Available online at www.tjnsa.com
J. Nonlinear Sci. Appl. 9 (2016), 736–756
Research Article

Journal of Nonlinear Science and Applications
Print: ISSN 2008-1898 Online: ISSN 2008-1901

Asymptotic behavior and a posteriori error estimates in Sobolev space for the generalized overlapping domain decomposition method for evolutionary HJB equation with nonlinear source terms. Part 1

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Communicated by Bessem Samet

Abstract

A posteriori error estimates for the generalized overlapping domain decomposition method with Dirichlet boundary conditions on the boundaries for the discrete solutions on subdomains of evolutionary HJB equation with nonlinear source terms are established using the semi-implicit time scheme combined with a finite element spatial approximation. Also the techniques of the residual a posteriori error analysis are used. Moreover, using Benssoussan–Lions' algorithm, an asymptotic behavior in H_0^1-norm is deduced. Furthermore, the results of some numerical experiments are presented to support the theory.

Keywords: A posteriori error estimates, GODDM, Dirichlet boundary conditions, algorithm, asymptotic behavior.
2010 MSC: 65N06, 65N12, 65F05.

1. Introduction

The paper deals with a posteriori error estimates in H^1-norm for the generalized overlapping domain decomposition method for the following evolutionary HJB problem:

$$\text{find } u(t,x) \in \left(L^2(0,T,D(\Omega)) \cap C^{\,2}\left(0,T,H^{-1}(\Omega)\right)\right)^M \text{such that}$$

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Received 2015-07-15

$$\begin{cases} \dfrac{\partial u^i}{\partial t} + \max\limits_{i=1,\ldots,M} \left(A^i u - f^i (u) \right) = 0, \text{ in } \Sigma, \\ \\ u^i = 0 \text{ in } \Gamma, \ u^i (x, 0) = u_0^i \text{ in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^2$, with sufficiently smooth boundary Γ and Σ is a set in $\mathbb{R} \times \mathbb{R}^2$ defined as $\Sigma = [0, \ T] \times \Omega$ with $T < +\infty$. A^i, $(i = 1, \ldots, M)$ are the elliptic operator defined by

$$A^i = \Delta + a_0^i \tag{1.2}$$

and the functions $a_0^i \in \left(L^2 (0, T, L^\infty (\Omega)) \cap C^{\ 0} (0, T, L^\infty (\Omega)) \right)^M, i = 1, \ldots, M$ are sufficiently smooth and satisfy

$$a_0^i(t, x) \geq \beta > 0, \ \beta \text{ is a constant}, \tag{1.3}$$

and where $f^1 (.), \ f^2 (.), \ldots, f^M (.)$ are nonlinear and Lipschitz functions (with Lipschitz constant $c \leq \beta$) satisfying

$$\begin{aligned} &f^i \in \left(L^2 \left(0, T, L^2 (\Omega) \right) \cap C^{\ 1} \left(0, T, H^{-1} (\Omega) \right) \right)^M, \\ &f^i > 0 \text{ and increasing.} \end{aligned} \tag{1.4}$$

K is an implicit convex set defined by

$$K = \begin{cases} \left(u^1, u^2, \ldots, u_h^M \right) \in L^2 \left(0, T, H_0^1 (\Omega) \right) \cap C^{\ 2} \left(0, T, H^{-1} (\Omega) \right), \ u^i (x) \leq l + u^{i+1}, \\ \\ u^i = 0 \text{ in } \Gamma, \ u^i (x, 0) = u_0^i \text{ in } \Omega. \end{cases} \tag{1.5}$$

The symbol $(.,.)_\Omega$ stands for the inner product in $L^2(\Omega)$.

The Schwarz alternating method can be used to solve elliptic boundary value problems on domains which consist of two or more overlapping subdomains. It was invented by Herman Amandus Schwarz in 1890. This method has been used for solving the stationary or evolutionary boundary value problems on domains which consist of two or more overlapping subdomains (see [1]–[5], [6], [9], [15]–[17], [18], [19]–[27]). The solution to these qualitative problems is approximated by an infinite sequence of functions resulting from solving a sequence of stationary or evolutionary boundary value problems in each of the subdomains. An extensive analysis of Schwarz alternating method for nonlinear elliptic boundary value problems can be found in [10]–[12], [13], [19]. Also the effectiveness of Schwarz methods for these problems, especially those in fluid mechanics, has been demonstrated in many papers. See the proceedings of the annual domain decomposition conference [14] and [20]–[22], [23]–[24], [26]. Moreover, a priori estimates of the errors for stationary problems is given in several papers; see for instance [21], [22] where a variational formulation of the classical Schwarz method is derived. In [20], geometry-related convergence results are obtained. In [13, 15, 16], an accelerated version of the GODDM has been treated. In addition, in [13], convergence for simple rectangular or circular geometries has been studied. However, a criterion to stop the iterative process has not been given. All these results can also be found in the recent books on domain decomposition methods [6], [18]. Recently in [15], [16], an improved version of the Schwarz method for highly heterogeneous media has been presented. The method uses new optimized boundary conditions specially designed to take into account the heterogeneity between the subdomains on the boundaries. A recent overview of the current state of the art on domain decomposition methods can be found in [1], [26].

In general, the a priori estimate for stationary problems is not suitable for assessing the quality of the approximate solutions on subdomains, since it depends mainly on the exact solution itself, which is unknown. An alternative approach is to use an approximate solution itself in order to find such an estimate. This approach, known as a posteriori estimate, became very popular in the 1990s with finite element methods; see the monographs [1], [28]. In [28], an algorithm for a nonoverlapping domain decomposition has been given. An a posteriori error analysis for the elliptic case has also been used by [1] to determine an optimal value of the penalty parameter for penalty domain decomposition methods for constructing fast solvers.

Quite a few works on maximum norm error analysis of overlapping nonmatching grids methods for elliptic problems are known in the literature (cf., e.g., [12]–[14]). To prove the main result of this paper, we proceed as in [2]. More precisely, we develop an approach which combines a geometrical convergence result, due to [9], and a lemma which consists of an error estimation in the maximum norm between the continuous and discrete Schwarz iterate.

In [2], the authors derived a posteriori error estimates for the generalized overlapping domain decomposition method (GODDM) with Robin boundary conditions on the boundaries for second order boundary value problems; they have shown that the error estimate in the continuous case depends on the differences of the traces of the subdomain solutions on the boundaries after a discretization of the domain by finite elements method. Also they used the techniques of the residual a posteriori error analysis to get an a posteriori error estimate for the discrete solutions on subdomains.

A numerical study of stationary and evolutionary free boundary problems of the finite element, combined with a finite difference, methods has been achieved in [2], [9]–[16], [25] and using the domain decomposition method combined with finite element method, has been treated in [6, 9, 7, 18]. Moreover, in a recent research [3], we have treated the overlapping domain decomposition method combined with a finite element approximation for elliptic quasi-variational inequalities related to impulse control problem with respect to the mixed boundary conditions for Laplace operator Δ, where a maximum norm analysis of an overlapping Schwarz method on nonmatching grids has been used. Then, in [6] we have extended the last result to the parabolic quasi variational inequalities with the similar conditions, and using the theta time scheme combined with a finite element spatial approximation, we have proved that the discretization on every subdomain converges in uniform norm. Furthermore, a result of asymptotic behavior in uniform norm has been given.

Moreover, in [9], we have been concerned with the system of parabolic quasi-variational inequalities (PQVIs) related to HJB equation with nonlinear source terms. Our goal is to show that evolutionary HJB equations can be properly approximated by Euler time scheme combined with a finite element spatial method, which turns out to be quasi-optimally accurate in uniform norm. Also we want to establish an asymptotic behavior in uniform norm similar to that in [10], where the stationary HJB equation with linear source terms have been investigated. So, we give the following estimate:

$$\left\|U_h^p - U^\infty\right\|_\infty = \max_{1\leq\ i\ \leq M} \left\|u_h^{i,p} - u^{i,\infty}\right\|_\infty \leq C^* \left[h^2 \left|\log h\right|^3 + \left(\frac{1+kc}{1+k\beta}\right)^p\right], \tag{1.6}$$

where C^* is a constant independent of both h and k, $U_h^{,p} = \left(u_h^1, \dots, u_h^p\right)$ is the discrete solution calculated at the moment-end $T = p\Delta t$ for an index of the time discretization $k = 1, \dots, p$, and U^∞ is the asymptotic continuous solution with respect to the right-hand side condition.

We prove an a posteriori error estimates for the generalized overlapping domain decomposition method with Dirichlet boundary conditions on the boundaries for the discrete solutions on subdomains of evolutionary HJB equation with nonlinear source terms using the Euler time scheme combined with a finite element spatial approximation, similar to that in [2], which investigated Laplace equation. Moreover, an asymptotic behavior in H_0^1-norm is deduced using Benssoussan–Lions' algorithms. Furthermore, the results of some numerical experiments are presented to support the theory.

The outline of the paper is as follows: In Section 2, we introduce some necessary notations and give a variational formulation of our model. In Section 3, an a posteriori error estimate is proposed for the convergence of the discrete solution using the semi-implicit time scheme combined with a finite element method on subdomains. In Section 4, we associate with the introduced discrete problem a fixed point mapping and use that in proving the existence of a unique discrete solution. Then in Section 5, an $H_0^1(\Omega)$-asymptotic behavior estimate for each subdomain is derived. Finally, in the same section the results of some numerical experiments are presented to support the theory.

2. Semi-continuous system of parabolic quasi-variational inequalities

Problem (1.1) can be stated using a system of continuous parabolic inequalities in the following way find $u^i \in \left(L^2\left(0,T,D\left(\Omega\right)\right) \cap C^{\ 2}\left(0,T,H^{-1}\left(\Omega\right)\right)\right)^M$ that is a solution to

$$\left\{ \frac{\partial u^i}{\partial t} + \Delta u^i + a_0^i(t,x)u^i \leq f^{\ i}\left(u^i\right) \text{ in } \Sigma, \right.$$

$$\left\{ \begin{array}{l} u^i \leq l + u^{i+1},\ u^{M+1} = u^1,\ i = 1, \ldots, M, \\ \\ \left(\dfrac{\partial u^i}{\partial t} + A^i u^i - f^i\left(u^i\right)\right)\left(u^i - \left(l + u^{i+1}\right)\right) = 0, \\ \\ u^i\left(0,x\right) = u_0^i \text{ in } \Omega, \\ \\ u^i = 0 \text{ in } \Gamma, \end{array} \right. \tag{2.1}$$

which is similar to that in [10] where stationary Hamilton–Jacobi–Bellman equations have been investigated.

