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Monotone hybrid methods for a common solution problem in Hilbert spaces

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Abstract

The purpose of this article is to investigate generalized mixed equilibrium problems and uniformly L-Lipschitz continuous asymptotically κ-strict pseudocontractions in the intermediate sense based on a monotone hybrid method. Strong convergence theorems of common solutions are established in the framework of Hilbert spaces.

Keywords: Asymptotically strict pseudocontraction, asymptotically nonexpansive mapping, generalized mixed equilibrium problem, solution, fixed point.
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1. Introduction-preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let $Proj_C$ be the metric projection onto C. From now on, we use $\rightharpoonup$ and $\rightarrow$ to stand for the weak convergence and strong convergence, respectively.

Let S be a mapping on C. In this paper, we use $F(S)$ to denote the set of fixed points of S. Recall that S is said to be closed at y if for any sequence $\{x_n\} \subset C$ such that $x_n \to x$ and $Sx_n \to y$, then $Sx = y$. Recall that S is said to be demiclosed at y if for any sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ and $Sx_n \to y$, then $Sx = y$. Recall that S is said to be nonexpansive iff

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in C.$$

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S is said to be asymptotically nonexpansive iff there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|S^n x - S^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, \ n \geq 1.$$

Asymptotically nonexpansive mapping was introduced by Goebel and Kirk [10] as a generalization of nonexpansive mappings. If C is weakly compact, then $F(T)$ is not empty.

S is said to be asymptotically nonexpansive in the intermediate sense iff it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|S^n x - S^n y\| - \|x - y\|) \leq 0. \tag{1.1}$$

Setting $\mu_n = \max\{0, \sup_{x,y\in C} (\|S^n x - S^n y\| - \|x - y\|)\}$, we see that $\mu_n \to 0$ as $n \to \infty$. Then (1.1) is reduced to the following:

$$\|S^n x - S^n y\| \leq \|x - y\| + \mu_n, \quad \forall x, y \in C. \tag{1.2}$$

Asymptotically nonexpansive mappings in the intermediate sense were introduced by Kirk [14] as a generalization of asymptotically nonexpansive mappings; see also [5].

S is said to be strictly pseudocontractive iff there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa \|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

Strict pseudocontractions is introduced by Brower and Petryshyn [4] as a generalization of the class of nonexpansive mappings. It is clear that every nonexpansive mapping is 0-strictly pseudocontractive.

S is said to be an asymptotically strict pseudocontraction iff there exist a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ and a constant $\kappa \in [0, 1)$ such that

$$\|S^n x - S^n y\|^2 \leq k_n \|x - y\|^2 + \kappa \|(I - S^n)x - (I - S^n)y\|^2, \quad \forall x, y \in C, \ n \geq 1.$$

Asymptotically strict pseudocontractions were introduced by Qihou [17] as a generalization of strict pseudocontractions. Note that both asymptotically strict pseudocontractions and strict pseudocontractions are Lipschitz continuous.

S is said to be an asymptotically strict pseudocontraction in the intermediate sense if there exist a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ and a constant $\kappa \in [0, 1)$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|S^n x - S^n y\|^2 - k_n \|x - y\|^2 - \kappa \|(I - S^n)x - (I - S^n)y\|^2) \leq 0. \tag{1.3}$$

Setting

$$\mu_n = \max\{0, \sup_{x,y\in C} (\|S^n x - S^n y\|^2 - k_n \|x - y\|^2 - \kappa \|(I - S^n)x - (I - S^n)y\|^2)\},$$

we see that $\mu_n \to 0$ as $n \to \infty$. Then (1.3) is reduced to the following:

$$\|S^n x - S^n y\|^2 \leq k_n \|x - y\|^2 + \kappa \|(I - S^n)x - (I - S^n)y\|^2 + \mu_n, \quad \forall x, y \in C, \ n \geq 1. \tag{1.4}$$

Asymptotically strict pseudocontractions in the intermediate sense were introduced by Sahu, Xu and Yao [21] as a generalization of asymptotically strict pseudocontractions.

