n+5} = \frac{x_{-3}(1+2A)^n}{(1+A)^{2n+1}(-1+A)^{n+1}}, \qquad z_{12n+6} = \frac{x_{-2}(-1)^{n+1}(1+3B)^n(1+B)^{2n+2}}{(1+2B)^{2n+1}},$$

$$z_{12n+7} = \frac{x_{-1}(1+2C)^n}{(1+C)^{2n+1}(-1+C)^{n+1}}, \qquad z_{12n+8} = -\frac{x_0(1+A)^{2n+2}(-1+A)^n}{(1+2A)^{n+1}},$$

where $A, C \neq \pm 1, \neq -\frac{1}{2}$ *and* $B_0 \neq -1, \neq -\frac{1}{2}, \neq -\frac{1}{3}$.

Theorem 5.2. *Let* $\{x_n, y_n, z_n\}_{n=-3}^{+\infty}$ *be a solution of system* (5.2) *and* $A, B, C \neq -1$. *Then for* $n = 0, 1, 2, \ldots,$ *we have*

$$x_{12n-3} = \frac{x_{-3}}{(-1-A)^{3n}}, \; x_{12n-2} = x_{-2}(-1-B)^{3n}, \; x_{12n-1} = \frac{x_{-1}}{(-1-C)^{3n}},$$

$$x_{12n} = x_0(-1-A)^{3n}, \; x_{12n+1} = \frac{z_{-3}}{(-1-B)^{3n+1}}, \; x_{12n+2} = z_{-2}(-1-C)^{3n+1},$$

$$x_{12n+3} = \frac{z_{-1}}{(-1-A)^{3n+1}}, \; x_{12n+4} = z_0(-1-B)^{3n+1}, \; x_{12n+5} = \frac{y_{-3}}{(-1-C)^{3n+2}},$$

$$x_{12n+6} = y_{-2}(-1-A)^{3n+2}, \; x_{12n+7} = \frac{y_{-1}}{(-1-B)^{3n+2}}, \; x_{12n+8} = y_0(-1-C)^{3n+2},$$

$$y_{12n-3} = \frac{y_{-3}}{(-1-C)^{3n}}, \quad y_{12n-2} = y_{-2}(-1-A)^{3n}, \; y_{12n-1} = \frac{y_{-1}}{(-1-B)^{3n}},$$

$$
\begin{aligned}
y_{12n} =& y_0(-1-C)^{3n},\ y_{12n+1} = \frac{x_{-3}}{(-1-A)^{3n+1}},\ y_{12n+2} = x_{-2}(-1-B)^{3n+1},\\
y_{12n+3} =& \frac{x_{-1}}{(-1-C)^{3n+1}},\ y_{12n+4} = x_0(-1-A)^{3n+1},\ y_{12n+5} = \frac{z_{-3}}{(-1-B)^{3n+2}},\\
y_{12n+6} =& z_{-2}(-1-C)^{3n+2},\ y_{12n+7} = \frac{z_{-1}}{(-1-A)^{3n+2}},\ y_{12n+8} = z_0(-1-B)^{3n+2},\\
z_{12n-3} =& \frac{z_{-3}}{(-1-B)^{3n}},\ z_{12n-2} = z_{-2}(-1-C)^{3n},\ z_{12n-1} = \frac{z_{-1}}{(-1-A)^{3n}},\\
z_{12n} =& z_0(-1-B)^{3n},\ z_{12n+1} = \frac{y_{-3}}{(-1-C)^{3n+1}},\ z_{12n+2} = y_{-2}(-1-A)^{3n+1},\\
z_{12n+3} =& \frac{y_{-1}}{(-1-B)^{3n+1}},\ z_{12n+4} = y_0(-1-C)^{3n+1},\ z_{12n+5} = \frac{x_{-3}}{(-1-A)^{3n+2}},\\
z_{12n+6} =& x_{-2}(-1-B)^{3n+2},\ z_{12n+7} = \frac{x_{-1}}{(-1-C)^{3n+2}},\ z_{12n+8} = x_0(-1-A)^{3n+2}.
\end{aligned}
$$

Lemma 5.3. *The solutions of system* (5.2) *are unbounded except in the cases given in the following two theorems.*

Theorem 5.4. *System* (5.2) *has a periodic solution of period twelve iff* $A = B = C = -2$ *and it has the form*

$$
\begin{aligned}
\{x_n\} &= \{x_{-3}, x_{-2}, x_{-1}, x_0, z_{-3}, z_{-2}, z_{-1}, z_0, y_{-3}, y_{-2}, y_{-1}, y_0, x_{-3}, x_{-2}, x_{-1}, x_0, \dots\},\\
\{y_n\} &= \{y_{-3}, y_{-2}, y_{-1}, y_0, x_{-3}, x_{-2}, x_{-1}, x_0, z_{-3}, z_{-2}, z_{-1}, z_0, y_{-3}, y_{-2}, y_{-1}, y_0, \dots\},\\
\{z_n\} &= \{z_{-3}, z_{-2}, z_{-1}, z_0, y_{-3}, y_{-2}, y_{-1}, y_0, x_{-3}, x_{-2}, x_{-1}, x_0, z_{-3}, z_{-2}, z_{-1}, z_0, \dots\}.
\end{aligned}
$$

Theorem 5.5. *System* (5.2) *has a periodic solution of period four iff* $x_{-i} = y_{-i} = z_{-i},\ i = 0, 1, 2, 3,\ A = -2$ *and it is of the form*

$$\{x_n\} = \{y_n\} = \{z_n\} = \{x_{-3}, x_{-2}, x_{-1}, x_0, x_{-3}, x_{-2}, \dots\}.$$

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