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Positive solutions of p-Laplacian fractional differential equations with integral boundary value conditions

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Abstract

In this work, we investigate the existence of solutions of p-Laplacian fractional differential equations with integral boundary value conditions. Using the five functionals fixed point theorem, the existence of multiple positive solutions is obtained for the boundary value problems. An example is also given to illustrate the effectiveness of our main result.

Keywords: Multiple positive solutions, p-Laplacian, the five functionals fixed point theorem.
2010 MSC: 34B15, 34B18.

1. Introduction

The equation with p-Laplacian operator arises in the modeling of different physical and natural phenomena, non-Newtonian mechanics, nonlinear flow laws and many other branches of engineering. The study of differential equations with p-Laplacian operator was initiated by many authors, one may see [1]–[8], [11], [12]–[14] and references therein.

In [6], Guo *et al.* discussed the existences of solution for the following boundary value problems for the p-Laplacian equation:

$$(\phi_p(u'(t)))' + a(t)f(t, u(t)) = 0, \ t \in (0, 1),$$

subject to one of the following boundary conditions:

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$$\phi_p(u'(0)) = \sum_{i=1}^{m-2} a_i \phi_p(u'(\xi_i)),\ u(1) = \sum_{i=1}^{m-2} b_i u(\xi_i),$$

or

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i),\ \phi_p(u'(1)) = \sum_{i=1}^{m-2} b_i \phi_p(u'(\xi_i)).$$

Using the five functionals fixed point theorem, they obtained the existence of multiple (at least three) positive solutions for above boundary value problems.

Chen *et al.* [3] showed the existence solutions by coincidence degree for the Caputo fractional p-Laplacian equations:

$$\begin{aligned} &D_{0^+}^{\beta}\phi_p(D_{0^+}^{\alpha}x(t)) = f(t, x(t), D_{0^+}^{\alpha}x(t)),\ 0 < t < 1,\\ &D_{0^+}^{\alpha}x(0) = D_{0^+}^{\alpha}x(1) = 0, \end{aligned}$$

where $0 < \alpha, \beta \leq 1$ and $1 < \alpha + \beta \leq 2$, $\phi_p(s) = |s|^{p-2}s,\ p > 1,\ f : [0, 1] \times R^2 \to R$ is continuous, $D_{0^+}^{\beta}$ and $D_{0^+}^{\alpha}$ are the Caputo fractional derivatives. They used $Lu = D_{0^+}^{\beta}\phi_p(D_{0^+}^{\alpha}x(t))$ with $D_{0^+}^{\alpha}x(0) = D_{0^+}^{\alpha}x(1) = 0$ and obtained dim ker$L = 1$.

Zhang *et al.* [14] discussed the eigenvalue problem for a class of singular p-Laplacian fractional differential equations involving the Riemann-Stieltjes integral boundary conditions

$$\begin{aligned} &-D_t^{\beta}\Big(\phi_p(D_t^{\alpha}x)\Big)(t) = \lambda f(t, x(t)),\ t \in (0, 1),\\ &x(0) = 0,\quad D_t^{\alpha}x(0) = 0,\quad x(1) = \int_0^1 x(s)\mathrm{d}A(s), \end{aligned}$$

where D_t^{β} and D_t^{α} are the standard Riemann-Liouville fractional derivatives with $1 < \alpha \leq 2,\ 0 < \beta \leq 1$, A is a function of bounded variation and $\int_0^1 x(s)\ \mathrm{d}A(s)$ denotes the Riemann-Stieltjes integral of x with respect to A, $f(t, x) : (0, 1) \times (0, \infty) \to [0, \infty)$ is continuous and may be singular at $t = 0, 1$ and $x = 0$. their results are derived based on the method of upper and lower solutions and the Schauder fixed point theorem.

Motivated by the aforementioned works, this paper is concerned with the existence of positive solutions for the coupled system of p-Laplacian fractional differential equations with integral boundary value conditions

$$D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha}u(t))) + f(t, u(t)) = 0,\ t \in (0, 1), \tag{1.1}$$

$$\phi_p(D_{0^+}^{\alpha}u(0))^{(i)} = 0,\ i = 1, 2, \dots, l-1, \tag{1.2}$$

$$\phi_p(D_{0^+}^{\alpha}u(1)) = \int_0^1 h(t)\phi_p(D_{0^+}^{\alpha}u(t))\mathrm{d}t, \tag{1.3}$$