From now on, $\mathbb{R}$ stands for the set of real numbers. Let B be a bifunction of $C \times C$ into $\mathbb{R}$. Consider the problem: find a p such that

$$B(p, y) \geq 0, \quad \forall y \in C. \tag{1.5}$$

The solution set of the problem is denoted by $EP(B)$. The problem was first introduced by Ky Fan [9]. In the sense of Blum and Oettli [3], the Ky Fan is also called an equilibrium problem.

Equilibrium problems are revealed as an powerful and effective mathematical modelling for studying real world problems which arise in many subjects, for instance, network, finance, transportation, elasticity and optimization; see [1], [2], [6], [7], [8], [11], [16], [22], [23], [24] and the references therein. It is known that the generalized mixed equilibrium problem is to find $p \in C$ such that

$$B(p, y) + \langle Ap, y - p\rangle + \varphi(y) - \varphi(p) \geq 0, \quad \forall y \in C, \tag{1.6}$$

where $\varphi : C \to \mathbb{R}$ is a real valued function and $A : C \to H$ is a nonlinear mapping. We use $GMEP(B, A, \varphi)$ to denote the solution set of the equilibrium problem. That is,

$$GMEP(B, A, \varphi) := \{p \in C : B(p, y) + \langle Ap, y - p\rangle + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C\}.$$

Next, we give some special cases:

If $A = 0$ and $\varphi = 0$, then problem (1.6) is equivalent to (1.5).

If $B = 0$, then problem (1.6) is equivalent to find $p \in C$ such that

$$\langle Ap, y - p\rangle + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \tag{1.7}$$

which is called the mixed variational inequality of Browder type.

If $A = 0$, then problem (1.6) is equivalent to find $p \in C$ such that

$$B(p, y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \tag{1.8}$$

which is called the mixed equilibrium problem.

If $\varphi = 0$, then problem (1.6) is equivalent to find $p \in C$ such that

$$B(p, y) + \langle Ap, y - p\rangle \geq 0, \quad \forall y \in C, \tag{1.9}$$

which is called the generalized equilibrium problem.

For solving the above equilibrium problems, let us assume that bifunction $B : C \times C \to \mathbb{R}$ satisfies the following conditions:

(B-1) B is monotone;
(B-2) $B(x, x) = 0, \forall x \in C$;
(B-3) for each $x \in C$, $y \mapsto B(x, y)$ is convex and weakly lower semi-continuous;
(B-4) $\limsup_{t\downarrow 0} B(tz + (1 - t)x, y) \leq B(x, y)$, $\forall x, y, z \in C$.

Recently, above problems have been extensively investigated based on iterative methods in different spaces by many authors; see, for instance, [8], [12], [13], [15], [19, 20, 25], and [26]–[29]. In this paper, motivated by the research mentioned above, we investigate fixed points of an asymptotically strict pseudo-contraction in the intermediate sense and generalized mixed equilibrium problem (1.6) based an a hybrid method.

Next, we give the following tools which play an important role in our paper.

Lemma 1.1 ([18], [21]). *Let $S : C \to C$ be a uniformly L-Lipschitz continuous and asymptotically strict pseudocontraction in the intermediate sense. Then the mapping $I - S$ is demiclosed at zero, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \bar{x}$ and $x_n - Sx_n \to 0$, then $\bar{x} \in F(S)$.*

Lemma 1.2 ([18], [21]). *Let C be a nonempty closed convex subset of H. Let $S : C \to C$ be an asymptotically κ-strict pseudocontraction in the intermediate sense. Then $F(S)$ is closed and convex.*

Lemma 1.3 ([3]). *Let C be a nonempty closed convex subset of H and let $B : C \times C \to \mathbb{R}$ be a bifunction with (B-1), (B-2), (B-3) and (B-4) . Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$B(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \Big\{z \in C : B(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\Big\}$$

for all $r > 0$ and $x \in H$. Then, the following hold:

(a) T_r *is single-valued firmly nonexpansive;*
(b) $EP(B)$ *is convex and closed;*
(c) $F(T_r) = EP(B)$.