2.1. The time discretization

We discretize problem (2.1) with respect to time by using the Euler scheme. Therefore, we search a sequence of elements $u^{i,k} \in \left(H_0^1\left(\Omega\right)\right)^M$, for $k = 1, \ldots, n$ which approach $u^i\left(x, t_k\right),\ t_k = k\Delta t$, with initial data $u^{i,0} = u_0^i$. Thus, we have

$$\left\{ \begin{array}{l} \dfrac{u^{i,k} - u^{i,k-1}}{\Delta t} + \Delta u^{i,k} + a_0^{i,k} u^i \leq f^{\ i,k}\left(u^{i,k}\right) \text{ in } \Sigma, \\ \\ u^{i,k} \leq l + u^{i+1,k},\ u^{M+1,k} = u^{1,k},\ i = 1, \ldots, M, \\ \\ u^{i,0}\left(x\right) = u_0^i \text{ in } \Omega,\ u^i = 0 \text{ on } \partial\Omega, \\ \\ u^i = 0 \text{ in } \Gamma. \end{array} \right. \tag{2.2}$$

First, we define the following mapping

$$\begin{aligned} T : \left(H_0^1\left(\Omega\right)\right)^M &\longrightarrow \left(H_0^1\left(\Omega\right)\right)^M \\ W \longrightarrow TW = \xi^{i,k} &= \left(\xi^{1,k}, \xi^{2,k}, \ldots, \xi^{M,k}\right) = \partial\left(F^{i,k}\left(w^i\right), l + w^{i+1}\right), \end{aligned} \tag{2.3}$$

where $\xi^{i,k}$ $i = 1, \ldots, M$ are solutions to the following problem

$$\left\{ \begin{array}{l} \dfrac{\xi^{i,k} - \xi^{i,k-1}}{\Delta t} + \Delta \xi^{i,k} + a_0^{i,k} \xi^{i,k} \leq f^{\ i}\ \left(\xi^{i,k}\right) \text{ in } \Sigma, \\ \\ \xi^{i,k} \leq l + w^{i+1,k},\ i = 1, \ldots, M, \\ \\ \xi^{M+1,k} = \xi^{1,k}. \end{array} \right. \tag{2.4}$$

2.2. Iterative semi-discrete algorithm

We choose $u^{i,0} = u_0^i$ to be a solution of the following stationary equation:

$$A^{i,0} u^i = g^{i,0}, \tag{2.5}$$

where $g^{i,0}$ is an M-regular function. Now we give the following semi-discrete algorithm:

$$U^k = TU^{k-1}, k = 1, \ldots, n, \tag{2.6}$$

where $U^k = \left(u^{1,k}, \ldots, u^{M,k}\right)$ is a solution of problem (2.2).

Remark 2.1. We denote

$$\mathbf{Q} = \left\{ W \in \left(H_0^1(\Omega)\right)^M, \text{ such that } 0 \leq W \leq U^0 \right\}, \tag{2.7}$$

where $U^0 = U_0 = \left(u_0^1, \ldots, u_0^M\right)$. Since $f^{i,k}(.) \geq 0$, and $u_h^{i,0} = u_{h0}^i \geq 0$, combining comparison results in variational inequalities with a simple induction, we obtain $u^{i,k} \geq 0$, i.e., $U^k \geq 0$, $\forall k = 1, \ldots, M$ and $TW \geq 0$.
Furthermore, by (2.6), (2.7), we have

$$U^1 = TU^0 \leq U^0.$$

As in the previous works [7], [10], the mapping T is a monotone increasing for the stationary HJB equation with nonlinear source term. Then it can be easily verified that

$$U^2 = TU^1 \leq TU^0 = U^1 \leq U^0,$$

thus, inductively

$$U^{k+1} = TU^k \leq U^k \leq \ldots \leq U^0, \ \forall k = 1, \ldots, n$$

and it can be seen also that the sequence $\left(u^k\right)_k$ stays in $\mathbf{Q}$.

According to assumption (1.4), we have that $f^i(.)$ is increasing, and using the remark above, we have for $k = 1, \ldots, n$

$$f^{\ i}\left(u^{i,k}\right) \leq f^{\ i}\left(u^{i,k-1}\right).$$

Then we can rewrite (2.2) as

$$\begin{cases} \dfrac{u^{i,k} - u^{i,k-1}}{\Delta t} - \Delta u^{i,k} + a_0^{i,k} u^i \leq f^{\ i}\left(u^{i,k-1}\right) \text{ in } \Sigma, \\ u^{i,k} \leq l + u^{i+1,k}, \ u^{M+1,k} = u^{1,k}, \ i = 1, \ldots, I, \\ u^i = 0 \text{ in } \Gamma. \end{cases} \tag{2.8}$$

Problem (2.8) can be reformulated via the following coercive discrete system of elliptic quasi-variational inequalities (EQVIs):

$$\begin{cases} b^i\left(u^{i,k}, v^i - u^{i,k}\right) \geq \left(F\left(u^{k-1,m+1}\right), v^i - u_h^{i,k}\right), \quad \text{in } \Sigma, \\ u^{i,k} \leq l + u^{i+1,k}, \ u^{M+1,k} = u^{1,k}, \ i = 1, \ldots, I, \\ u^i = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.9}$$

where

$$\begin{cases} b^i\left(u^{i,k}, v^i - u^{i,k}\right) = \lambda\left(u^{i,k}, v^i - u^{i,k}\right) + a\left(u^{i,k}, v^i - u^{i,k}\right), \\ F\left(u^{k-1,m+1}\right) = f\left(u^{i,k-1}\right) + \lambda u_h^{i,k-1}, \\ \lambda = \dfrac{1}{\Delta t} = \dfrac{1}{k} = \dfrac{T}{n}, \ k = 1, \ldots, n \end{cases} . \tag{2.10}$$

and $a^i(.,.)$ are the following bilinear forms that we associate to A^i defined in (1.2)

$$a^i(.,.) = \underset{\Omega}{dint}\left(\sum_{j,k=1}^{2} \frac{\partial^2}{\partial\, x_j \partial\, x_k} + a_0^i(t,x)\right) dx. \tag{2.11}$$

2.3. The semi-continuous generalized overlapping domain decomposition

Let Ω be a bounded domain in $\mathbb{R}^2$ with a piecewise $C^{1,1}$ boundary $\partial\Omega$. We split the domain Ω into two overlapping subdomains Ω_1 and Ω_2 such that

$$\Omega_1 \cap \Omega_2 = \Omega_{12},\ \partial\Omega_s \cap \Omega_t = \Gamma_s,\ s \neq t \text{ and } s,t = 1,2.$$

We need the spaces

$$V_s = H^1(\Omega) \cap H^1(\Omega_s) = \left\{v \in H^1(\Omega_i) :\ v_{\partial\Omega_i \cap \partial\Omega} = 0\right\}$$

and

$$W_s = H_0^{\frac{1}{2}}(\Gamma_s) = \left\{v_{\Gamma_s},\ v \in V_s \ \text{ and } v = 0 \text{ on } \ \partial\Omega_s \backslash \Gamma_s\right\}, \tag{2.12}$$

which is a subspace of

$$H^{\frac{1}{2}}(\Gamma_s) = \left\{\psi \in L^2(\Gamma_s) : \psi = \varphi_{\Gamma_s} \ \text{ for } \varphi \in V_s,\ s = 1,2\right\}$$

equipped with the norm

$$\|\varphi\|_{W_s} = \inf_{v \in V_s v = \varphi \text{ on } \Gamma_s} \|v\|_{1,\Omega}. \tag{2.13}$$

We define continuous counterparts of the continuous Schwarz sequences defined in (2.9), respectively by

$$\text{find } u_1^{i,k,m+1} \in \left(H_0^1(\Omega)\right)^M,\ m = 0,1,2,\ldots, \text{ to be solutions of}$$

$$\begin{cases} b^i\left(u_1^{i,k,m+1}, v^i - u_1^{i,k,m+1}\right) \geq \left(F\left(u_1^{i,k-1,m+1}\right), v^i - u_1^{i,k,m+1}\right)_{\Omega_1}, \\ u_1^{i,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega = \partial\Omega_1 - \Gamma_1, \\ \dfrac{\partial u_1^{i,k,m+1}}{\partial\eta_1} + \alpha_1 u_1^{i,k,m+1} = \dfrac{\partial u_2^{i,k,m}}{\partial\eta_1} + \alpha_1 u_2^{i,k,m} \text{ on } \Gamma_1 \end{cases} \tag{2.14}$$

and $u_2^{i,k,m+1} \in \left(H_0^1(\Omega)\right)^M$, to be solutions of

$$\begin{cases} b\left(u_2^{i,k,m+1}, v^i - u_2^{i,k,m+1}\right) \geq \left(F\left(u_2^{i,k-1,m+1}\right), v^i - u_2^{i,k,m+1}\right)_{\Omega_2}, \quad m = 0,1,2,\ldots, \\ u_2^{i,k,m+1} = 0, \quad \text{on } \partial\Omega_2 \cap \partial\Omega = \partial\Omega_2 - \Gamma_2, \\ \dfrac{\partial u_2^{i,k,m+1}}{\partial\eta_2} + \alpha_2 u_2^{i,n+1,m+1} = \dfrac{\partial u_1^{i,k,m}}{\partial\eta_2} + \alpha_2 u_1^{i,k,m},\ \text{on } \Gamma_2, \end{cases} \tag{2.15}$$

where η_s is the exterior normal to Ω_s and α_s is a real parameter, $s = 1,2$.