$$u^{(j)}(0) = 0,\quad j = 1, 2, \dots, n-1, \tag{1.4}$$

$$u(0) = \int_0^1 k(t)u(t)\mathrm{d}t, \tag{1.5}$$

where $\phi_p(u) = |u|^{p-2}u$, $p > 1$. $D_{0^+}^{\beta}$ and $D_{0^+}^{\alpha}$ are the Caputo fractional derivatives, $l-1 < \beta \leq l,\ n-1 < \alpha \leq n,\ l \geq 1,\ n \geq 1$ and $l + n - 1 < \alpha + \beta \leq l + n$. Using the five functionals fixed point theorem, the existence of multiple positive solutions is obtained for the aforementioned boundary value problems.

We will suppose the following conditions are satisfied:

(H_1) $k(t),\ h(t) \in L^1[0, 1],\ k(t) \geq 0,\ h(t) \geq 0,\ 0 < \int_0^1 k(t)\mathrm{d}t < 1$ and $0 < \int_0^1 h(t)\mathrm{d}t < 1$;

(H_2) $f(t, u) : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous.

2. Background and definitions

For the convenience, we present some necessary basic knowledge and definitions about fractional calculus theory, which can be found in [9, 10].

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s,$$

provided that the right side is pointwise defined on $(0,\infty)$.

Definition 2.2. For a continuous function $y : (0,\infty) \to R$, the Caputo derivative of fractional order $\alpha > 0$ is defined as

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s)\mathrm{d}s,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0,\infty)$.

Lemma 2.3. *Let $\alpha > 0$ and $u \in AC^{N-1}[0,1]$. Then the fractional differential equation $D_{0+}^{\alpha} u(t) = 0$ has*

$$u(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{N-1} t^{N-1}, \quad c_i \in R,\ i = 1,2,\ldots,N$$

as the unique solution, where N is the smallest integer greater than or equal to α.

Let K be a cone in real Banach space E and γ, φ, θ be nonnegative continuous convex functional on K and let ω, ψ be nonnegative continuous concave functional on K. Then for nonnegative numbers h, a, b, d and c, we define the following convex sets

$$\begin{aligned}
P(\gamma, c) &= \{x \in K | \gamma(x) < c\},\\
P(\gamma, \omega, a, c) &= \{x \in K | a \le \omega(x), \gamma(x) \le c\},\\
Q(\gamma, \varphi, d, c) &= \{x \in K | \varphi(x) \le d, \gamma(x) \le c\},\\
P(\gamma, \theta, \omega, a, b, c) &= \{x \in K | a \le \omega(x), \theta(x) \le b, \gamma(x) \le c\},\\
Q(\gamma, \varphi, \psi, h, d, c) &= \{x \in K | h \le \psi(x), \varphi(x) \le d, \gamma(x) \le c\}.
\end{aligned}$$

Theorem 2.4. *Let K be a cone in real Banach space E. Suppose there exist nonnegative continuous concave functionals ω and ψ on K, and nonnegative continuous convex functionals γ, φ, and θ on K such that for some positive numbers c and m,*

$$\omega(x) \le \varphi(x), \text{ and } \|x\| \le m\gamma(x) \text{ for all } x \in \overline{P(\gamma, c)}.$$

Suppose further that $T : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is completely continuous and there exist $h, d, a, b \ge 0$ with $0 < d < a$ such that each of the following is satisfied:

(C_1) *$\{x \in P(\gamma, \theta, \omega, a, b, c) | \omega(x) > a\} \ne \emptyset$ and $\omega(Tx) > a$ for $x \in P(\gamma, \theta, \omega, a, b, c)$,*

(C_2) *$\{x \in Q(\gamma, \varphi, \psi, h, d, c) | \varphi(x) < d\} \ne \emptyset$ and $\varphi(Tx) < d$ for $x \in Q(\gamma, \varphi, \psi, h, d, c)$,*

(C_3) *$\omega(Tx) > a$ provided $x \in P(\gamma, \omega, a, c)$ with $\theta(Tx) > b$,*

(C_4) *$\varphi(Tx) < d$ provided $x \in Q(\gamma, \varphi, d, c)$ with $\psi(Tx) < h$.*

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$\varphi(x_1) < d,\ a < \omega(x_2) \ \text{ and } \ d < \varphi(x_3) \ \text{ with } \ \omega(x_3) < a.$$

3. Preliminary lemmas

Lemma 3.1. *The boundary value problems (1.1)–(1.5) has a solution $u(t)$ if and only if $u(t)$ solves the equation*