2. Main results

Theorem 2.1. *Let* C *be a nonempty closed convex subset of* H. *Let* B *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *enjoys (B-1), (B-2), (B-3) and (B-4). Let* $\varphi : C \to \mathbb{R}$ *be a lower semicontinuous and convex function and let* $A : C \to H$ *be a continuous and monotone mapping. Let* S *be a uniformly* L*-Lipschitz continuous and asymptotically* κ*-strict pseudocontraction in the intermediate sense on* C. *Assume that the common solution set* $GMEP(B, A, \varphi) \cap F(S)$ *is nonempty and bounded. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0, 1]$. *Let* $\{r_n\}$ *be a real sequence in* $(0, \infty)$. *Let* $\{x_n\}$ *be a sequence generated as follows:*

$$\begin{cases} x_1 \in H, C_1 = C, \\ B(u_n, u) + \varphi(u) - \varphi(u_n) + \langle Au_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ y_n = (1 - \alpha_n)\big(\beta_n u_n + (1 - \beta_n)S^n u_n\big) + \alpha_n x_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \leq (k_n - 1)\sup\{\|x_n - w\|^2 : w \in GMEP(B, A, \varphi) \cap F(S)\} \\ \qquad\qquad + \mu_n + \|x_n - w\|^2\}, \\ x_{n+1} = Proj_{C_{n+1}} x_1, \quad n \geq 1, \end{cases}$$

where $Proj_{C_{n+1}}$ *is the metric projection from* H *onto* C_{n+1}. *Assume that the control sequences* $\{r_n\}$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following restrictions:* $0 \leq \alpha_n \leq a < 1$, $\liminf_{n\to\infty} r_n > 0$ *and* $0 < \kappa \leq \beta_n \leq b < 1$, *where* a *and* b *are two real numbers. Then sequence* $\{x_n\}$ *converges strongly to* $\bar{x} = P_{GMEP(B,A,\varphi)\cap F(S)} x_1$.

Proof. First, we define $G(p, y) = B(p, y) + \langle Ap, y - p\rangle + \varphi(y) - \varphi(p)$, $\forall p, y \in C$. Next, we prove that bifunction G satisfies (B-1), (B-2), (B-3) and (B-4). Therefore, generalized mixed equilibrium problem is equivalent to the following equilibrium problem:

$$\text{find } p \in C \text{ such that } G(p, y) \geq 0, \forall y \in C.$$

Next, we prove G is monotone. Since A is a continuous and monotone operator, we find from the definition of G that

$$\begin{aligned} G(y, z) + G(z, y) &= B(y, z) + \langle Ay, z - y\rangle + \varphi(z) - \varphi(y) + B(z, y) \\ &\quad + \langle Az, y - z\rangle + \varphi(y) - \varphi(z) \\ &= B(z, y) + \langle Az, y - z\rangle + B(y, z) + \langle Ay, z - y\rangle \\ &\leq \langle Az - Ay, y - z\rangle \\ &\leq 0. \end{aligned}$$

It is clear that G satisfies (B-2). Next, we show that for each $x \in C$, $y \longmapsto G(x, y)$ is convex and lower semicontinuous. For each $x \in C$, for all $t \in (0, 1)$ and for all $y, z \in C$, since φ is convex, we have

$$\begin{aligned} G(x, ty + (1 - t)z) &= B(x, ty + (1 - t)z) + \langle Ax, ty + (1 - t)z - x\rangle + \varphi(ty + (1 - t)z) - \varphi(x) \\ &\leq t\big(B(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle\big) \\ &\quad + (1 - t)\big(B(x, z) + \varphi(z) - \varphi(x) + \langle Ax, z - x\rangle\big) \\ &= (1 - t)G(x, z) + tG(x, y). \end{aligned}$$