In the next section, our main interest will be to obtain an a posteriori error estimate. We need to stop the iterative process as soon as the required global precision is reached. Namely, by applying Green formula in (1.2) with the new boundary conditions of generalized Schwarz alternating method, we get

$$\begin{aligned} \left(-\Delta u_1^{i,k,m+1}, v_1^i - u_1^{i,k,m+1}\right)_{\Omega_1} &= \left(\nabla u_1^{i,k,m+1}, \nabla\left(v_1^i - u_1^{i,k,m+1}\right)\right)_{\Omega_1} \\ &\quad - \left(\frac{\partial u_1^{i,k,m+1}}{\partial\eta_1}, v_1^i - u_1^{i,k,m+1}\right)_{\partial\Omega_1 - \Gamma_1} + \left(\frac{\partial u_1^{i,k,m+1}}{\partial\eta_1}, v_1^i - u_1^{i,k,m+1}\right)_{\Gamma_1} \\ &= \left(\nabla u_1^{i,k,m+1}, \nabla\left(v_1^i - u_1^{i,k,m+1}\right)\right)_{\Omega_1} - \left(\frac{\partial u_1^{i,k,m+1}}{\partial\eta_1}, v_1^i - u_1^{i,k,m+1}\right)_{\Gamma_1}. \end{aligned}$$

Thus we can deduce

$$\begin{aligned}\left(-\Delta u_1^{i,k,m+1}, v_1^i - u_1^{i,k,m+1}\right)_{\Omega_1} &= \left(\nabla u_1^{i,k,m+1}, \nabla\left(v_1^i - u_1^{i,k,m+1}\right)\right)_{\Omega_1}\\ &\quad - \left(\frac{\partial u_1^{i,k,m+1}}{\partial \eta_1}, v_1^i - u_1^{i,k,m+1}\right)_{\partial\Omega_1 - \Gamma_1} + \left(\frac{\partial u_1^{i,k,m+1}}{\partial \eta_1}, v_1^i - u_1^{i,k,m+1}\right)_{\Gamma_1}\\ &= \left(\nabla u_1^{i,k,m+1}, \nabla\left(v_1^i - u_1^{i,k,m+1}\right)\right)_{\Omega_1}\\ &\quad - \left(\frac{\partial u_2^{i,k,m}}{\partial \eta_2} + \alpha_1 u_2^{i,k,m} - \alpha_1 u_1^{i,k,m+1}, v_1^i - u_1^{i,k,m+1}\right)_{\Gamma_1}\\ &= \left(\nabla u_1^{i,k,m+1}, \nabla\left(v_1^i - u_1^{i,k,m+1}\right)\right)_{\Omega_1} + \left(\alpha_1 u_1^{i,k,m+1}, v_1^i - u_1^{i,k,m+1}\right)_{\Gamma_1}\\ &= \left(\nabla u_1^{i,k,m+1}, \nabla\left(v_1^i - u_1^{i,k,m+1}\right)\right)_{\Omega_1} + \left(\alpha_1 u_1^{i,k,m+1}, v_1^i - u_1^{i,k,m+1}\right)_{\Gamma_1}\\ &\quad - \left(\frac{\partial u_2^{i,k,m}}{\partial \eta_1} + \alpha_1 u_2^{i,k,m}, v_1^i - u_1^{i,k,m+1}\right)_{\Gamma_1}.\end{aligned}$$

So, problem (2.14) is equivalent to:

$$\text{find } u_1^{i,k,m+1} \in (V_1)^M \text{ such that}$$

$$\begin{aligned}&b^i(u_1^{i,k,m+1}, v_1^i - u_1^{i,k,m+1}) + \left(\alpha_1 u_1^{i,n+1,m+1}, v_1^i - u_1^{i,k,m+1}\right)_{\Gamma_1}\\ &\quad\geq \left(F(u^{i,k-1}), v_1^i - u_1^{i,k,m+1}\right)_{\Omega_1} + \left(\frac{\partial u_2^{i,k,m}}{\partial \eta_1} + \alpha_1 u_2^{i,k,m}, v_1^i - u_1^{i,k,m+1}\right)_{\Gamma_1}, \forall v_1 \in V_1;\end{aligned} \tag{2.16}$$

for (2.15) and $u_2^{i,k,m+1} \in V_2$, we want

$$\begin{aligned}&b^i\left(u_2^{i,k,m+1}, v_2^i - u_2^{i,k,m+1}\right) + \left(\alpha_2 u_2^{i,k,m+1}, v_2^i - u_2^{i,k,m+1}\right)_{\Gamma_2}\\ &\quad\geq \left(F(u^{i,k-1}), v_2^i - u_2^{i,k,m+1}\right)_{\Omega_2} + \left(\frac{\partial u_1^{i,k,m}}{\partial \eta_2} + \alpha_2 u_1^{i,k,m}, v_2^i - u_2^{i,k,m+1}\right)_{\Gamma_2}.\end{aligned} \tag{2.17}$$

3. A posteriori error estimate in the continuous case

Since it is numerically easier to compare the subdomain solutions on the boundaries Γ_1 and Γ_2 rather than on the overlap Ω_{12}, we need to introduce two auxiliary problems defined on nonoverlapping subdomains of Ω. This idea allows us to obtain an a posteriori error estimate by following the steps of Otto and Lube [24]. These auxiliary problems are needed for an analysis and not for the computation, to get the estimate.

To define these auxiliary problems we need to split the domain Ω into two sets of disjoint subdomains: (Ω_1, Ω_3) and (Ω_2, Ω_4) such that

$$\Omega = \Omega_1 \cup \Omega_3, \text{ with } \Omega_1 \cap \Omega_3 = \varnothing \qquad \Omega = \Omega_2 \cup \Omega_4, \text{ with } \Omega_2 \cap \Omega_4 = \varnothing.$$

Let $(u_1^{i,k,m}, u_2^{i,k,m})$ be a solution of problems (2.14) and (2.15). We define the couple $(u_1^{k,m}, u_3^{k,m})$ over (Ω_1, Ω_3) to be the solution to the following nonoverlapping problems:

$$\begin{cases} \dfrac{u_1^{i,k,m+1} - u_1^{i,k-1,m+1}}{\Delta t} - \Delta u_1^{i,k,m+1} + a_0^{i,k} u_1^{i,k,m+1} \geq f^{\ i} \ \left(u^{i,m+1,k-1}\right) \text{ in } \Omega_1, \\ u_1^{i,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \ k = 1, \ldots, n, \ i = 1, \ldots, M \\ \dfrac{\partial u_1^{i,k,m+1}}{\partial \eta_1} + \alpha_1 u_1^{i,k,m+1} = \dfrac{\partial u_2^{i,k,m}}{\partial \eta_1} + \alpha_1 u_2^{i,k,m}, \text{ on } \Gamma_1 \end{cases} \tag{3.1}$$

and

$$\begin{cases} \dfrac{u_3^{i,k,m+1} - u^{i,k-1,m+1}}{\Delta t} - \Delta u_3^{i,k,m+1} + a_0^{i,k} u_3^{i,k,m+1} \geq f^{\ i} \left(u^{i,m+1,k-1}\right) \text{in } \Omega_3, \\ u_3^{i,k,m+1} = 0, \quad \text{on } \partial\Omega_3 \cap \partial\Omega, \\ \dfrac{\partial u_3^{i,k,m+1}}{\partial \eta_3} + \alpha_3 u_3^{i,k,m+1} = \dfrac{\partial u_1^{i,k,m}}{\partial \eta_3} + \alpha_3 u_1^{i,k,m}, \text{ on } \Gamma_1. \end{cases} \tag{3.2}$$

One can take $\epsilon_1^{i,n+1,m} = u_2^{i,n+1,m} - u_3^{i,n+1,m}$ on Γ_1, the difference between the overlapping and the nonoverlapping solutions $u_2^{i,n+1,m}$ and $u_3^{i,n+1,m}$ (in problems (2.14)–(2.15) and (3.1)–(3.2), respectively) in Ω_3. Both the overlapping and the nonoverlapping problems converge see [24] that is, $u_2^{i,k,m}$ and $u_3^{i,k,m}$ tend to u_2^i (resp. u_3^i). Then $\epsilon_1^{i,k,m}$ should tend to zero when m tends to infinity in V_2.

By taking

$$\begin{aligned} \Lambda_3^{i,k,m} &= \frac{\partial u_2^{i,k,m}}{\partial \eta_1} + \alpha_1 u_2^{i,k,m}. \\ \Lambda_1^{i,k,m} &= \frac{\partial u_1^{i,k,m}}{\partial \eta_3} + \alpha_3 u_1^{i,k,m}. \\ \Lambda_3^{i,k,m} &= \frac{\partial u_3^{i,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i,k,m} + \frac{\partial \epsilon_1^{i,k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{i,k,m}. \\ \Lambda_1^{i,k,m} &= \frac{\partial u_1^{i,k,m}}{\partial \eta_3} + \alpha_3 u_1^{i,k,m}. \end{aligned} \tag{3.3}$$

and using the Green formula, (3.1) and (3.2) can be transformed into the following systems of elliptic variational inequalities

$$\begin{aligned} & b_1 \left(u_1^{i,k,m+1}, v_1^i - u_1^{i,k,m+1}\right) + \left(\alpha_1 u_1^{i,k,m+1}, v_1^i - u_1^{i,k,m+1}\right)_{\Gamma_1} \\ & \quad \geq \left(F^i(u^{i,m+1,k-1}), v_1^i - u_1^{i,k,m+1}\right)_{\Omega_1} + \left(\Lambda_3^{i,k,m}, v_1^i - u_1^{i,k,m+1}\right)_{\Gamma_1}, \forall v_1^i \in (V_1)^M \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} & b_3^i \left(u_3^{i,k,m+1}, v_3^i - u_3^{i,k,m+1}\right) + \left(\alpha_3 u_3^{i,k,m+1}, v_3^i - u_3^{i,k,m+1}\right)_{\Gamma_1} \\ & \quad \geq \left(F^i \left(u^{i,m+1,k-1}\right), v_3^i - u_3^{i,k,m+1}\right)_{\Omega 3} + \left(\Lambda_1^{i,k,m}, v_3^i - u_3^{i,k,m+1}\right)_{\Gamma_1}, \forall v_3^i \in V_3. \end{aligned} \tag{3.5}$$

On the other hand by taking

$$\theta_1^{i,k,m} = \frac{\partial \epsilon_1^{i,k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{i,k,m}, \tag{3.6}$$

we get

$$\Lambda_3^{i,k,m} = \frac{\partial u_3^{i,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i,k,m} + \frac{\partial \left(u_2^{i,k,m} - u_3^{i,k,m}\right)}{\partial \eta_1} + \alpha_1 \left(u_2^{i,k,m} - u_3^{i,k,m}\right) \tag{3.7}$$

$$= \frac{\partial u_3^{i,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i,k,m} + \frac{\partial \epsilon_1^{i,k,m}}{\partial \eta_1} + \alpha_1 \epsilon_1^{i,k,m}$$

$$= \frac{\partial u_3^{i,k,m}}{\partial \eta_1} + \alpha_1 u_3^{i,k,m} + \theta_1^{i,k,m}.$$

Using [10], we have

$$\begin{aligned}
\Lambda_3^{i,k,m+1} &= \frac{\partial u_3^{i,k,m+1}}{\partial \eta_1} + \alpha_1 u_3^{i,k,m+1} + \theta_1^{i,k,m+1} \\
&= -\frac{\partial u_3^{i,k,m+1}}{\partial \eta_3} + \alpha_1 u_3^{i,k,m+1} + \theta_1^{i,k,m+1} \\
&= \alpha_3 u_3^{i,k,m+1} - \frac{\partial u_1^{i,k,m}}{\partial \eta_3} - \alpha_3 u_1^{i,k,m} + \alpha_1 u_3^{i,k,m+1} + \theta_1^{i,k,m+1} \\
&= (\alpha_1 + \alpha_3) u_3^{i,k,m+1} - \Lambda_1^{i,k,m} + \theta_1^{i,k,m+1}
\end{aligned} \tag{3.8}$$

and by the last equation in (3.7), we have

$$\begin{aligned}
\Lambda_1^{i,k,m+1} &= -\frac{\partial u_1^{i,k,m+1}}{\partial \eta_1} + \alpha_3 u_1^{i,k,m+1} \\
&= \alpha_1 u_1^{i,k,m+1} - \frac{\partial u_2^{i,k,m}}{\partial \eta_1} - \alpha_1 u_2^{i,k,m} + \alpha_3 u_1^{i,k,m+1} + \alpha_3 u_1^{i,k,m+1} \\
&= (\alpha_1 + \alpha_3) u_1^{i,k,m+1} - \Lambda_3^{i,k,m} + \theta_3^{i,k,m+1}.
\end{aligned} \tag{3.9}$$

From (3.9), we can write the next algorithm and two lemmas which will be needed for obtaining an a posteriori error estimate for the problems (3.4), (3.5).