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} r(s)\mathrm{d}s + b_0, \tag{3.1}$$

where

$$r(s) = \phi_q\left(a_0 - \frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f\mathrm{d}\tau\right), \tag{3.2}$$

$$a_0 = \frac{\int_0^1 (1-\tau)^{\beta-1} f\mathrm{d}\tau - \int_0^1 h(t)\left[\int_0^t (t-\tau)^{\beta-1} f\mathrm{d}\tau\right]\mathrm{d}t}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}, \tag{3.3}$$

$$b_0 = \frac{\int_0^1 k(t)\left[\int_0^t (t-s)^{\alpha-1} r(s)\mathrm{d}s\right]\mathrm{d}t}{\Gamma(\alpha)\left[1-\int_0^1 k(t)\mathrm{d}t\right]}. \tag{3.4}$$

$\phi_q(s)$ is the inverse function of $\phi_p(s)$, a.e., $\phi_q(s) = |s|^{q-2}s$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From (1.1), we get

$$D_{0^+}^{\beta}(\phi_p(D_{0^+}^{\alpha}u(t))) = -f(t,u(t)).$$

For $t \in [0,1]$, integrate from 0 to t, in view of Lemma 2.3, we have

$$\phi_p(D_{0^+}^{\alpha}u(t)) = -\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} f(\tau,u(\tau))\mathrm{d}\tau + a_0 + a_1 t + \cdots + a_{l-1}t^{l-1}.$$

By (1.2), we obtain $a_1 = \cdots = a_{l-1} = 0$. i.e.,

$$\phi_p(D_{0^+}^{\alpha}u(t)) = -\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} f(\tau,u(\tau))\mathrm{d}\tau + a_0. \tag{3.5}$$

From (1.3), we get

$$a_0 = \frac{\int_0^1 (1-\tau)^{\beta-1} f\mathrm{d}\tau - \int_0^1 h(t)\left[\int_0^t (t-\tau)^{\beta-1} f\mathrm{d}\tau\right]\mathrm{d}t}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}.$$

In view of (3.5), we have

$$D_{0^+}^{\alpha}u(t) = \phi_q\left(a_0 - \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} f\mathrm{d}\tau\right).$$

For $t \in [0,1]$, integrate from 0 to t, in view of Lemma 2.3, we get

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} r(s)\mathrm{d}s + b_0 + b_1 t + \cdots + b_{n-1}t^{n-1}.$$

By (1.4), we get $b_1 = \cdots = b_{n-1} = 0$. i.e.,

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} r(s)\mathrm{d}s + b_0.$$

From (1.5), we get

$$b_0 = \frac{\int_0^1 k(t)\left[\int_0^t (t-s)^{\alpha-1} r(s)\mathrm{d}s\right]\mathrm{d}t}{\Gamma(\alpha)\left[1-\int_0^1 k(t)\mathrm{d}t\right]}.$$

So we have

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} r(s)\mathrm{d}s + b_0.$$

The proof is complete. □

Lemma 3.2. *Every solution satisfied in* Lemma 3.1 *is non-negative and non-decreasing function for all* $t \in [0,1]$.

Proof. For $f(t,u(t)) \geq 0,\ h(t) \geq 0$ and $0 < \int_0^1 h(t)\mathrm{d}t < 1$, in view of (3.2), we get

$$\begin{aligned}
r(s) &= \phi_q\left(a_0 - \frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f \mathrm{d}\tau\right),\\
&= \phi_q\left(\frac{\int_0^1 (1-\tau)^{\beta-1} f\mathrm{d}\tau - \int_0^1 h(t)\left[\int_0^t (t-\tau)^{\beta-1} f \mathrm{d}\tau\right]\mathrm{d}t}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]} - \frac{\int_0^s (s-\tau)^{\beta-1} f\mathrm{d}\tau}{\Gamma(\beta)}\right)\\
&\geq \phi_q\left(\frac{\int_0^1 (1-\tau)^{\beta-1} f\mathrm{d}\tau - \int_0^1 h(t)\left[\int_0^1 (1-\tau)^{\beta-1} f \mathrm{d}\tau\right]\mathrm{d}t}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]} - \frac{\int_0^1 (1-\tau)^{\beta-1} f\mathrm{d}\tau}{\Gamma(\beta)}\right)\\
&= \phi_q\left(\frac{\int_0^1 (1-\tau)^{\beta-1} f\mathrm{d}\tau}{\Gamma(\beta)} - \frac{\int_0^1 (1-\tau)^{\beta-1} f\mathrm{d}\tau}{\Gamma(\beta)}\right) = 0,
\end{aligned}$$