So, $y \longmapsto G(x, y)$ is convex. Similarly, we find that $y \longmapsto G(x, y)$ is also lower semicontinuous. Next, we show G satisfies (B-4), that is,

$$\limsup_{t\downarrow 0} G(tz + (1 - t)x, y) \leq G(x, y), \forall x, y, z \in C.$$

Since A is continuous and φ is lower semicontinuous, we have

$$\begin{aligned}
\limsup_{t\downarrow 0} G(tz+(1-t)x,y) &= \limsup_{t\downarrow 0} B(tz+(1-t)x,y) \\
&\quad + \limsup_{t\downarrow 0} \big(\varphi(y)-\varphi(tz+(1-t)x)big) \\
&\quad + \limsup_{t\downarrow 0} \langle A\big(tz+(1-t)x\big), y-\big(tz+(1-t)x\big)\rangle \\
&\le B(x,y)+\varphi(y)-\varphi(x)+\langle Ax, y-x\rangle \\
&= G(x,y).
\end{aligned}$$

Now, we are in a position to show that C_n is closed and convex. It is easy to see that C_n is closed. We only show that C_n is convex. Note that $C_1 = H$ is convex. Suppose that C_i is convex for some positive integer $i \ge 1$. Next, we show that C_{i+1} is convex for the same i. Put

$$\Theta_n = \sup\{\|x_n - w\|^2 : w \in GMEP(B,A,\varphi)\cap F(S)\}.$$

Since

$$\|y_i - w\|^2 \le (k_i-1)\Theta_i + \mu_i + \|x_i - w\|^2$$

is equivalent to

$$2\langle x_i - y_i, w\rangle \le \|x_i\|^2 - \|y_i\|^2 + (k_i-1)\Theta_i + \mu_i,$$

we take w_1 and w_2 in C_{i+1} and put $\bar{w} = tw_1 + (1-t)w_2$ to find that $w_1 \in C_i$, $w_2 \in C_i$,

$$2\langle x_i - y_i, w_1\rangle \le \|x_i\|^2 - \|y_i\|^2 + (k_i-1)\Theta_i + \mu_i$$

and

$$2\langle x_i - y_i, w_2\rangle \le \|x_i\|^2 - \|y_i\|^2 + (k_i-1)\Theta_i + \mu_i.$$

Using the above two inequalities, we obtain that $\|y_i - w\|^2 \le \|x_i - w\|^2 + (k_i-1)\Theta_i + \mu_i$. By use the convexity of C_i, we find that $\bar{w} \in C_i$. This shows that $\bar{w} \in C_{i+1}$. This yields that C_n is convex and closed. Next, we find that $GMEP(B,A,\varphi)\cap F(S) \subset C_n$. It is clear that $GMEP(B,A,\varphi)\cap F(S) \subset C_1 = H$. Assume that $GMEP(B,A,\varphi)\cap F(S) \subset C_h$ for some integer $h \ge 1$. We intend to claim that $GMEP(B,A,\varphi)\cap F(S) \subset C_{h+1}$ for the same h. For any $p \in GMEP(B,A,\varphi)\cap F(S) \subset C_h$, we have

$$\begin{aligned}
\|y_h - p\|^2 &\le \alpha_h\|x_h - p\|^2 + (1-\alpha_h)\|\beta_h(u_h - p) + (1-\beta_h)(S^h u_h - p)\|^2 \\
&\le \alpha_h\|x_h - p\|^2 + (1-\alpha_h)\Big(\beta_h\|u_h - p\|^2 \\
&\quad + (1-\beta_h)\big(k_h\|u_h - p\|^2 + \kappa\|u_h - S^h u_h\|^2 + \mu_h\big) - \beta_h(1-\beta_h)\|u_h - S^h u_h\|^2\Big) \\
&\le (1-\alpha_h)(k_h-1)\|x_h - p\|^2 + (1-\alpha_h)(1-\beta_h)(\kappa-\beta_h)\|u_h - S^h u_h\|^2 \\
&\quad + (1-\alpha_h)(1-\beta_h)\mu_h + \|x_h - p\|^2 \\
&\le (k_h-1)\Theta_h + \mu_h + \|x_h - p\|^2.
\end{aligned}$$