3.1. Semi-discrete algorithm

The sequences $(u_1^{i,k,m}, u_3^{i,k,m})_{m\in\mathbb{N}}$, solutions of (3.4), (3.5) verify the following algorithm:
Step 1: $k = 0$.

Step 2: Let $\Lambda_s^{i,k,0} \in W_1^*$ be an initial value, $s = 1, 3$ (W_1^* is the dual of W_1).

Step 3; Given $\Lambda_t^{k,m} \in W^*$ solve for $s, t = 1, 3, s \neq t$: Find $u_s^{i,k,m+1} \in V_s$ solution of

$$\begin{aligned}
b_s^i(u_s^{i,k,m+1}, v_s^i - u_s^{i,k,m+1}) + \left(\alpha_s u_s^{i,k,m+1}, v_s^i\right)_{\Gamma s} \\
\geq \left(F^i(u^{i,k-1,m+1}), v_s^i\right)_{\Omega s} + \left(\Lambda_t^{i,k,m+1}, v_s^i\right)_{\Gamma s}, \forall v_s \in V_s.
\end{aligned}$$

Step 4: Compute

$$\theta_1^{i,k,m+1} = \frac{\partial \epsilon_1^{i,k,m+1}}{\partial \eta_1} + \alpha_1 \epsilon_1^{i,k,m+1}.$$

Step 5: Compute new data $\Lambda_t^{n+1,m} \in W^*$ solve for $s, t = 1, 3$, from

$$\left(\Lambda_s^{i,k,m+1},\varphi\right)_{\Gamma_i}=\left((\alpha_s+\alpha_t)u_i^{i,k,m+1},v_s\right)_{\Gamma s}-\left(\Lambda_t^{k,m+1},\varphi\right)_{\Gamma_s}+\left(\theta_t^{k,m+1},\varphi\right)_{\Gamma t},\forall\varphi\in W_s, s\neq t.$$

Step 6: Set $m=m+1$ go to **Step 3.**

Step 7: Set $k=k+1$ go to **Step 2.**

Lemma 3.1. *Let* $u_s^{i,k}=u_{\Omega s}^{i,k}$, $e_s^{i,k,m+1}=u_s^{i,k,m+1}-u_s^{i,k}$ *and* $\eta_s^{i,k,m+1}=\Lambda_s^{i,k,m+1}-\Lambda_s^{i,k}$. *Then for* $s,t=1,3, s\neq t$, *we have*

$$b_s^i\left(e_s^{i,k,m+1},v_s^i-e_s^{i,k,m+1}\right)+\left(\alpha_s e_s^{i,k,m+1},v_s^i-e_s^{i,k,m+1}\right)_{\Gamma s}=\left(\eta_t^{i,k,m},v_s^i-e_s^{i,k,m+1}\right)_{\Gamma_s},\forall v_s^i\in V_s^i \tag{3.10}$$

and

$$\left(\eta_s^{i,k,m+1},\varphi\right)_{\Gamma_s}=\left((\alpha_s+\alpha_t)e_s^{i,k,m+1},v_s^i\right)_{\Gamma_s}-\left(\eta_t^{i,k,m},\varphi\right)_{\Gamma_s}+\left(\theta_t^{i,k,m+1},\varphi\right)_{\Gamma s},\forall\varphi\in W_1. \tag{3.11}$$

Proof. 1. We have

$$\begin{cases} b_s^i\left(u_s^{k,m+1},v_s^i-u_s^{i,k,m+1}\right)+\left(\alpha_s u_s^{i,k,m+1},v_s^i-u_s^{i,k,m+1}\right)_{\Gamma_s}, \\ \geq\left(F^i(u_s^{i,k-1,m+1}),v_s^i-u_s^{k,m+1}\right)_{\Omega_s}, \\ +\left(\Lambda_t^{i,k,m},v_s^i-u_s^{k,m+1}\right)_{\Gamma s},\forall v_s^i\in V_s \end{cases}$$

and

$$\begin{cases} b_s^i\left(u_s^{i,k},v_s^i-u_s^{i,k}\right)+\left(\alpha_s u_s^{i,k},v_s-u_s^{i,k}\right)_{\Gamma_s} \\ \geq\left(F^i(u_s^{i,k-1}),v_s^i-u_s^{i,k}\right)_{\Omega_s}, \\ +\left(\Lambda_t^{i,k},v_s^i-u_s^{i,k}\right)_{\Gamma_s},\forall v_s^i\in V_s. \end{cases}$$

Since $b^i\left(.,.\right)$ is a coercive bilinear form, we can deduce

$$b_s^i\left(u_s^{i,k,m+1}-u_s^{i,k+1},v_s^i\right)+\left(\alpha_s u_s^{i,k,m+1}-u_s^{i,k+1},v_s^i\right)_{\Gamma_s}\geq\left(\Lambda_t^{i,k,m}-\Lambda_s^{i,k},v_s^i\right)_{\Gamma_s},\forall v_s^i\in V_s$$

and so

$$b_s^i\left(e_s^{i,k,m+1},v_s^i-e_s^{i,k,m+1}\right)+\left(\alpha_s e_s^{i,k,m+1},v_s^i-e_s^{i,k,m+1}\right)_{\Gamma_s}\geq\left(\eta_s^{i,k,m},v_1^i-e_s^{i,k,m+1}\right)_{\Gamma_s},\forall v_s^i\in V_s.$$

2. We have $\lim\limits_{m\to+\infty}\epsilon_1^{i,k+1,m}=\lim\limits_{m\to+\infty}\theta_1^{i,k+1,m}=0$. Then

$$\Lambda_s^{i,k}=(\alpha_1+\alpha_3)u_s^{i,k}-\Lambda_t^{i,k}.$$

Therefore

$$\begin{aligned}\eta_s^{i,k,m+1}&=\Lambda_s^{i,k,m+1}-\Lambda_s^{i,k}\\ &=(\alpha_1+\alpha_3)u_s^{i,k,m+1}-\Lambda_t^{i,k,m}+\theta_t^{i,k,m+1}-(\alpha_1+\alpha_3)u_i^{i,k}+\Lambda_j^{i,k}\\ &=(\alpha_1+\alpha_3)(u_1^{i,k,m+1}-u_s^{i,k})-(\Lambda_t^{i,k,m}-\Lambda_t^{i,k})+\theta_t^{i,k,m+1}.\end{aligned}$$

$\square$

Lemma 3.2. *By letting C be a generic constant which has different values at different places, one gets for $s,t=1,3,s\neq t$*

$$\left(\eta_s^{i,k,m-1}-\alpha_s e_s^{i,k,m},w^i\right)_{\Gamma_1}\leqslant C\left\|e_s^{i,k,m}\right\|_{1,\Omega s}\left\|w^i\right\|_{W_1} \tag{3.12}$$

and

$$\left(\alpha_s w_s^i+\theta_1^{i,k,m+1},e_s^{i,k,m+1}\right)_{\Gamma_1}\leqslant C\left\|e_s^{i,k,m+1}\right\|_{1,\Omega_s}\left\|w^i\right\|_{W_1}. \tag{3.13}$$

Proof. Using Lemma 3.1 and the fact of the inverse of the trace mapping $Tr_i^{-1}:W_1\longrightarrow V_s$ is continuous, we have for $s,t=1,3,s\neq t$

$$\begin{aligned}\left(\eta_s^{i,k,m-1}-\alpha_s e_s^{i,k,m},w^i\right)_{\Gamma s}&=b_s^i(e_t^{i,k,m},Tr^{-1}w^i)=\left(\nabla e_s^{i,k,m},\nabla Tr^{-1}w^i\right)_{\Omega_i}\\&\quad+\left(\alpha e_i^{i,k,m},Tr^{-1}w^i\right)_{\Omega_i}+\lambda\left(e_s^{i,k,m},Tr^{-1}w^i\right)_{\Omega_i}\\&\leqslant\left|e_s^{i,k,m}\right|_{1,\Omega_s}\left|Tr^{-1}w^i\right|_{1,\Omega_s}+\|\alpha\|_\infty\left\|e_s^{i,k,m}\right\|_{0,\Omega_s}\left\|Tr^{-1}w^i\right\|_{0,\Omega_s}\\&\quad+|\lambda|\left\|e_s^{i,k,m}\right\|_{0,\Omega_s}\left\|Tr^{-1}w^i\right\|_{0,\Omega_s}\\&\leqslant C\left\|e_s^{i,k,m}\right\|_{1,\Omega_i}\left\|w^i\right\|_{W_1}.\end{aligned}$$

For the second estimate, we have

$$\begin{aligned}\left(\alpha_s w_s^i+\theta_1^{i,k,m+1},e_s^{i,k,m+1}\right)_{\Gamma_s}&=\left(\alpha_s w_s+\theta_1^{i,k,m+1},e_s^{i,k,m+1}\right)_{\Gamma_s}\\&\leqslant\left\|\alpha_s w_s+\theta_1^{i,k,m+1}\right\|_{0,\Gamma_1}\left\|e_s^{i,k,m+1}\right\|_{0,\Gamma_1}\\&\leqslant\left(|\alpha_s|\left\|w_s^i\right\|_{0,\Gamma_1}+\left\|\theta_1^{i,k,m+1}\right\|_{0,\Gamma_1}\right)\left\|e_s^{i,k+1,m+1}\right\|_{0,\Gamma_1}\\&\leq\max\left(|\alpha_s|,\left\|\theta_1^{i,k,m+1}\right\|_{0,\Gamma_1}\right)\left\|w_s^i\right\|_{0,\Gamma_1}\left\|e_s^{i,k,m+1}\right\|_{0,\Gamma_1}\\&\leq C\left\|e_s^{i,k,m+1}\right\|_{0,\Gamma_1}\left\|w_s^i\right\|_{0,\Gamma_1}\leqslant C\left\|e_s^{i,k,m+1}\right\|_{0,\Gamma_1}\left\|w_s^i\right\|_{W_1}.\end{aligned}$$