thus,

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} r(s)\mathrm{d}s + b_0$$

is nondecreasing. $u(0)$ is minimum of $u(t)$ for $t \in [0,1]$. In view of $k(t) \geq 0,\ 0 < \int_0^1 k(t)\mathrm{d}t < 1$, and

$$u(0) = b_0 = \frac{\int_0^1 k(t)\left[\int_0^t (t-s)^{\alpha-1} r(s)\mathrm{d}s\right]\mathrm{d}t}{\Gamma(\alpha)\left[1-\int_0^1 k(t)\mathrm{d}t\right]},$$

so $u(0) \geq 0$. Thus, we obtain $u(t) \geq 0$ for $t \in [0,1]$. The proof is complete. □

4. Main results

Let E be the real Banach space $C[0,1]$ with the maximum norm and define the cone $K \subset E$ by

$$K = \{u \mid u \in E \text{ and } u(t) \text{ are non-negative, nondecreasing function on } [0,1]\}.$$

Define the operator T on K by

$$Tu(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} r(s)\mathrm{d}s + b_0, \tag{4.1}$$

where $r(s),\ b_0$, is defined by (3.2) and (3.4). Obviously, u is a solution of equation (1.1)–(1.5) if and only if u is a fixed point of operator T. In order to obtain the result, we define the nonnegative continuous concave functionals ω, ψ and the nonnegative continuous convex functionals θ, φ, γ on K by

$$\omega(u) = \min_{t\in[t_1,t_2]} u(t) = u(t_1), \tag{4.2}$$

$$\psi(u) = \min_{t\in[\frac{1}{\lambda},1]} u(t) = u\left(\frac{1}{\lambda}\right), \tag{4.3}$$

$$\theta(u) = \max_{t\in[t_1,t_2]} u(t) = u(t_2), \tag{4.4}$$

$$\varphi(u) = \max_{t\in[\frac{1}{\lambda},1]} u(t) = u(1), \tag{4.5}$$

$$\gamma(u) = \max_{t\in[0,1]} u(t) = u(1), \tag{4.6}$$

where $0 < t_1 < t_2 < 1,\quad 0 < \frac{1}{\lambda} < 1$. It is easy to see that, for each $u \in K$,

$$\omega(u) = u(t_1) \le u(1) = \varphi(u) \text{ and } \|u\| = \gamma(u).$$

Now, for the convenience, we introduce the following notations. Let

$$\begin{aligned}
e_1 &= \frac{1}{\Gamma(\alpha+1)\left[1-\int_0^1 k(t)\mathrm{d}t\right]}, & e_2 &= \frac{1}{\Gamma(\beta+1)\left[1-\int_0^1 h(t)\mathrm{d}t\right]},\\
e_3 &= \frac{t_1^\alpha}{\Gamma(\alpha+1)} + \frac{\int_0^1 k(t)t^\alpha \mathrm{d}t}{\Gamma(\alpha+1)\left[1-\int_0^1 k(t)\mathrm{d}t\right]}, & e_5 &= \frac{1-(1-\frac{1}{\lambda})^\beta}{\Gamma(\beta+1)\left[1-\int_0^1 h(t)\mathrm{d}t\right]},\\
e_4 &= \frac{\int_{t_1}^{t_2} h(t)((1-t)^\beta-(1-t_2)^\beta)\mathrm{d}t}{\Gamma(\beta+1)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}, & e_6 &= \frac{(1-\frac{1}{\lambda})^\beta}{\Gamma(\beta+1)\left[1-\int_0^1 h(t)\mathrm{d}t\right]},\\
e_7 &= \left(\frac{t_1}{t_2}\right)^\alpha, & e_8 &= \lambda^\alpha.
\end{aligned}$$

The following theorem is main result in this paper.