This implies that $GMEP(B,A,\varphi)\cap F(S) \subset C_n$. Since $x_n = P_{C_n}x_1$, we have $\|x_1 - x_n\| \le \|x_1 - p\|$. In particular, one has

$$\|x_1 - x_n\| \le \|x_1 - Proj_{GMEP(B,A,\varphi)\cap F(S)}x_1\|.$$

This obtains the boundedness of $\{x_n\}$. It follows that

$$0 \le \langle x_1 - x_n, x_n - x_{n+1}\rangle \le \|x_1 - x_n\|\|x_1 - x_{n+1}\| - \|x_1 - x_n\|^2.$$

This implies $\|x_n - x_1\| \le \|x_{n+1} - x_1\|$. Hence, we obtain that $\lim_{n\to\infty}\|x_n - x_1\|$ exists. Since

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\
&= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n + x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\
&= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\
&\le \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2,
\end{aligned}$$

we find that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{2.1}$$

By use of $x_{n+1} = Proj_{C_{n+1}} x_1 \in C_{n+1}$, we have

$$\|y_n - x_{n+1}\|^2 \leq (k_n - 1)\Theta_n + \mu_n + \|x_n - x_{n+1}\|^2.$$

Hence

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{2.2}$$

This together with (2.1) yield that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{2.3}$$

Since $\|x_n - y_n\| = (1-\alpha_n)\|x_n - \big(\beta_n u_n + (1-\beta_n)S^n u_n\big)\|$, we find from (2.3) and the restriction imposed on sequence $\{\alpha_n\}$ that

$$\lim_{n\to\infty} \|x_n - \big(\beta_n u_n + (1-\beta_n)S^n u_n\big)\| = 0. \tag{2.4}$$

For $p \in GMEP(B, A, \varphi) \cap F(S)$, one has

$$\|u_n - p\|^2 \leq \langle T_{r_n} x_n - T_{r_n} p, x_n - p\rangle = \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2).$$

Hence, we have

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - x_n\|^2. \tag{2.5}$$

It follows that

$$\begin{aligned}
\|y_n - p\|^2 &\leq \alpha_h \|x_n - p\|^2 + (1-\alpha_n)\big(\beta_n\|u_n - p\|^2 + (1-\beta_n)\|S^n u_n - S^n p\|^2 \\
&\quad - \beta_n(1-\beta_n)\|u_n - p - (S^n u_n - S^n p)\|^2\big) \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|u_n - p\|^2 + (1-\alpha_n)(1-\beta_n)\big(k_n\|u_n - p\|^2 \\
&\quad + \kappa\|u_n - p - (S^n u_n - S^n p)\|^2 + \mu_n\big) \\
&\quad - (1-\alpha_n)\beta_n(1-\beta_n)\|u_n - p - (S^n u_n - S^n p)\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)k_n\|u_n - p\|^2 \\
&\quad - (1-\alpha_n)(1-\beta_n)(\beta_n - \kappa)\|u_n - p - (S^n u_n - S^n p)\|^2 \\
&\leq (k_n - 1)\|x_n - p\|^2 - (1-\alpha_n)k_n\|u_n - x_n\|^2 + \|x_n - p\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
(1-\alpha_n)k_n\|u_n - x_n\|^2 &\leq (k_n - 1)\|x_n - p\|^2 - \|y_n - p\|^2 + \|x_n - p\|^2 \\
&\leq (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| + (k_n - 1)\|x_n - p\|^2.
\end{aligned}$$