Thus, it can be deduced that

$$|\alpha_s|\left\|w_s\right\|_{0,\Gamma_1}+\left\|\theta_1^{i,k,m+1}\right\|_{0,\Gamma_1}\leqslant\max\left(|\alpha_s|,\left\|\theta_1^{i,k,m+1}\right\|_{0,\Gamma_1}\right)\left\|w_s^i\right\|_{0,\Gamma_1}.$$

□

Proposition 3.3. *For the sequences $\left(u_1^{i,k,m},u_3^{i,k,m}\right)_{m\in\mathbb{N}}$, solutions of (3.4), (3.5), we have the following a posteriori error estimation*

$$\left\|u_1^{i,k,m+1}-u_1^{i,k}\right\|_{1,\Omega_1}+\left\|u_3^{i,k,m}-u_3^{k}\right\|_{3,\Omega_3}\leqslant C\left\|u_1^{i,k,m+1}-u_3^{i,k,m}\right\|_{W_1}.$$

Proof. From (3.9), (3.11) and by taking $v_s^i=v_1^i-u^{i,k,m+1}$ in (3.4), we have

$$\begin{aligned}&b_1^i\left(e_1^{i,k,m+1},v_1^i\right)+b_3^i\left(e_3^{i,k,m},v_3^i\right)\\&=\left(\eta_3^{i,k,m}-\alpha_1 e_1^{i,k,m+1},v_1^i\right)_{\Gamma_1}+\left(\eta_1^{i,k,m-1}-\alpha_3 e_3^{i,k,m},v_3^i\right)_{\Gamma_1}\\&=\left(\eta_3^{i,k+1,m}-\alpha_1 e_1^{i,k+1,m+1},v_1^i\right)_{\Gamma_1}+\left(\eta_1^{i,k+1,m-1}-\alpha_3 e_3^{i,k+1,m},v_3^i\right)_{\Gamma_1}\\&\quad+\left(\eta_1^{i,k,m-1}-\alpha_3 e_3^{i,k+1,m},v_1^i\right)_{\Gamma_1}-\left(\eta_1^{i,k+1,m-1}-\alpha_3 e_3^{i,k+1,m},v_1^i\right)_{\Gamma_1}.\end{aligned}$$

Thus, we have

$$
\begin{aligned}
& b_1^i\left(e_1^{i,k,m+1}, v_1^i\right) + b_3^i\left(e_3^{i,k,m}, v_3^i\right) \\
&= \left(\eta_3^{i,n+1,m} - \alpha_1 e_1^{i,k+1,m+1} + \eta_1^{i,k+1,m-1} - \alpha_3 e_3^{i,k+1,m}, v_1^i\right)_{\Gamma_1} \qquad (3.14) \\
&\quad + \left(\eta_1^{i,k+1,m-1} - \alpha_3 e_3^{i,k+1,m}, v_3^i - v_1^i\right)_{\Gamma_1} \\
&= \left((\alpha_1+\alpha_3) e_3^{i,k+1,m} + \theta_1^{i,k+1,m} - \alpha_1 e_1^{i,k+1,m+1} - \alpha_3 e_3^{i,k+1,m}, v_1^i\right)_{\Gamma_1} \\
&\quad + \left(\eta_1^{i,k+1,m-1} - \alpha_3 e_3^{i,k+1,m}, v_3^i - v_1^i\right)_{\Gamma_1} \\
&= \left(\alpha_1(e_3^{i,k+1,m} - e_1^{i,k+1,m+1}) + \theta_1^{i,k+1,m}, v_1\right)_{\Gamma_1} + \left(\eta_1^{i,k+1,m-1} - \alpha_3 e_3^{i,k+1,m}, v_3^i - v_1^i\right)_{\Gamma_1}. \qquad (3.15)
\end{aligned}
$$

Taking $v_1^i = e_1^{i,k+1,m+1}$ and $v_3^i = e_3^{i,k+1,m}$ in (3.15), then using $\frac{1}{2}(a+b) \leqslant a^2+b^2$ and Lemma 3.2, we get

$$
\begin{aligned}
& \frac{1}{2}\left(\left\|u_1^{i,k,m+1} - u_1^{i,k+1}\right\|_{1,\Omega_1} + \left\|u_3^{i,k,m} - u_3^{i,k+1}\right\|_{3,\Omega_3}\right)^2 \\
&\quad \leqslant \left\|u_1^{i,k,m+1} - u_1^{i,k}\right\|_{1,\Omega_1}^2 + \left\|u_3^{i,k,m} - u_3^{i,k}\right\|_{3,\Omega_3}^2 \\
&\quad \leq \left\|e_1^{i,k,m+1}\right\|_{1,\Omega_1}^2 + \left\|e_3^{i,k,m}\right\|_{3,\Omega_3}^2 \\
&\quad \leq \left(\nabla e_1^{i,k,m+1}, \nabla e_1^{i,k,m+1}\right)_{\Omega_1} + \left(a_0^i e_1^{i,k,m+1}, e_1^{i,k,m+1}\right)_{\Omega_3} \\
&\qquad + \left(\nabla e_3^{i,k,m}, \nabla e_3^{n+1,m}\right)_{\Omega_1} + \left(a_0^i e_3^{i,k,m}, e_3^{i,k,m}\right)_{\Omega_3} \\
&\quad \leqslant \left(\nabla e_1^{i,k,m+1}, \nabla e_1^{k,m+1}\right)_{\Omega_1} + \left\|a_0^i\right\|_\infty \left(e_1^{i,k,m+1}, e_1^{i,k,m+1}\right)_{\Omega_1} \\
&\qquad + \left(\nabla e_3^{i,k,m}, \nabla e_3^{i,k,m}\right)_{\Omega_1} + \left\|a_0^i\right\|_\infty \left(e_3^{i,k,m}, e_3^{i,k,m}\right)_{\Omega_3}.
\end{aligned}
$$

Then

$$
\begin{aligned}
& \frac{1}{2}\left(\left\|u_1^{i,k,m+1} - u_1^{i,k+1}\right\|_{1,\Omega_1} + \left\|u_3^{i,k,m} - u_3^{i,k+1}\right\|_{3,\Omega_3}\right)^2 \\
&\quad \leqslant \max\left(1, \left\|a_0^i\right\|_\infty\right)\left(b_1^i\left(e_1^{i,k,m+1}, e_1^{i,k,m+1}\right) + b_3^i\left(e_3^{i,k,m}, e_3^{i,k,m}\right)\right) \\
&\quad = \max\left(1, \left\|a_0^i\right\|_\infty\right)\left(\alpha_1(e_3^{i,k,m} - e_1^{i,k,m+1}) + \theta_1^{i,k,m}, e_1^{i,k,m+1}\right)_{\Gamma_1} + \left(\eta_1^{i,k,m-1} - \alpha_3 e_3^{i,k,m}, e_3^{i,k,m} - e_1^{i,k,m+1}\right)_{\Gamma_1} \\
&\quad \leqslant C_1 \left\|e_1^{i,k,m+1}\right\|_{1,\Omega_1} \left\|e_3^{i,k,m} - e_1^{i,k,m+1}\right\|_{W_1} + C_1 \left\|e_3^{i,k,m}\right\|_{3,\Omega_3} \left\|e_3^{i,k,m} - e_1^{i,k,m+1}\right\|_{W_1} \\
&\quad \leqslant C_1 \left[\left\|e_1^{i,k,m+1}\right\|_{1,\Omega_1} + \left\|e_3^{i,k,m}\right\|_{3,\Omega_3}\right] \left\|e_3^{i,k,m} - e_1^{i,k,m+1}\right\|_{W_1},
\end{aligned}
$$

thus

$$
\left\|e_1^{i,k+1,m+1}\right\|_{1,\Omega_1} + \left\|e_3^{i,k+1,m}\right\|_{3,\Omega_3} \leqslant \left\|e_1^{i,k+1,m+1} - e_3^{i,k+1,m}\right\|_{W_1}.
$$

Therefore

$$
\left\|u_1^{i,k+1,m+1} - u_1^{k+1}\right\|_{1,\Omega_1} + \left\|u_3^{k+1,m} - u_3^{k+1}\right\|_{3,\Omega_3} \leqslant 2C_1 \left\|u_1^{n+1,m+1} - u_3^{n+1,m}\right\|_{W_1}.
$$

□

In a similar way, we define another nonoverlapping auxiliary problem over (Ω_2, Ω_4), and we get the same result.