Theorem 4.1. *Suppose that the conditions* (H_1) *and* (H_2) *hold. In addition, assume there exist nonnegative numbers* d, a *and* c *such that* $0 < h < e_8 h \le d < a < \frac{a}{e_7} \le b \le c$, *and* $f(t,u)$ *satisfies the following growth conditions:*

(H_3) $f(t,u) \le \frac{1}{e_2}\phi_p\left(\frac{c}{e_1}\right)$, *for* $t \in [0,1]$ *and* $u \in [0,c]$,

(H_4) $f(t,u) > \frac{1}{e_4}\phi_p\left(\frac{a}{e_3}\right)$, *for* $t \in [t_1,t_2]$ *and* $u \in [a,b]$,

(H_5) $f(t,u) < \frac{\phi_p\left(\frac{d}{e_1}\right)-\frac{e_5}{e_2}\phi_p\left(\frac{c}{e_1}\right)}{e_6}$, *for* $t \in [\frac{1}{\lambda},1]$ *and* $u \in [h,d]$.

Then the boundary value problems (1.1)–(1.5) *have at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$\|u_i\| < c, \text{ for } i = 1,2,3,$$
$$\|u_1\| < d, \ a < \omega(u_2), \ \text{ and } \|u_3\| > d, \ \omega(u_3) < a.$$

Proof. First, we show that $T : \overline{P(\gamma,c)} \to \overline{P(\gamma,c)}$ is a completely continuous operator.

Let $u \in K$, in view of (4.1), we have $Tu(t)$ are nonnegative and nondecreasing, consequently, $T : K \to K$. Applying the Arzela-Ascoli Theorem and standard arguments, we conclude that T is a completely continuous operator. If $u \in \overline{P(\gamma,c)}$, in view of (4.1), we have

$$Tu(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} r(s)\mathrm{d}s + b_0,$$

so $\| Tu \| = Tu(1)$, where $r(s)$ and b_0 are defined in (3.2) and (3.4). By (3.2) and condition (H_3), we have

$$\begin{aligned}
r(s) &= \phi_q\left(a_0 - \frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f \mathrm{d}\tau\right) \le \phi_q\left(\frac{\int_0^1 (1-\tau)^{\beta-1} f \mathrm{d}\tau}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}\right)\\
&\le \phi_q\left(\frac{1}{e_2}\phi_p\left(\frac{c}{e_1}\right)\frac{\int_0^1 (1-\tau)^{\beta-1}\mathrm{d}\tau}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}\right) = \phi_q\left(\frac{1}{e_2}\phi_p\left(\frac{c}{e_1}\right)\frac{1}{\Gamma(\beta+1)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}\right) = \frac{c}{e_1}.
\end{aligned}$$

By (4.1), we have

$$Tu(1) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} r(s)\mathrm{d}s + b_0$$

$$
\begin{aligned}
&= \frac{\int_0^1 (1-s)^{\alpha-1} r(s)\mathrm{d}s}{\Gamma(\alpha)} + \frac{\int_0^1 k(t)\left[\int_0^t (t-s)^{\alpha-1} r(s)\mathrm{d}s\right]\mathrm{d}t}{\Gamma(\alpha)\left[1-\int_0^1 k(t)\mathrm{d}t\right]} \\
&\le \frac{c}{e_1}\left(\frac{1}{\Gamma(\alpha+1)} + \frac{\int_0^1 k(t)\mathrm{d}t}{\Gamma(\alpha+1)\left[1-\int_0^1 k(t)\mathrm{d}t\right]}\right) \\
&= \frac{c}{e_1}\left(\frac{1}{\Gamma(\alpha+1)\left[1-\int_0^1 k(t)\mathrm{d}t\right]}\right) = c.
\end{aligned}
$$

Thus, $\|Tu\| \le c$. Consequently, we show $T : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$.

Next, we show that conditions $(C_1) - (C_4)$ in Theorem 2.4 are satisfied for T. It is easy to see that

$$\{u \in P(\gamma, \theta, \omega, a, b, c) | \omega(u) > a\} \neq \emptyset,$$

$$\{u \in Q(\gamma, \varphi, \psi, h, d, c) | \varphi(u) < d\} \neq \emptyset.$$

To prove that the second part of condition (C_1) holds, let $u \in P(\gamma, \theta, \omega, a, b, c)$, in view of (4.2) and (4.4), we get

$$\omega(u) = u(t_1) \ge a, \quad \theta(u) = u(t_2) \le b.$$

It implies that $a \le u(t) \le b$ for $t \in [t_1, t_2]$. From (3.2) and condition (H_4), we get