By use of (2.3), we find that $\lim_{n\to\infty} \|u_n - x_n\| = 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q$. It follows $u_{n_i} \rightharpoonup q$. Using the restriction imposed on $\{r_n\}$, we may assume there exists a positive real number r such that $r_n \geq r$. Hence,

$$\lim_{n\to\infty} \frac{\|u_n - x_n\|}{r_n} = 0. \tag{2.6}$$

Next, we show that $q \in F(S)$. Note that

$$\begin{aligned}
\|x_n - \big(\beta_n x_n + (1-\beta_n)S^n x_n\big)\| &\leq \|x_n - \big(\beta_n u_n + (1-\beta_n)S^n u_n\big)\| + \beta_n\|u_n - x_n\| + (1-\beta_n)\|S^n u_n - S^n x_n\| \\
&\leq \|x_n - \big(\beta_n u_n + (1-\beta_n)S^n u_n\big)\| + L\|u_n - x_n\|.
\end{aligned}$$

It follows from (2.4) that

$$\lim_{n\to\infty} \|x_n - \big(\beta_n x_n + (1-\beta_n)S^n x_n\big)\| = 0. \tag{2.7}$$

Since

$$\begin{aligned}\|S^n x_n - x_n\| &\le \|S^n x_n - \big(\beta_n x_n + (1-\beta_n)S^n x_n\big)\| + \|\big(\beta_n x_n + (1-\beta_n)S^n x_n\big) - x_n\| \\ &\le \beta_n\|S^n x_n - x_n\| + \|\big(\beta_n x_n + (1-\beta_n)S^n x_n\big) - x_n\|,\end{aligned}$$

one sees from the restriction imposed on $\{\beta_n\}$ and (2.7) that

$$\lim_{n\to\infty} \|S^n x_n - x_n\| = 0. \tag{2.8}$$

Since S is Lipschitz continuous, one has

$$\begin{aligned}\|x_n - Sx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - S^{n+1}x_{n+1}\| + \|S^{n+1}x_{n+1} - S^{n+1}x_n\| + \|S^{n+1}x_n - Sx_n\| \\ &\le (1+L)\|x_n - x_{n+1}\| + \|x_{n+1} - S^{n+1}x_{n+1}\| + L\|S^n x_n - x_n\|.\end{aligned}$$

It follows from (2.1) and (2.8) that

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

Using Lemma 1.1, we obtain that $q \in F(S)$.

Next, we prove $q \in GMEP(B,A,\varphi) = EP(G)$. By use of (B-1), one has $\frac{1}{r_n}\langle u-u_n, u_n - x_n\rangle \ge G(u,u_n)$. Replacing n by n_i, we arrive at $\langle u - u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle \ge G(u,u_{n_i})$. Using assumption (B-3), we get from (2.6) that $G(u,q) \le 0$, $\forall u \in C$. For any t with $0 < t \le 1$ and $u \in C$, let $u_t = tu + (1-t)q$. Since $u \in C$ and $q \in C$, we have $u_t \in C$ and hence $G(u_t,q) \le 0$. It follows that

$$0 = G(u_t,u_t) \le tG(u_t,u) + (1-t)G(u_t,q) \le tG(u_t,u),$$

which yields that $G(u_t,u) \ge 0$, $\forall u \in C$. Letting $t \downarrow 0$, we obtain from assumption (B-4) that

$$G(q,u) \ge 0, \quad \forall u \in C.$$

This implies that $q \in EP(G) = GMEP(B,A,\varphi)$. This shows that $q \in GMEP(B,A,\varphi) \cap F(S)$. Since $\bar{x} = P_{GMEP(B,A,\varphi)\cap F(S)}x_1$, we obtain that

$$\|x_1 - \bar{x}\| \le \|x_1 - q\| \le \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \le \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \le \|x_1 - \bar{x}\|.$$

It follows that

$$\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - q\| = \|x_1 - \bar{x}\|.$$

Hence, $\{x_{n_i}\}$ converges strongly to $\bar{x}$. It follows that the sequence $\{x_n\}$ converges strongly to $\bar{x} = P_{GMEP(B,A,\varphi)\cap F(S)}x_1$. This completes the proof. □

Next, we give some subresults based on Theorem 2.1. First, we consider solutions of equilibrium problem (1.5).