Proposition 3.4. *For the sequences* $(u_2^{i,k,m}, u_4^{i,k,m})_{m\in\mathbb{N}}$, *we have the following a posteriori error estimation*

$$\left\|u_2^{i,k,m+1} - u_2^{i,k}\right\|_{2,\Omega_2} + \left\|u_4^{i,k,m} - u_4^{i,k}\right\|_{4,\Omega_4} \leqslant C\left\|u_2^{i,k,m+1} - u_4^{i,k,m}\right\|_{W_2}. \tag{3.16}$$

Proof. The proof is very similar to the proof of Proposition 3.3. □

Theorem 3.5. *Let* $u_s^{i,k} = u_{\Omega_s}^{i,k}$. *For the sequences* $(u_1^{i,k,m}, u_2^{i,k,m})_{m\in\mathbb{N}}$, *solutions of problems* (3.1), (3.2), *we have*

$$\begin{aligned}&\left\|u_1^{i,k,m+1} - u_1^{i,k}\right\|_{1,\Omega_1} + \left\|u_2^{i,k,m} - u_2^{i,k}\right\|_{2,\Omega_2}\\ &\quad\leqslant C\left(\left\|u_1^{i,k,m+1} - u_2^{i,k,m}\right\|_{W_1} + \left\|u_2^{i,k,m} - u_1^{i,k,m-1}\right\|_{W_2} + \left\|e_1^{i,k,m}\right\|_{W_1} + \left\|e_2^{i,k,m-1}\right\|_{W_2}\right).\end{aligned}$$

Proof. We use two nonoverlapping auxiliary problems over (Ω_1, Ω_3) and over (Ω_2, Ω_4), respectively. From the previous two propositions, we have

$$\begin{aligned}&\left\|u_1^{i,k,m+1} - u_1^{i,k}\right\|_{1,\Omega_1} + \left\|u_2^{i,k,m} - u_2^{i,k}\right\|_{2,\Omega_2}\\ &\quad\leqslant \left\|u_1^{i,k,m+1} - u_1^{i,k}\right\|_{1,\Omega_1} + \left\|u_3^{i,k,m} - u_3^{i,k}\right\|_{3,\Omega_3}\\ &\qquad + \left\|u_2^{i,k,m} - u_2^{i,k+1}\right\|_{2,\Omega_2} + \left\|u_4^{i,k,m-1} - u_4^{i,k+1}\right\|_{4,\Omega_4}\\ &\quad\leqslant C\left\|u_1^{i,k,m+1} - u_3^{i,k+1,m}\right\|_{W_1} + C\left\|u_2^{i,k,m} - u_4^{i,k,m-1}\right\|_{W_2}\\ &\quad\leqslant C\left(\left\|u_1^{i,k,m+1} - u_2^{i,k,m} + \epsilon_1^{i,k+1,m}\right\|_{W_1} + \left\|u_2^{i,k,m} - u_1^{i,k,m-1} + \epsilon_2^{i,k,m-1}\right\|_{W_2}\right).\end{aligned}$$

Then

$$\begin{aligned}&\left\|u_1^{i,k,m+1} - u_1^{i,k}\right\|_{1,\Omega_1} + \left\|u_2^{i,k,m} - u_2^{i,k}\right\|_{2,\Omega_2}\\ &\quad\leqslant C\left(\left\|u_1^{i,k,m+1} - u_2^{i,k,m} + \epsilon_1^{i,k,m}\right\|_{W_1} + \left\|u_2^{i,k,m} - u_1^{i,k,m-1} + \epsilon_2^{i,k,m-1}\right\|_{W_2} + \left\|\epsilon_1^{i,k,m}\right\|_{W_1} + \left\|\epsilon_2^{i,k,m-1}\right\|_{W_2}\right).\end{aligned}$$

□

4. A posteriori error estimate in the discrete case

4.1. The space discretization

Let Ω be decomposed into triangles and τ_h denote the set of all those elements where $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform. We consider the usual basis of affine functions φ_i $i = \{1, \ldots, m(h)\}$ defined by $\varphi_i(M_j) = \delta_{ij}$, where M_j is a vertex of the considered triangle.

We discretize in space, i.e., we approach the space H_0^1 by a finite dimensional space $V^h \subset H_0^1$. In the second step, we discretize the problem with respect to time using the semi-implicit scheme. Therefore, we search a sequence of elements $u_h^n \in V^h$ which approaches $u_h(t_n, .)$, $t_n = n\Delta t$, $k = 1, \ldots, n$, with initial data $u_h^0 = u_{0h}$.

Let $u_h^{i,k,m+1} \in V_h$ be a solution of the discrete problem associated with (3.1), $u_{s,h}^{m+1} = u_{h\Omega_s}^{m+1}$. We construct the sequences $(u_{s,h}^{i,k,m+1})_{m\in\mathbb{N}}, u_{s,h}^{i,k,m+1} \in K_{i,h}, (s = 1, 2)$ solutions of the discrete problems associated with (3.1), (3.2) where K_h is the set given by

$$K_h = \begin{cases} \left(u_h^1, \dots, u_h^M\right) \in \left(L^2\left(0, T, H_0^1(\Omega)\right) \cap C\left(0, T, H_0^1(\Omega)\right)\right)^M, \\ u_h^i(x) \le r_h\left(l + u^{i+1}\right), \\ u_h^i = 0 \text{ in } \Gamma, \ u_h^i(t_0) = u_{h,0}^i \text{ in } \Omega, \end{cases} \tag{4.1}$$

where r_h is the usual interpolation operator defined by

$$r_h v = \sum_{i=1}^{m(h)} v(M_j)\, \varphi_i(x). \tag{4.2}$$

In a manner similar to that of the previous section, we introduce two auxiliary problems. We define for (Ω_1, Ω_3) the following full-discrete problems:

$$\text{find } u_{1,h}^{i,k,m+1} \in K_h \text{ such that}$$

$$\begin{cases} b_1^i\left(u_{1,h}^{i,k,m+1}, \tilde{v}_{1,h} - u_{1,h}^{i,k,m+1}\right) + \left(\alpha_{1,h} u_{1,h}^{i,k,m+1}, \tilde{v}_{1,h} - u_{1,h}^{i,k,m+1}\right)_{\Gamma_1} \\ \ge \left(F(u_{1,h}^{i,k-1,m+1}), \tilde{v}_{1,h} - u_{1,h}^{i,k,m+1}\right)_{\Omega_1}, \quad \tilde{v}_{1,h} \in K_h, \\ u_{1,h}^{i,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \\ \dfrac{\partial u_{1,h}^{i,k,m+1}}{\partial \eta_1} + \alpha_1 u_{1,h}^{i,k,m+1} = \dfrac{\partial u_{2,h}^{i,k,m}}{\partial \eta_1} + \alpha_1 u_{2,h}^{i,k,m}, \ \text{on } \Gamma_1. \end{cases} \tag{4.3}$$

By taking the trial function $\tilde{v}_{1,h} = v_{1,h}^i - u_{1,h}^{i,k,m+1}$ in (4.2), we get

$$\begin{cases} b_1^i\left(u_{1,h}^{i,k,m+1}, v_{1,h}^i\right) + \left(\alpha_{1,h} u_{1,h}^{i,k,m+1}, v_{1,h}^i\right)_{\Gamma_1} \\ \le \left(F(u_{1,h}^{i,k-1,m+1}), v_{1,h}^i\right)_{\Omega_1}, \quad v_{1,h}^i \in K_h, \\ u_{1,h}^{i,k,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \\ \dfrac{\partial u_{1,h}^{i,k,m+1}}{\partial \eta_1} + \alpha_1 u_{1,h}^{i,k,m+1} = \dfrac{\partial u_{2,h}^{i,k,m}}{\partial \eta_1} + \alpha_1 u_{2,h}^{i,k,m}, \ \text{on } \Gamma_1. \end{cases} \tag{4.4}$$

Similarly, we get

$$\begin{cases} b_1^i\left(u_{3,h}^{i,k,m+1}, v_{1,h}^i\right) + \left(\alpha_{3,h} u_{3,h}^{i,k,m+1}, v_{1,h}^i\right)_{\Gamma_1} \le \left(F^i(u_{3,h}^{i,k-1,m+1}), v_{1,h}^i\right)_{\Omega_3}, \\ u_{3,h}^{i,k,m+1} = 0, \quad \text{on } \partial\Omega_3 \cap \partial\Omega, \\ \dfrac{\partial u_{3,h}^{i,k,m+1}}{\partial \eta_3} + \alpha_3 u_{3,h}^{i,k,m+1} = \dfrac{\partial u_1^{i,k,m}}{\partial \eta_3} + \alpha_3 u_{1,h}^{i,k,m}, \ \text{on } \Gamma_1. \end{cases} \tag{4.5}$$

For (Ω_2, Ω_4) , we have

$$\begin{cases} b_1^i\left(u_{2,h}^{i,k,m+1}, v_{2,h}^i\right) + \left(\alpha_{2,h} u_{2,h}^{i,k,m+1}, v_{2,h}^i\right)_{\Gamma_1} \leq \left(F^i(u_{2,h}^{i,k-1,m+1}), v_{2,h}^i\right)_{\Omega_2}, \\ u_{2,h}^{i,k,m+1} = 0, \quad \text{on } \partial\Omega_2 \cap \partial\Omega, \\ \dfrac{\partial u_{2,h}^{i,k,m+1}}{\partial \eta_2} + \alpha_2 u_{2,h}^{i,k,m+1} = \dfrac{\partial u_{1,h}^{i,k,m}}{\partial \eta_2} + \alpha_2 u_{1,h}^{i,k,m}, \text{ on } \Gamma_2 \end{cases}$$

and

$$\begin{cases} b_1^i(u_{4,h}^{i,k,m+1}, v_{4,h}^i) + \left(\alpha_{4,h} u_{4,h}^{i,k,m+1}, v_{4,h}^i\right)_{\Gamma_1} \leq \left(F^i(u_{4,h}^{i,k-1}), v_{4,h}^i\right)_{\Omega_4}, \\ u_{4,h}^{n+1,m+1} = 0, \quad \text{on } \partial\Omega_1 \cap \partial\Omega, \\ \dfrac{\partial u_{4,h}^{n+1,m+1}}{\partial \eta_4} + \alpha_4 u_{4,h}^{n+1,m+1} = \dfrac{\partial u_{2,h}^{n+1,m}}{\partial \eta_4} + \alpha_4 u_{2,h}^{n+1,m}, \text{ on } \Gamma_2. \end{cases} \tag{4.6}$$

Theorem 4.1 ([8])**.** *The solution of the system of QVI* (4.3), (4.4), *and* (4.5) *is the maximum element of the discrete subsolutions set.*

We can obtain discrete counterparts of Propositions 3.3 and 3.4 by almost the same analysis as in the section above (i.e., passing from continuous spaces to discrete subspaces and from continuous sequences to discrete ones). Therefore,

$$\left\| u_1^{i,k,m+1} - u_1^{i,k} \right\|_{1,\Omega_1} + \left\| u_3^{i,k,m} - u_3^{i,k} \right\|_{1,\Omega_3} \leqslant C \left\| u_1^{i,k,m+1} - u_3^{i,k,m} \right\|_{W_1} \tag{4.7}$$

and

$$\left\| u_2^{i,k,m+1} - u_2^{i,k} \right\|_{1,\Omega_2} + \left\| u_4^{i,k,m} - u_4^{i,k} \right\|_{1,\Omega_4} \leqslant C \left\| u_2^{i,k,m+1} - u_4^{i,k,m} \right\|_{W_2}. \tag{4.8}$$

As in the proof of Theorem 3.5, we get the following discrete estimates:

$$\begin{aligned} &\left\| u_{1,h}^{i,k,m+1} - u_{1,h}^{i,k} \right\|_{1,\Omega_1} + \left\| u_{2,h}^{i,k,m} - u_{2,h}^{i,k} \right\|_{1,\Omega_2} \\ &\quad \leqslant C \left(\left\| u_{1,h}^{i,k,m+1} - u_{2,h}^{i,k,m} \right\|_{W_1} + \left\| u_{2,h}^{i,k,m} - u_{1,h}^{i,k,m-1} \right\|_{W_2} + \left\| e_{1,h}^{i,k+1,m} \right\|_{W_1} + \left\| e_{2,h}^{i,k+1,m-1} \right\|_{W_2} \right). \end{aligned}$$

Next we will obtain an error estimate between the approximated solution $u_{s,h}^{i,k,m+1}$ and the semidiscrete solution $u^{i,k}$. We introduce some necessary notations.