$$
\begin{aligned}
r(s) &= \phi_q\left(a_0 - \frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f\mathrm{d}\tau\right) \ge \phi_q\left(a_0 - \frac{1}{\Gamma(\beta)}\int_0^1 (1-\tau)^{\beta-1} f\mathrm{d}\tau\right) \\
&= \phi_q\left(\frac{\int_0^1 h(t)\mathrm{d}t \cdot \int_0^1 (1-\tau)^{\beta-1} f\mathrm{d}\tau - \int_0^1 h(t)\int_0^t (t-\tau)^{\beta-1} f\mathrm{d}\tau\mathrm{d}t}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}\right) \\
&\ge \phi_q\left(\frac{\int_0^1 h(t)\int_t^1 (1-\tau)^{\beta-1} f\mathrm{d}\tau\mathrm{d}t}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}\right) \ge \phi_q\left(\frac{\int_{t_1}^{t_2} h(t)\int_t^{t_2} (1-\tau)^{\beta-1} f\mathrm{d}\tau\mathrm{d}t}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}\right) \\
&> \phi_q\left(\frac{\phi_p\left(\frac{a}{e_3}\right)}{e_4}\frac{\int_{t_1}^{t_2} h(t)\int_t^{t_2} (1-\tau)^{\beta-1}\mathrm{d}\tau\mathrm{d}t}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}\right) \\
&\ge \phi_q\left(\frac{\phi_p\left(\frac{a}{e_3}\right)}{e_4}\frac{\int_{t_1}^{t_2} h(t)((1-t)^{\beta} - (1-t_2)^{\beta})\mathrm{d}t}{\Gamma(\beta+1)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}\right) \ge \frac{a}{e_3}.
\end{aligned}
$$

By (4.1), we have

$$
\begin{aligned}
Tu(t_1) &= \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1} r(s)\mathrm{d}s + b_0 \\
&= \frac{\int_0^{t_1} (t_1-s)^{\alpha-1} r(s)\mathrm{d}s}{\Gamma(\alpha)} + \frac{\int_0^1 k(t)\left[\int_0^t (t-s)^{\alpha-1} r(s)\mathrm{d}s\right]\mathrm{d}t}{\Gamma(\alpha)\left[1-\int_0^1 k(t)\mathrm{d}t\right]} \\
&> \frac{a}{e_3}\left(\frac{\int_0^{t_1} (t_1-s)^{\alpha-1}\mathrm{d}s}{\Gamma(\alpha)} + \frac{\int_0^1 k(t)\left[\int_0^t (t-s)^{\alpha-1}\mathrm{d}s\right]\mathrm{d}t}{\Gamma(\alpha)\left[1-\int_0^1 k(t)\mathrm{d}t\right]}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{a}{e_3}\left(\frac{t_1^{\alpha}}{\Gamma(\alpha+1)} + \frac{\int_0^1 k(t)t^{\alpha}\mathrm{d}t}{\Gamma(\alpha+1)\left[1-\int_0^1 k(t)\mathrm{d}t\right]}\right)\\
&= a.
\end{aligned}
$$

Thus, in view of (4.2), we get $\omega(Tu) = Tu(t_1) > a$.

To show that the second part of condition (C_2) holds, let $u \in Q(\gamma,\varphi,\psi,h,d,c)$, in view of (4.3) and (4.5), we get

$$
\psi(u) = u(\frac{1}{\lambda}) \geq h, \quad \varphi(u) = u(1) \leq d.
$$

It implies that $h \leq u(t) \leq d$ for $t \in [\frac{1}{\lambda}, 1]$, and $0 \leq u \leq c$ for $t \in [0,1]$. In view of conditions (H_3), (H_5) and (3.2), we get