Corollary 2.2. *Let C be a nonempty closed convex subset of H. Let B be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (B-1), (B-2), (B-3) and (B-4) and let $S : C \to C$ be a uniformly L-Lipschitz continuous and asymptotically κ-strict pseudocontraction in the intermediate sense. Assume that $EP(F)$ is nonempty and bounded. Let $\{\alpha_n\}$ be a sequence in $[0,1]$. Let $\{r_n\}$ be a real sequence in $(0,\infty)$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in H, \\ C_1 = C, \\ F(u_n,u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \ge 0, \quad \forall u \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)u_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\| \le \|x_n - w\|\}, \\ x_{n+1} = Proj_{C_{n+1}}x_1, \quad n \ge 1. \end{cases}$$

Assume that the control sequences $\{r_n\}$ *and* $\{\alpha_n\}$ *satisfy the following restrictions:* $\liminf_{n\to\infty} r_n > 0$ *and* $0 \le \alpha_n \le a < 1$, *where* a *is a real number. Then sequence* $\{x_n\}$ *converges strongly to* $\bar{x} = Proj_{EP(F)}x_1$.

Proof. Putting $\varphi = 0$, $A = 0$ and $S = I$, we find from Theorem 2.1 the desired conclusion. □

Next, we give a result on equilibrium problem (1.9).

Corollary 2.3. *Let* C *be a nonempty closed convex subset of* H. *Let* B *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies (B-1), (B-2), (B-3) and (B-4) and* $A : C \to H$ *a continuous and monotone mapping. Assume that* $EP(F, A)$ *is nonempty and bounded. Let* $\{\alpha_n\}$ *be a sequence in* $[0, 1]$. *Let* $\{r_n\}$ *be a real sequence in* $(0, \infty)$. *Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in H, \\ C_1 = C, \\ F(u_n, u) + \langle Au_n, u - u_n\rangle + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \ge 0, \quad \forall u \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)u_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\| \le \|x_n - w\|\}, \\ x_{n+1} = Proj_{C_{n+1}}x_1, \quad n \ge 1. \end{cases}$$

Assume that the control sequences $\{r_n\}$ *and* $\{\alpha_n\}$ *satisfy the following restrictions:* $\liminf_{n\to\infty} r_n > 0$, $0 \le \alpha_n \le a < 1$, *where* a *is a real number. Then sequence* $\{x_n\}$ *converges strongly to* $\bar{x} = Proj_{EP(F,A)}x_1$.

Finally, we give a result on fixed point of asymptotically κ-strict pseudocontraction in the intermediate sense.

Corollary 2.4. *Let* C *be a nonempty closed convex subset of* H. *Let* $S : C \to C$ *be a uniformly L-Lipschitz continuous and asymptotically* κ*-strict pseudocontraction in the intermediate sense with a nonempty and bounded fixed-point set. Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences in* $[0, 1]$. *Let* $\{x_n\}$ *be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)\big(\beta_n x_n + (1 - \beta_n)S^n x_n\big), \\ C_{n+1} = \{w \in C_n : \|y_n - w\|^2 \le (k_n - 1)\sup\{\|x_n - w\|^2 : w \in F(S)\} + \mu_n + \|x_n - w\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \ge 1. \end{cases}$$

Assume that the control sequences $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following restrictions:* $0 \le \alpha_n \le a < 1$ *and*
(b) $0 < \kappa \le \beta_n \le b < 1$, *where* a *and* b *are two real numbers. Then sequence* $\{x_n\}$ *converges strongly to* $\bar{x} = Proj_{F(S)}x_1$.

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