We denote

$$\varepsilon_h = \{E \in T : T \in \tau_h \text{ and } E \notin \partial\Omega\}$$

and for every $T \in \tau_h$ and $E \in \varepsilon_h$, we define

$$\omega_T = \left\{T' \in \tau_h : T' \cap T \neq \varnothing\right\}, \quad \omega_E = \left\{T' \in \tau_h : T' \cap E \neq \varnothing\right\}.$$

The right-hand side f of (2.1) is not necessarily a continuous function across two neighboring elements of τ_h having E as a common side, $[f]$ denotes the jump of f across E and η_E the normal vector of E.

We have the following theorem which gives an a posteriori error estimate for the discrete GODDM.

Theorem 4.2. *Let* $u_s^k = u^k \mid_{\Omega_s}$ *and the sequences* $\left(u_{1,h}^{i,k,m+1}, u_{2,h}^{i,k,m}\right)_{m\in\mathbb{N}}$ *be solutions of problems* (3.4) *and* (3.1). *Then there exists a constant* C *independent of* h *such that*

$$\left\| u_{1,h}^{i,k,m+1} - u_1^{i,k} \right\|_{1,\Omega_1} + \left\| u_{2,h}^{i,k,m} - u_2^{i,k} \right\|_{1,\Omega_2} \leqslant C \left\{ \sum_{i=1}^{2} \underset{T\in\tau_h}{sum} \left(\eta_i^T\right) + \eta_{\Gamma_s} \right\},$$

where

$$\eta_{\Gamma_s} = \left\| u_{h,s}^{i,k,*} - u_{h,t}^{i,k,*-1} \right\|_{W_{h,s}} + \left\| \epsilon_{i,h}^{i,k,*} \right\|_{W_{h,s}}$$

and

$$\eta_s^T = h_T \left\| F\left(u_{h,s}^{i,k,*}\right) + u_{h,s}^{i,k-1} + \Delta u_{h,s}^{i,k,*} - \left(1 + \lambda a_{h0}^{i,k}\right) u_{h,s}^{i,k} \right\|_{0,T} + \sum_{E \in \varepsilon_h} h_E^{\frac{1}{2}} \left\| \left[\frac{\partial u_{h,s}^{i,k,*}}{\partial \eta_E} \right] \right\|_{0,E},$$

where C is Lipschitz constant of the right-hand side and the symbol $*$ corresponds to $m+1$ when $s=1$ and to m when $s=2$.

Proof. The proof is based on the technique of the residual a posteriori estimation see [24] and on Theorem 4.1. We give the main steps. By the triangle inequality, we have

$$\sum_{s=1}^{2} \left\| u_s^{i,k} - u_{h,s}^{i,k,*} \right\|_{1,\Omega_s} \leqslant \sum_{s=1}^{2} \left\| u_s^{i,k} - u_{h,s}^{i,k} \right\|_{1,\Omega_s} + \sum_{s=1}^{2} \left\| u_{h,s}^{i,k} - u_{i,h}^{*} \right\|_{1,\Omega_s}. \tag{4.9}$$

The second term on the right-hand side of (4.8) is bounded, so

$$\sum_{s=1}^{2}\sum_{i=1}^{2} \left\| u_{h,s}^{i,k} - u_{h,s}^{*} \right\|_{1,\Omega_s} \leqslant C \sum_{s=1}^{2} \eta_{\Gamma_s}.$$

To bound the first term on the right-hand side of (4.8), we use the residual equation and apply the technique of the residual a posteriori error estimation [1], and we get for $v_h^i \in K_h$

$$\begin{cases} b^i\left(u_s^{i,k} - u_{h,s}^{i,k}, v_s^i\right) = b^i(u_s^{i,k} - u_{h,s}^{i,k}, v_s^i - v_{h,s}^i\Big) \\ \le \sum\limits_{T \subset \Omega_s} \underset{T}{dint} \left(F^i\left(u_{h,s}^{i,k}\right) + u_{h,s}^{i,k-1} + \lambda \Delta u_{h,s}^{i,k} - \left(1 + \lambda a_{h0}^{i,k}\right) u_{h,s}^{i,k}\right) \left(v_s^i - v_{h,s}^i\right) ds \\ - \sum\limits_{E \subset \Omega_s} \underset{E}{dint} \left[\dfrac{\partial u_{h,s}^{i,k}}{\partial \eta_E}\right] \left(v_s^i - v_{h,s}^i\right) ds - \sum\limits_{E \subset \Gamma_s} \underset{E}{dint} \dfrac{\partial u_{h,s}^{i,k}}{\partial \eta_E} \left(v_s^i - v_{h,s}^i\right) ds' \\ + \sum\limits_{E \subset \Omega_s} \underset{T}{dint} \left(F^i\left(u_s^{i,k}\right) - F^i\left(u_{h,s}^{i,k}\right)\right) (v_s^i - v_{h,s}^i) d\sigma + \left(\dfrac{\partial u_{h,s}^{i,k}}{\partial \eta_s}, v_s^i - v_{h,s}^i\right)_{\Gamma_s}. \end{cases}$$

Since $F^i\left(u_{h,s}^{i,k}\right)$ is an approximation of $F^i\left(u_s^{i,k}\right)$, we have

$$\begin{aligned} \sum_{s=1}^{2} b^i\left(u_s^{i,k} - u_{h,s}^{i,k}, v_s^i\right) \le{}& \sum_{s=1}^{2}\sum_{T \subset \Omega_s} \left\| F^i\left(u_{h,s}^{i,k}\right) + u_{h,s}^{i,k-1} + \lambda \Delta u_{h,s}^{i,k} - \left(1 + \lambda a_{h0}^{i,k}\right) u_{h,s}^{i,k} \right\|_{0,T} \left\| v_s^i - v_{h,s}^i \right\|_{0,T} \\ &+ \sum_{s=1}^{2} \sum_{E \subset \Omega_s} \left\| \left[\frac{\partial u_{h,s}^{i,k}}{\partial \eta_E}\right] \right\|_{0,E} \left\| v_s^i - v_{h,s}^i \right\|_{0,E} + \sum_{s=1}^{2}\sum_{E \subset \Gamma_s} \left\| \frac{\partial u_{h,s}^{i,k}}{\partial \eta_E} \right\|_{0,E} \left\| v_s^i - v_{h,s}^i \right\|_{0,E} \\ &+ \sum_{s=1}^{2}\sum_{T \subset \Omega_s} c \left\| u_s^{i,k} - u_{h,s}^{i,k} \right\|_{0,T} \left\| v_s^i - v_{h,s}^i \right\|_{0,T} + \sum_{s=1}^{2}\sum_{T \subset \Omega_s} \left\| \frac{\partial u_{h,s}^{i,k}}{\partial \eta_s} \right\|_{0,T} \left\| v_s^i - v_{h,s}^i \right\|_{0,T}. \end{aligned} \tag{4.10}$$

Using the fact

$$\left\| u_s^{i,k} - u_{h,s}^{i,k} \right\|_{1,\Omega_s} \leqslant \sup_{v_s^i \in K} \frac{b^i\left(u_s^{i,k} - u_{h,s}^{i,k}, v_s^i + ch_s^T\right)}{\| v_s^i + ch_s^T \|_{1,\Omega_i}},$$

we get

$$\sum_{s=1}^{2} b^i\left(u_s^{i,k}-u_{h,s}^{i,k},v_s^i+ch_s^T\right)\leq\sum_{s=1}^{2}\left(\sum_{T\subset\Omega_s}\eta_s^T\right)\sum_{s=1}^{2}\left\|v_s^i\right\|_{1,\Omega_s}. \tag{4.11}$$

Finally, by combining (4.8), (4.9), and (4.11) the required result follows. □

5. An asymptotic behavior for the problem

5.1. A fixed point mapping associated with discrete problem

We define for $i=1,2,3,4$ the following mapping

$$\begin{aligned} &T_h:K_h\longrightarrow H_0^1\left(\Omega_s\right)\\ &W_s\longrightarrow TW_s=\xi_{h,s}^{k,m+1}=\partial_h\left(F\left(w_s\right)\right), \end{aligned} \tag{5.1}$$

where $\xi_{h,s}^k$ is the solution to the following problem

$$\begin{aligned} &b_s^i(\xi_{s,h}^{i,k,m+1},v_s^i-\xi_{s,h}^{i,k,m+1})+\left(\alpha_{s,h}\xi_{s,h}^{i,k,m+1},v_{s,h}^i-\xi_{s,h}^{i,k,m+1}\right)_{\Gamma s}\geq\left(F(w_s),v_{s,h}-\xi_{s,h}^{i,k,m+1}\right)_{\Omega_s},\\ &\xi_{s,h}^{i,k,m+1}=0,\quad \text{on } \partial\Omega_s\cap\partial\Omega,\\ &\frac{\partial\xi_{s,h}^{k,m+1}}{\partial\eta_s}+\alpha_i\xi_{s,h}^{k,m+1}=\frac{\partial\xi_{t,h}^{k,m}}{\partial\eta_s}+\alpha_s\xi_{t,h}^{k,m},\ \text{on } \Gamma_s,\ s=1,\ldots,4,\ t=1,2. \end{aligned} \tag{5.2}$$

5.2. An iterative discrete algorithm

Choosing $u_{h,s}^{i,0}=u_{h0,s}^i\in\left(H_0^1\left(\Omega_s\right)\cap C\left(\Omega_s\right)\right)^M,\ i=1,\ldots,M$, we get a solution of the following discrete equation

$$\Delta u_{h,s}^{i,0}+a_0^iu_{h,s}^{i,0}=g_h^{i,0}, \tag{5.3}$$

where $g^{i,0}$ is a regular function. Now we give the following discrete algorithm

$$u_{s,h}^{i,k,m+1}=T_hu_{s,h}^{i,k-1,m+1},k=1,\ldots,n,\ i=1,\ldots,M,s=1,\ldots,4,$$

where $u_{hs}^{i,k}$ is a solution to problem (5.2).