$$
\begin{aligned}
r(s) &= \phi_q\left(a_0 - \frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} f \mathrm{d}\tau\right) \leq \phi_q(a_0)\\
&= \phi_q\left(\frac{\int_0^1 (1-\tau)^{\beta-1} f\mathrm{d}\tau - \int_0^1 h(t)\left[\int_0^t (t-\tau)^{\beta-1} f\mathrm{d}\tau\right]\mathrm{d}t}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}\right)\\
&\leq \phi_q\left(\frac{\int_0^1 (1-\tau)^{\beta-1} f\mathrm{d}\tau}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}\right) \leq \phi_q\left(\frac{\int_0^{\frac{1}{\lambda}} (1-\tau)^{\beta-1} f\mathrm{d}\tau}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]} + \frac{\int_{\frac{1}{\lambda}}^1 (1-\tau)^{\beta-1} f\mathrm{d}\tau}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}\right)\\
&< \phi_q\left(\frac{1}{e_2}\phi_p\left(\frac{c}{e_1}\right)\frac{\int_0^{\frac{1}{\lambda}} (1-\tau)^{\beta-1}\mathrm{d}\tau}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]} + \frac{\phi_p\left(\frac{d}{e_1}\right) - \frac{e_5}{e_2}\phi_p\left(\frac{c}{e_1}\right)}{e_6}\frac{\int_{\frac{1}{\lambda}}^1 (1-\tau)^{\beta-1}\mathrm{d}\tau}{\Gamma(\beta)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}\right)\\
&\leq \phi_q\left(\frac{1}{e_2}\phi_p\left(\frac{c}{e_1}\right)\frac{1-(1-\frac{1}{\lambda})^{\beta}}{\Gamma(\beta+1)\left[1-\int_0^1 h(t)\mathrm{d}t\right]} + \frac{\phi_p\left(\frac{d}{e_1}\right) - \frac{e_5}{e_2}\phi_p\left(\frac{c}{e_1}\right)}{e_6}\frac{(1-\frac{1}{\lambda})^{\beta}}{\Gamma(\beta+1)\left[1-\int_0^1 h(t)\mathrm{d}t\right]}\right)\\
&= \frac{d}{e_1}.
\end{aligned}
$$

By (4.1), we have

$$
\begin{aligned}
Tu(1) &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} r(s)\mathrm{d}s + b_0\\
&= \frac{\int_0^1 (1-s)^{\alpha-1} r(s)\mathrm{d}s}{\Gamma(\alpha)} + \frac{\int_0^1 k(t)\left[\int_0^t (t-s)^{\alpha-1} r(s)\mathrm{d}s\right]\mathrm{d}t}{\Gamma(\alpha)\left[1-\int_0^1 k(t)\mathrm{d}t\right]}\\
&< \frac{d}{e_1}\left(\frac{\int_0^1 (1-s)^{\alpha-1}\mathrm{d}s}{\Gamma(\alpha)} + \frac{\int_0^1 k(t)\left[\int_0^t (t-s)^{\alpha-1}\mathrm{d}s\right]\mathrm{d}t}{\Gamma(\alpha)\left[1-\int_0^1 k(t)\mathrm{d}t\right]}\right)\\
&= \frac{d}{e_1}\left(\frac{1}{\Gamma(\alpha+1)} + \frac{\int_0^1 k(t)\mathrm{d}t}{\Gamma(\alpha+1)\left[1-\int_0^1 k(t)\mathrm{d}t\right]}\right)\\
&= \frac{d}{e_1}\left(\frac{1}{\Gamma(\alpha+1)\left[1-\int_0^1 k(t)\mathrm{d}t\right]}\right) = d.
\end{aligned}
$$

So, we have $\varphi(Tu) = Tu(1) < d$. To see that (C_3) is satisfied, let $u \in P(\gamma,\omega,a,c)$ with $\theta(Tu) > b$, that is, $Tu(t_2) > b$. By (4.1), we have

$$Tu(t_1) = \frac{\int_0^{t_1}(t_1-s)^{\alpha-1}r(s)\mathrm{d}s}{\Gamma(\alpha)} + b_0.$$

For

$$\frac{\int_0^{t_1}(t_1-s)^{\alpha-1}r(s)\mathrm{d}s}{\int_0^{t_2}(t_2-s)^{\alpha-1}r(s)\mathrm{d}s} = \frac{t_1^\alpha\int_0^1(1-v)^{\alpha-1}r(t_1v)\mathrm{d}v}{t_2^\alpha\int_0^1(1-v)^{\alpha-1}r(t_2v)\mathrm{d}v} \geq \frac{t_1^\alpha}{t_2^\alpha} = e_7,$$

so, we get

$$Tu(t_1) \geq \frac{e_7\int_0^{t_2}(t_2-s)^{\alpha-1}r(s)\mathrm{d}s}{\Gamma(\alpha)} + b_0 \geq \ e_7\left\{\frac{\int_0^{t_2}(t_2-s)^{\alpha-1}r(s)\mathrm{d}s}{\Gamma(\alpha)} + b_0\right\} = e_7Tu(t_2).$$

Thus, we have

$$\omega(Tu) = Tu(t_1) \geq e_7Tu(t_2) > e_7b \geq a.$$

Finally, to show (C_4), we take $u \in Q(\gamma,\varphi,d,c)$ with $\psi(Tu) < h$, that is $Tu\left(\frac{1}{\lambda}\right) < h$. By (4.1), we have

$$Tu(1) = \frac{\int_0^1(1-s)^{\alpha-1}r(s)\mathrm{d}s}{\Gamma(\alpha)} + b_0.$$