Proposition 5.1. *Let $\xi_{hs}^{i,k}$ be a solution to problem (5.2) with the right-hand side $F^i\left(w_s^i\right)$ and the boundary condition $\dfrac{\partial\xi_{s,h}^{i,k,m+1}}{\partial\eta_i}+\alpha_i\xi_{h,s}^{i,k,m+1}$, $\tilde{\xi}_h^{i,k}$the solution for $\tilde{F}^i\left(\tilde{w}_s^i\right)$ and $\dfrac{\partial\tilde{\xi}_{i,h}^{k,m+1}}{\partial\eta_i}+\alpha_i\tilde{\xi}_{i,h}^{k,m+1}$. The mapping T_h is a contraction in K_h with the rate of contraction $\dfrac{\lambda+c}{\beta+\lambda}$. Therefore, T_h admits a unique fixed point which coincides with the solution to problem (5.2).*

Proof. We note that

$$\|W\|_{H_0^1(\Omega_i)}=\|W\|_1.$$

Set

$$\phi=\frac{1}{\beta+\lambda}\left\|F\left(w_s^i\right)-F\left(\tilde{w}_s^i\right)\right\|_1.$$

Then, $\xi_{h,s}^{i,k,m+1} + \phi$ is a solution of

$$\begin{cases} b\left(\xi_{h,s}^{i,k,m+1} + \phi, v_{h,s}^i + \phi\right) \leq \left(F(w_s^i) + a_{0i}^i \phi, v_{h,s}^i + \phi\right), \\ \xi_{h,s}^{i,k,m+1} = 0, \quad \text{on } \partial\Omega_s \cap \partial\Omega, \\ \dfrac{\partial \xi_{h,s}^{i,k,m+1}}{\partial \eta_s} + \alpha_s \xi_{h,s}^{i,k,m+1} = \dfrac{\partial \xi_{h,t}^{i,k,m}}{\partial \eta_s} + \alpha_s \xi_{h,s}^{i,k,m}, \text{ on } \Gamma_s, \ s = 1, ..., 4, \ t = 1, 2. \end{cases}$$

From an assumption in [18], we have

$$\begin{aligned} F(w_s^i) &\leq F\left(\tilde{w}_s^i\right) + \left\|F(w_s^i) - F\left(\tilde{w}_s^i\right)\right\|_1 \\ &\leq F\left(\tilde{w}_s^i\right) + \frac{a_0^i}{\beta + \lambda} \left\|F(w_i) - F\left(\tilde{w}_s^i\right)\right\|_1 \\ &\leq F\left(\tilde{w}_s^i\right) + a_0^i \phi. \end{aligned}$$

We know by [8] that if $F^i(w_s^i) \geqq F^i\left(\tilde{w}_s^i\right)$, then $\xi_{h,s}^{i,k,m+1} \geqq \tilde{\xi}_{h,s}^{i,k,m+1}$. Thus

$$\xi_{h,s}^{i,k,m+1} \leq \tilde{\xi}_{h,s}^{i,k,m+1} + \phi.$$

But the roles of w_s^i and $\tilde{w}_s^i$ are symmetric, thus we have a similar proof, i.e.,

$$\tilde{\xi}_{h,s}^{i,k,m+1} \leq \xi_{h,s}^{i,k,m+1} + \phi,$$

yields

$$\begin{aligned} \left\|T\left(w_s^i\right) - T\left(\tilde{w}_s^i\right)\right\|_\infty &\leq \frac{1}{\beta + \lambda} \left\|F(w_s^i) - F\left(\tilde{w}_s^i\right)\right\|_1 \\ &= \frac{1}{\beta + \lambda} \left\|f^i\left(w_s^i\right) + \lambda w_i - f^i\left(\tilde{w}_s^i\right) - \lambda \tilde{w}_i\right\|_1 \\ &\leq \frac{\lambda + c}{\beta + \lambda} \left\|w_i - \tilde{w}_i\right\|_1. \end{aligned}$$

□

Proposition 5.2. *Under the previous hypotheses and notations, we have the following estimate:*

$$\left\|u_{h,s}^{i,n,m+1} - u_{h,s}^{i,\infty,m+1}\right\|_1 \leq \left(\frac{1 + c\,(\Delta t)}{1 + \beta\,(\Delta t)}\right)^n \left\|u_{h,s}^{i,\infty,m+1} - u_{s,h_0}^i\right\|_1, \ k = 0, \ldots, n, \tag{5.4}$$

where $u_{h,s}^{i,\infty,m+1}$ is an asymptotic continuous solution and $u_{h_0}^i$ is a solution of (5.3)*.*

Proof. The proof is similar to that in [7] which has treated an evolutionary HJB equation with nonlinear source terms. □

Theorem 5.3. *Under the previous hypotheses and notations, we have*

$$\sum_{s=1}^{2} \left\|u_{h,s}^{i,n,m+1} - u^{i,\infty}\right\|_1 \leq C \left(\sum_{s=1}^{2} \sum_{T \in \tau_h} \left(\eta_s^T + \eta_{\Gamma_s}\right) + \left(\frac{1 + c\,(\Delta t)}{1 + \beta\,(\Delta t)}\right)^n\right). \tag{5.5}$$

Proof. Using Theorem 4.2 and Proposition 5.2, we get (5.5). □

6. Numerical example

In this section we give a simple numerical example. We consider the following evolutionary HJB equation:

$$\begin{cases} \max\limits_{1\leq\ i\ \leq\ 2}\left(\dfrac{\partial u^i}{\partial t}+A^iu^i-f^i\right)=0,\ \text{in } \Omega\times[0,T]\,,\\ u(0,t)\ \text{in } \Omega=0,\end{cases}$$

where $\Omega=]0,1[\,,\ u\,(0,x)=0,\ T=1$ and

$$A^1u=\frac{\partial^2u}{\partial x^2},\ A^2u=\frac{\partial^2u}{\partial x^2}+u,\quad f^1\,(u)=f^2\,(u)=\max\left(A^1u,A^2u\right).$$

The exact solution to the problem is

$$u\,(x,t)=\left(x^4-x^5\right)\sin\left(10x\right)\cos\left(20\pi t\right).$$

For the finite element approximation, we take uniform partition and linear conforming element. For the domain decomposition, we use the following decomposition

$$\Omega_1=]0,\ 0.55[\,,\ \Omega_2=]0.45,\ 1[\,.$$

We compute the bilinear Euler scheme combined with Galerkin solution in Ω and we apply the generalized overlapping domain decomposition method to compute the bilinear sequences $u_{h,s}^{i,k,m+1}$, $(s=1,2)$ to be able to look the behavior of the constant C, where the space steps are $h=\frac{1}{10},\ \frac{1}{100}$ and $\frac{1}{1000}$, and the steps of the time discretization are $\Delta t=\frac{1}{10},\ \frac{1}{50}$ and $\frac{1}{100}$. We denote

$$E_s=\left\|u_s^{i,k}-u_{h,s}^{i,k,m}\right\|_{1,\Omega_s}\ \text{and}\ T_1=\left\|u_{h,1}^{i,k,m+1}-u_{h,2}^{i,k,m}\right\|_{W_h^1}$$

and

$$T_2=\left\|u_{h,2}^{i,k,m}-u_{h,1}^{i,k,m-1}\right\|_{W_h^2}.$$

The generalized overlapping domain decomposition method with $\alpha_1=\alpha_2=0.55$ converges. The iterations stop when the relative error between two subsequent iterates is less than 10^{-6}, and we get

$$\Delta t\ =\ \frac{1}{10},$$

h	$1/10$	$1/100$	$1/1000$
E_s	$0.5081043\,(-4)$	$0.264825\,(-6)$	$0.4725905\,(-6)$
E_s	$0.6265874\,(-4)$	$03852017\,(-6)$	$0.3837247\,(-6)$
T_1	$0.9650827\,(-4)$	$0.573981\,(-6)$	$0.1286211\,(-6)$
T_2	$0.892843\,(-4)$	$0.6418371\,(-6)$	$0.9430526\,(-6)$
Iterations	8	14	20

$$\Delta t\ =\ \frac{1}{20}$$

h	$1/10$	$1/100$	$1/1000$
E_s	$0.4759595\,(-3)$	$0.8496273\,(-4)$	$0.9482601\,(-4)$
E_s	$0.5083649\,(-3)$	$0.7892758\,(-4)$	$0.8542894\,(-4)$
T_1	$0.7592478\,(-3)$	$0.927307\,(-4)$	$0.9785809\,(-4)$
T_2	$0.8584208\,(-3)$	$0.855012\,(-4)$	$0.9438526\,(-4)$
Iterations	8	14	20

$$\Delta t\ =\ 1/40$$

h	1/10	1/100	1/1000
E_s	0.9276183 (−2)	0.2937842 (−3)	0.8297682 (−4)
E_s	0.8524725 (−2)	0.2572064 (−3)	0.87085497 (−4)
T_1	0.9793482 (−2)	0.6079027 (−3)	0.5433127 (−4)
T_2	0.7582921 (−2)	0.51975802 (−3)	0.517528 (−4)
Iterations	8	14	20

Finally, we can deduce the asymptotic behavior

$$As = \sum_{s=1}^{2} \left\| u_{h,s}^{i,n,m+1} - u^{i,\infty} \right\|_1 \text{ for } \Delta t = 1/1000 \text{ ie., } n = 1000$$

as the following result

h	1/10	1/100	1/1000
As	0.5218747 (−3)	0.2519226 (−4)	0.1500514 (−4)
Iterations	8	14	20

In the tables above we also see that the iteration number is roughly related to h and Δt, and the order of convergence is in a good agreement with our estimates (5.5). Using adequate assumption, we can prove that

$$u^{i,\infty} \leq u_{h,s}^{i,1000,m+1} + \sum_{s=1}^{2} \sum_{T \in \tau_h} \left(\eta_s^T + \eta_{\Gamma_s} \right) + 1,$$

where $c = \beta = 1$ without the discrete maximum principle assumption [12].

Conclusion

In this paper, a posteriori error estimates for the generalized overlapping domain decomposition method with Dirichlet boundary conditions on the boundaries for the discrete solutions on subdomains of evolutionary HJB equation with nonlinear source terms are proved using Euler time scheme combined with a finite element spatial approximation. Also the techniques of the residual a posteriori error analysis are used. Then a result of asymptotic behavior in Sobolev space is deduced using Benssoussan–Lions' algorithm. Furthermore the results of some numerical experiments are presented to support the theory. In the second part an optimal error estimate with an asymptotic behavior will be given for an evolutionary HJB equation with linear source terms and respect to the same proposed boundary conditions, using the discontinuous Galerkin methods coupled with a theta time discretization scheme and the numerical example will be shown to prove that the new presented scheme is efficient.

Acknowledgement

The author wishes to thank deeply the anonymous referees and the handling editor for their useful remarks.

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