For

$$\frac{\int_0^1(1-s)^{\alpha-1}r(s)\mathrm{d}s}{\int_0^{\frac{1}{\lambda}}(\frac{1}{\lambda}-s)^{\alpha-1}r(s)\mathrm{d}s} = \frac{\int_0^1(1-s)^{\alpha-1}r(s)\mathrm{d}s}{\left(\frac{1}{\lambda}\right)^\alpha\int_0^1(1-v)^{\alpha-1}r(\frac{v}{\lambda})\mathrm{d}v} \leq \lambda^\alpha = e_8,$$

thus, we have

$$Tu(1) \leq \frac{e_8\int_0^{\frac{1}{\lambda}}(\frac{1}{\lambda}-s)^{\alpha-1}r(s)\mathrm{d}s}{\Gamma(\alpha)} + b_0 \leq e_8\left\{\frac{\int_0^{\frac{1}{\lambda}}(\frac{1}{\lambda}-s)^{\alpha-1}r(s)\mathrm{d}s}{\Gamma(\alpha)} + b_0\right\} \leq e_8Tu\left(\frac{1}{\lambda}\right).$$

Therefore, the hypotheses of Theorem 2.4 are satisfied. So the boundary value problems (1.1)–(1.5) have at least three positive solutions u_1, u_2 and u_3 such that

$$\|u_i\| < c \text{ for } i = 1,2,3,$$
$$\|u_1\| < d,\ a < \omega(u_2)\ \text{ and } \|u_3\| > d,\ \omega(u_3) < a.$$

The proof is complete. □

5. Example

In this section, we present a simple example to explain our result.

Consider the following boundary value problems

$$\begin{cases} D_{0^+}^{0.82}(\phi_p(D_{0^+}^{0.7}u(t))) + f(t,u(t)) = 0,\ t \in (0,1), \\ \phi_p(D_{0^+}^{0.7}u(1)) = \int_0^1(1-t)^{0.11}\phi_p(D_{0^+}^{0.7}u(t))\mathrm{d}t, \\ u(0) = \int_0^1 t^{0.1}u(t)\mathrm{d}t, \end{cases}$$

where

$$f(t,u) = \begin{cases} 0.2, & t \in [0,1],\quad 0 \leq u \leq 105, \\ 3(u-105)+0.2, & t \in [0,1],\quad 105 < u \leq 106, \\ 3.2+0.1\sqrt{u-106}, & t \in [0,1],\quad 106 < u \leq 870, \\ 3.2+0.1\sqrt{764}, & t \in [0,1],\quad 870 < u. \end{cases}$$

And we notice that $\alpha = 0.7,\ \beta = 0.82$. If we take $t_1 = 0.05,\ t_2 = 0.99,\ \lambda = 10$, then $0 < t_1 < t_2 < 1$.

It follows from a direct calculation that

$$e_1 = 12.106009,\ e_2 = 10.771129,\ e_3 = 6.860732,\ e_4 = 4.845449,$$
$$e_5 = 0.891513,\quad e_6 = 9.879617,\quad e_7 = 0.123690,\quad e_8 = 5.001872.$$

In addition, if we take $p = 2,\ h = 1,\ d = 105,\ a = 106,\ b = 870$, and $c = 920$, if $f(t,u)$ satisfies the following growth conditions:

$$f(t,u) \le \frac{1}{e_2}\phi_p\left(\frac{c}{e_1}\right) = 7.055464, \qquad t \in [0,1], \qquad u \in [0,920],$$
$$f(t,u) > \frac{1}{e_4}\phi_p\left(\frac{a}{e_3}\right) = 3.188610, \qquad t \in [0.05,0.99],\ u \in [106,870],$$
$$f(t,u) < \frac{\phi_p\left(\frac{d}{e_1}\right) - \frac{e_5}{e_2}\phi_p\left(\frac{c}{e_1}\right)}{e_6} = 0.241238,\quad t \in [0.1,1], \qquad u \in [1,105],$$

then all conditions of Theorem 4.1 are satisfied. Therefore, by Theorem 4.1, we know that the boundary value problems have at least three positive solutions u_1, u_2 and u_3 such that

$$\|u_i\| < 920 \text{ for } i = 1,2,3,$$
$$\|u_1\| < 105,\ \omega(u_2) > 106,\ \text{ and } \|u_3\| > 105,\ \omega(u_3) < 106.$$

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